Hopf Bifurcation Thm: Case $n=2$

Consider a system of two eqns w/ a parameter γ:

$$\begin{cases} \dot{x} = f(x,y;\gamma), \\ \dot{y} = g(x,y;\gamma), \end{cases} \quad . \quad . \quad . \quad . \quad (1)$$

where f & g are C' functions with respect to differentiation in x, y, and γ. Suppose that (1) has steady states

$$(x^*, y^*) \text{ where } \begin{cases} x^* = x^*(\gamma) \\ y^* = y^*(\gamma) \end{cases}$$

$\forall \gamma \in \mathbb{R}$. The Jacobian matrix at (x^*, y^*) is

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x^*,y^*)}$$

and suppose J has eigenvalues $\lambda(\gamma) = a(\gamma) \pm i b(\gamma)$. Also suppose that there is a value $\hat{\gamma}$, called the bifurcation value, such that

$$a(\hat{\gamma}) = 0, \quad b(\hat{\gamma}) \neq 0$$

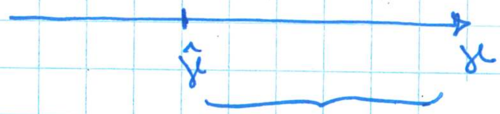
and $a(\hat{\gamma})$ changes sign as γ increases from values below $\hat{\gamma}$ to values above (i.e. $da/d\gamma \neq 0$ at $\gamma = \hat{\gamma}$).

Given these conditions, the following possibilities arise:

1. At $\gamma = \hat{\gamma}$ a centre is created & thus also infinitely many neutrally stable concentric periodic orbits surrounding $(x^*(\hat{\gamma}), y^*(\hat{\gamma}))$.

2. case (a) & case (b):

case (a)

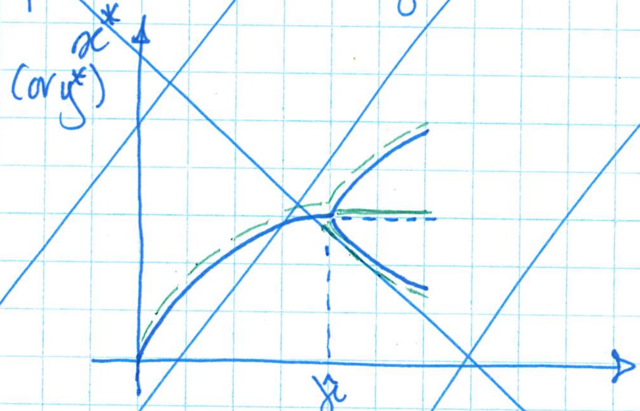


For γ values here, a single periodic orbit exists (limit cycle).
$\Rightarrow$ supercritical Hopf

case (b)

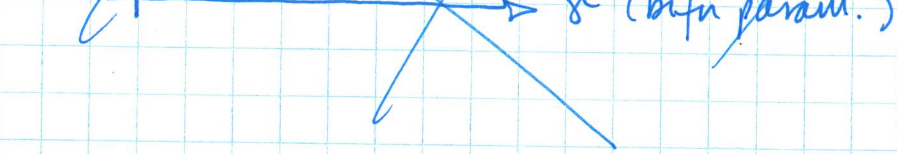
For γ values here, a single periodic orbit exists (limit cycle).
$\Rightarrow$ subcritical Hopf

