Lecture 8

Tuesday, October 1, 2019 3:31 PM

Guest Lecturer

Predator - Prey Model:

convenient form

$$\begin{cases} \dot{n} = f(n)\,[g(n) - p] & \leftarrow \text{prey} \\ \dot{p} = \beta\,[f(n) - \alpha]\,p & \leftarrow \text{predator} \end{cases} \qquad f(n) = \frac{n}{1+n} \;; \quad g(n) = (1+n)\left(1 - \frac{n}{\alpha}\right)$$

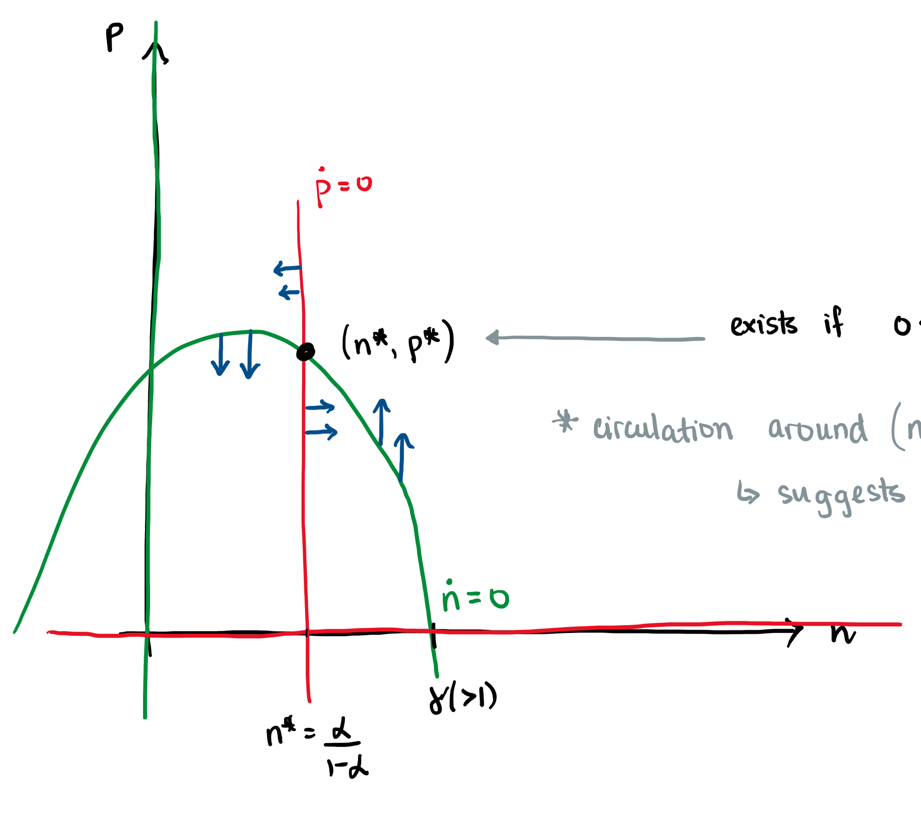
Nullclines

$\dot{n} = 0 \iff f(n) = 0$ or $p = g(n)$ (Sub in & solve for n.)

$\hookrightarrow n = 0$

$\dot{p} = 0 \iff p = 0$ or $f(n) = \alpha$

$\hookrightarrow n = \frac{\alpha}{1-\alpha}$



exists if $0 < n^* < \alpha$, in $\mathbb{R}^2_+$

* circulation around (n^*, p^*)

$\hookrightarrow$ suggests that there is a complex component

$$A = \begin{pmatrix} \partial_n \dot{n} & \partial_p \dot{n} \\ \partial_n \dot{p} & \partial_p \dot{p} \end{pmatrix} = \begin{pmatrix} f'[g-p] + fg' & -f \\ \beta f' p & \beta[f-a] \end{pmatrix}$$

- We want to know $A(n^*, p^*)$
- We know $p^* = g(n^*)$ and $f(n^*) = \alpha$.