Bifurcation Diagrams

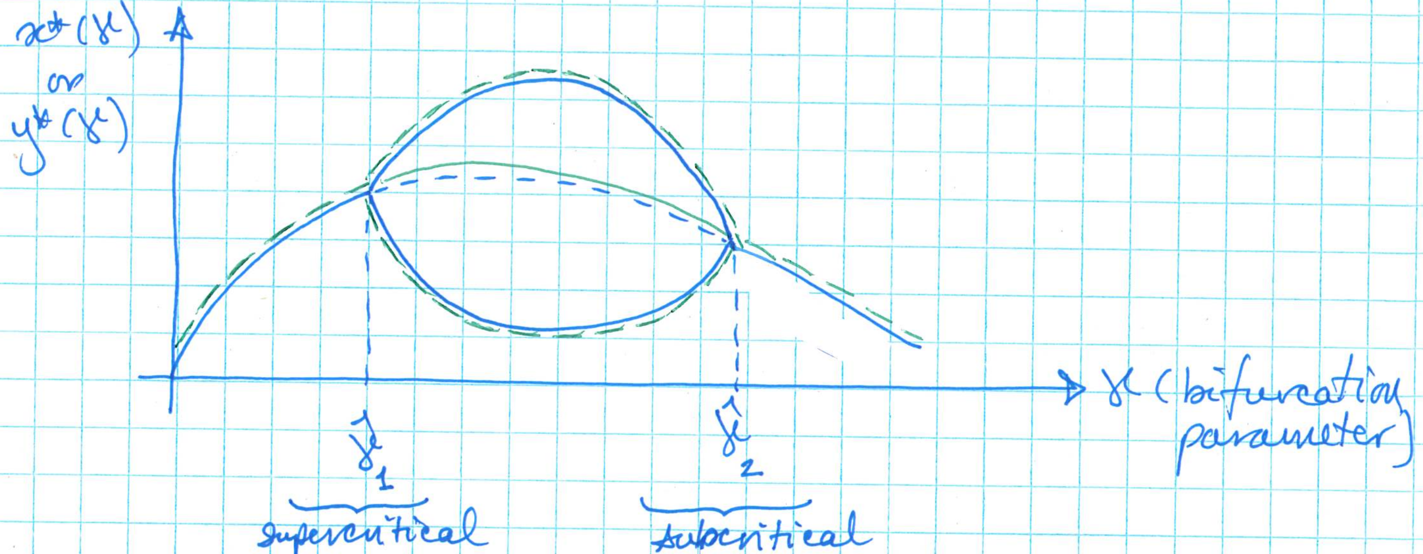


supercritical Hopf (or pitchfork bifn)

— stable steady-state $\rightarrow$ stable limit cycle
-- unstable " " $\rightarrow$ unstable " "
(stable steady-state)



Bifurcation Diagrams



— stable steady-state → stable limit cycle → stable steady state

— unstable steady state → unstable limit cycle → unstable steady state

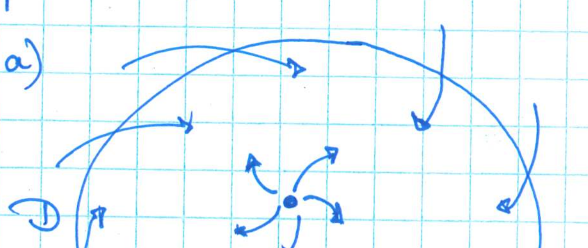
The Poincaré-Bendixson Theory

Existence of periodic orbits in the plane ($n=2$):

The Poincaré-Bendixson Theorem

If, for $t \geq t_0$, a trajectory is bounded and does not approach any singular point, then it is either a closed periodic orbit or approaches a closed periodic orbit as $t \to \infty$.

a)

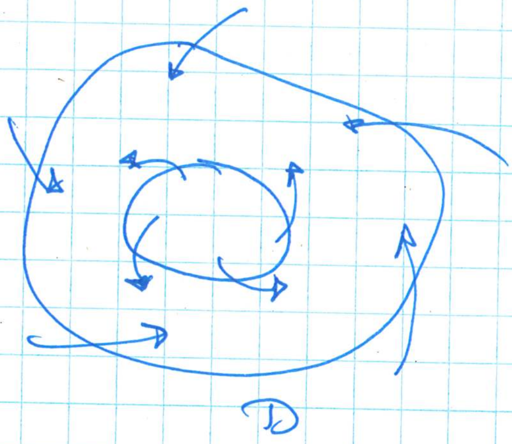


• = unstable node or focus

D = bounded region into which all flow enters

P-B Thm says a limit cycle must exist in D.

b)



P-B Thm says a limit cycle must exist (D contains no steady states).

In order to rule out the presence of a limit cycle, we can use the following two criteria:

1. Bendixson's criterion: Suppose D is a simply-connected region of the plane (i.e. D has no holes). If

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \qquad (\text{where } \dot{x} = f(x,y) \text{ and } \dot{y} = g(x,y))$$

is not identically 0 (i.e. not 0 $\forall (x,y) \in D$) and does not change sign in D, then there are no closed orbits in D.

2. Dulac's criterion: Suppose D is a simply-connected region in the plane & suppose $\exists$ a function $B(x,y)$, continuously differentiable in D, such that

$$\frac{\partial (Bf)}{\partial x} + \frac{\partial (Bg)}{\partial y}$$

is not identically zero & does not change sign in D. Then, there are no closed orbits in D.

Examples

1. Use Bendixson's negative criterion to indicate whether or not limit cycles can be ruled out in the system

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + dxy \end{cases} \quad \text{in } D = \{(x,y) \mid x > 0, y > 0\}$$

$(a, b, c, d \text{ all } \geq 0)$

Ans:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = a - by - c + dx$$

$$= (a-c) + (dx - by) = Q$$

$\therefore$ x + y are independent, it is impossible to guarantee that Q does not change sign in the positive quadrant. So we cannot use Bendixson's negative criterion to rule out limit cycles.