So

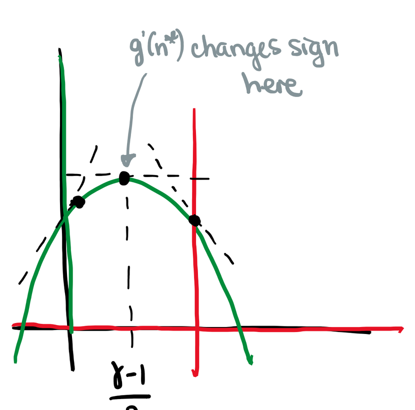
$$A(n^*, p^*) = \begin{bmatrix} f'(\underbrace{g-p}_{=0}) + fg' & -f \\ \beta f' p & \beta\underbrace{[f-a]}_{=0} \end{bmatrix}$$

$$= \begin{bmatrix} f(n^*)g'(n^*) & -f(n^*) \\ \beta f'(n^*)p & 0 \end{bmatrix}$$

$\leftarrow$ messy

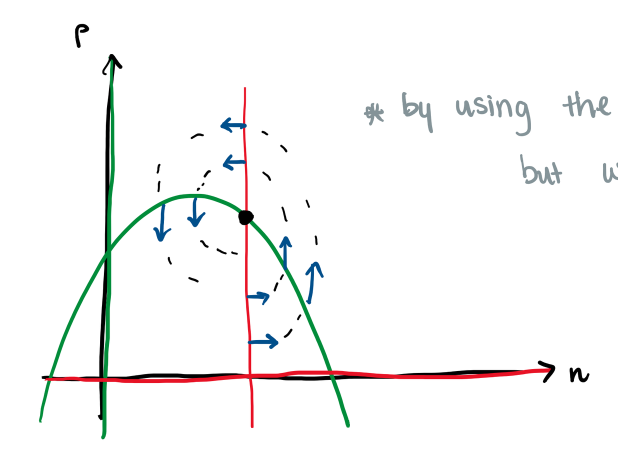
$\therefore$ use $Tr(A)$ & $Det(A)$ to make it easier

- $Tr(A(n^*, p^*)) = \alpha \overbrace{g'(n^*)}^{\text{slope of } g(n) \text{ at } (n^*, p^*)}$ $\Rightarrow$ changes sign when (n^*, p^*) is at vertex of $g(n)$
- $Det(A(n^*, p^*)) = \alpha\beta \underbrace{f'(n^*)}_{>0} \underbrace{p^*}_{>0}$ $\Rightarrow$ $Det(A) > 0$



Eigenvalues of $A(n^*, p^*)$ as n^* increase

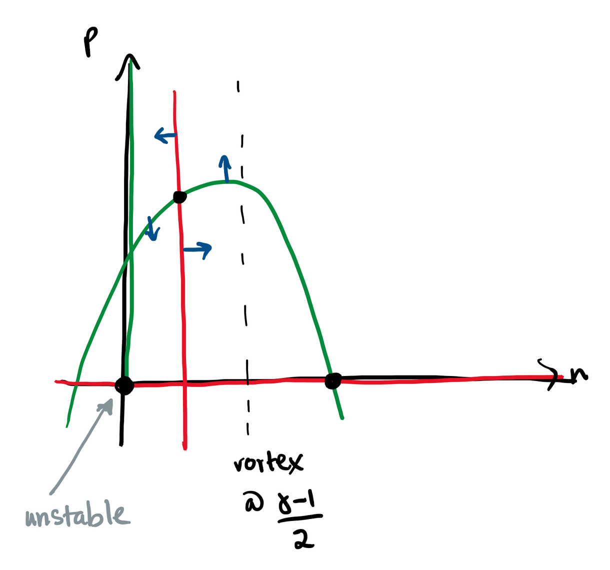
$\therefore$ eigenvalues cross the imaginary axis



* by using the flow lines, one can see its behavior but we don't know its stability.

(knows it spirals but don't know if it's stable)

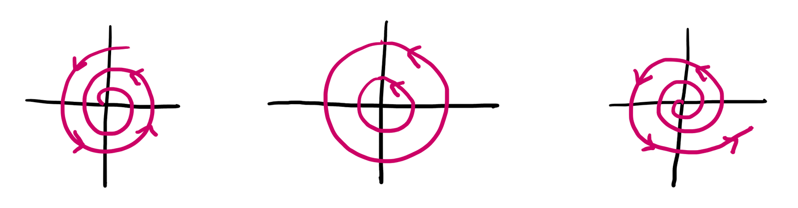
* analyzing A (the Jacobian) only gives local information



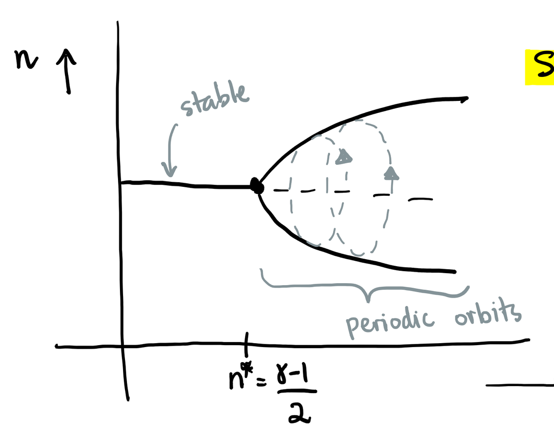
Question: What can happen when imaginary eigenvalues cross the real axis?