(P-B Thm example)

Ex **Brusselator** (hypothetical chem reacΘ)

$$\begin{cases} x' = a - bx + x^2y - x \\ y' = bx - x^2 y \end{cases} \qquad \text{require} \begin{cases} b > 1 + a^2 \\ a > 0 \end{cases}$$

SS: $\left(a, \frac{b}{a}\right)$

$$J\Big|_{SS} = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}$$

$$\begin{cases} \mathrm{Det}(J) = a^2 > 0 \\ \mathrm{Tr}(J) = b - 1 - a^2 > 0 \end{cases} \qquad \therefore \text{unstable}$$

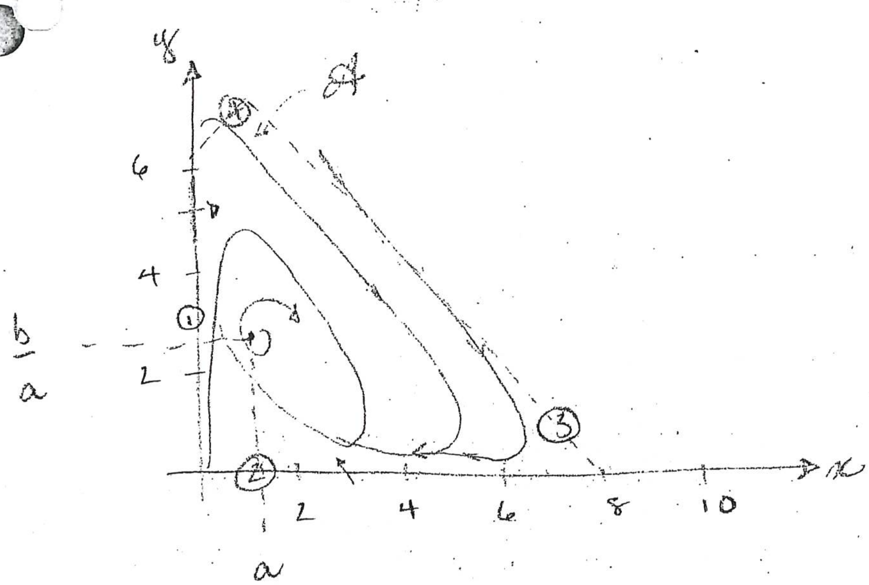
Explore with pplane:

$a=1,\ b=3$

$0 < x < 4.5$

$0 < y < 6$

Are there periodic orbits?



$\mathcal{A}$: quadrilateral

1. $x = 0$
2. $y = 0$
3. $y = A - x$
4. $y = B + x$

Along 1:

$x' = a > 0$ $\therefore$ trajectories enter $\mathcal{A}$

$(y' = 0)$

Along 2

$y' = bx > 0$ $\therefore$ trajectories enter $\mathcal{A}$

Along 3

Consider $L_3(x,y) = y + x - A = 0$ on 3, & $\mathcal{A}$ on the side where $L_3 < 0$. On 3,

$$\frac{dL_3}{dt} = y' + x' = a - x < 0 \text{ for } x > a$$

So if the top vertex of $\mathcal{A}$ occurs at $x > a$, we have $L_3' < 0$ on 4 & flow is into $\mathcal{A}$.

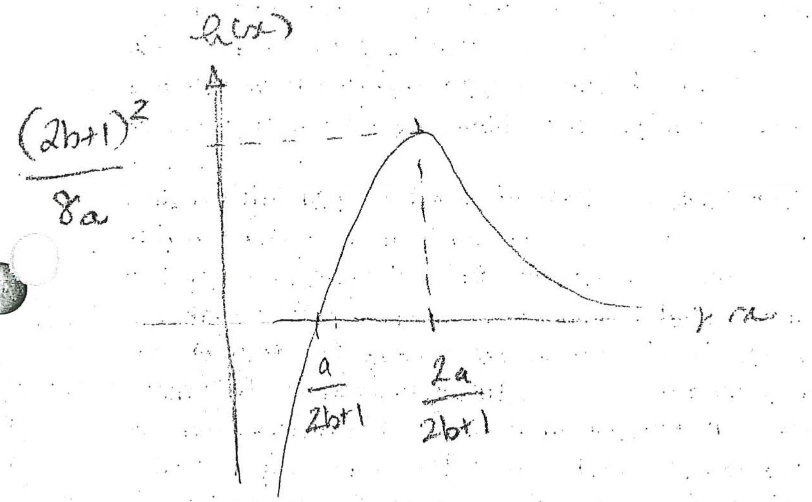
Along (4)

Consider $L_4(x,y) = y - x - B = 0$ on (4), & $\mathcal{A}$ on the side where $L_4 < 0$. On (4),

$$\frac{dL_4}{dt} = y' - x' = 2bx - 2x^2 y - a + x$$

So

$$L' < 0 \quad \text{if} \quad y > \frac{(2b+1)x - a}{2x^2} = h(x)$$



$h(x) = 0$ at $x = \frac{a}{2b+1}$

Set $B = \frac{(2b+1)^2}{8a}$. Then, along (4)

$$y = B + x = \frac{(2b+1)^2}{8a} + x > h(x) \quad \text{so } L_4' < 0$$

and flow is into $\mathcal{A}$.

Now set A.

Intersection of ③ + ④ is at

$$B + x = A - x \iff 2x = A - B$$
$$\iff x = \frac{A-B}{2}$$

Require $x > a$, so

$$\frac{A-B}{2} > a \iff A > 2a + B = 2a + \frac{(2b+1)^2}{8a}$$

Any choice is fine. Then we have satisfied the requirements of the P-B Thm & we can assert that there must be a periodic orbit in $\mathcal{R}$.

→ GO BACK TO P① for B's CRITERION