Linear System

- stable spiral $\rightarrow$ centre $\rightarrow$ unstable spiral



Non-Linear System



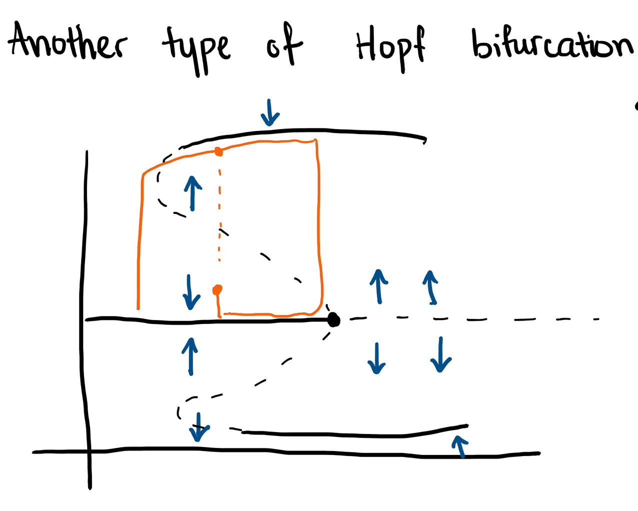
$\longrightarrow$ n^* decreasing (stable $\rightarrow$ unstable)

Super Critical Hopf Bifurcation - stable eq. loses stability giving way to a (locally) stable periodic orbit.

Another type of Hopf bifurcation (also requires non-linearity)

Subcritical Hopf Bifurcation

unstable periodic orbit collides w/ stable equilibrium, which loses stability.



— the orange line demonstrate how when we change the parameter & it becomes unstable, we need to change the parameter further back to reach a stable state again.

- - - even when we change the parameter back to the original, we can't go back to the stable state.

Example of Supercritical Hopf:

In polar coordinates:

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + br^2$$

μ, b are parameters

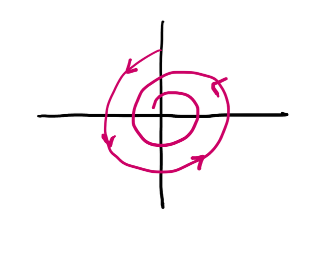
μ is a bifurcation parameter.

$\omega > 0$

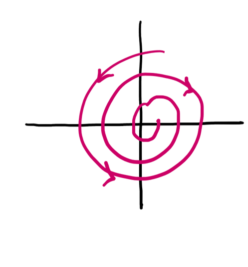
$\dot{r} = 0$ if $r(\mu - r^2) = 0 \iff r = 0, \sqrt{\mu}$

$\dot{\theta} > 0 \quad \forall\; \theta, r$

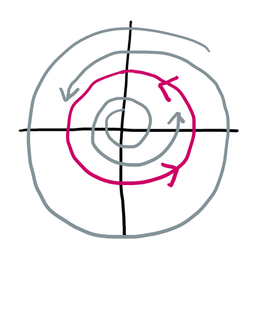
Case $\mu < 0$: stable spiral ($\dot{r} < 0 \;\; \forall\; r > 0$)



Case $\mu = 0$ $\quad \dot{r} = -r^3$ (marginally) stable spiral



Case $\mu > 0$ $\quad$ Stable orbit at $r = \sqrt{\mu}$



Check eigenvalues cross imaginary axis.

$$x = r\cos\theta$$
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$= (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta$$
$$= (\mu - (x^2+y^2))x - y(\omega + b(x^2+y^2))$$
$$= \mu x - \omega y + \text{cubic terms}$$

Similarly

$$\dot{y} = \omega x + \mu y + \text{cubic terms}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \mu & -\omega \\ \omega & -\mu \end{pmatrix}}_{\uparrow} \begin{pmatrix} x \\ y \end{pmatrix} + \text{cubic}$$

eigenvalues $\mu \pm i\omega$

Subcritical example

$$\dot{r} = \mu r + r^3 - r^5$$
$$\theta = \omega + br^2$$