Lecture 9

Thursday, October 3, 2019 3:31 PM

Hopf Bifurcation Theorem

case $n = 2$

Consider a system of two equations w/ a parameter γ

$$\begin{cases} \dot{x} = f(x, y; \gamma) \\ \dot{y} = g(x, y; \gamma) \end{cases} \quad \dots (1)$$

where f & g are C^1 with respect to differentiation in x, y, & γ.
continuous in first-order differentials

Suppose (1) has steady states

$$(x^*, y^*) = (x^*(\gamma), y^*(\gamma)) \quad \forall \gamma \in \mathbb{R}.$$

The Jacobian matrix for system (1) evaluated at the steady state is

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{x^*, y^*}$$

and suppose that J has eigenvalues

$$\lambda(\gamma) = a(\gamma) \pm ib(\gamma)$$

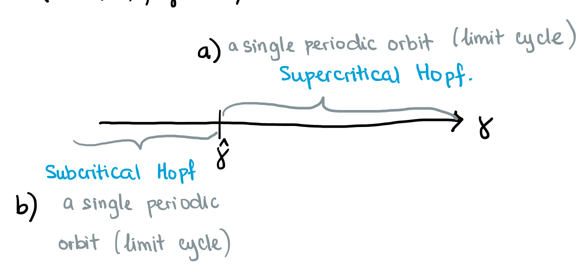
Also suppose that there is a value $\hat{\gamma}$, called the bifurcation value, such that

$$A(\hat{\gamma}) = 0$$
$$B(\hat{\gamma}) \neq 0$$
$$\frac{da}{d\gamma}\bigg|_{\hat{\gamma}} \neq 0 \quad \therefore \text{ (a changes sign as } \gamma \text{ increases past } \hat{\gamma})$$

Given these conditions, the following possibilities arises:

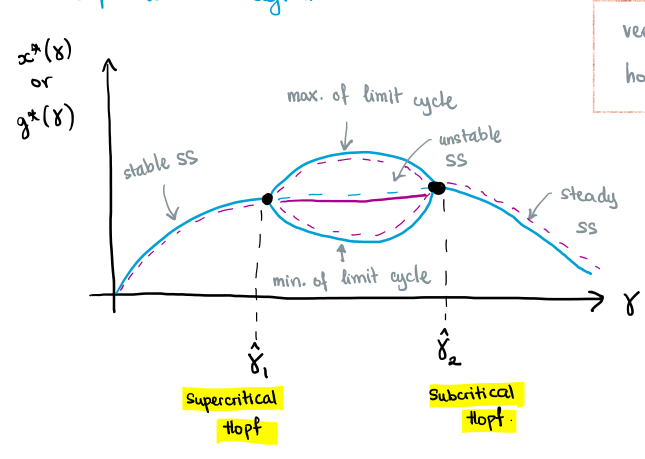
1. At $\gamma = \hat{\gamma}_1$, a centre is created & thus also infinitely many neutrally-stable concentric periodic orbits surrounding $(x^*(\hat{\gamma}), y^*(\hat{\gamma}))$

2.



Bifurcation Diagram

vertical axis = steady states
horizontal axis = bifurcation parameter



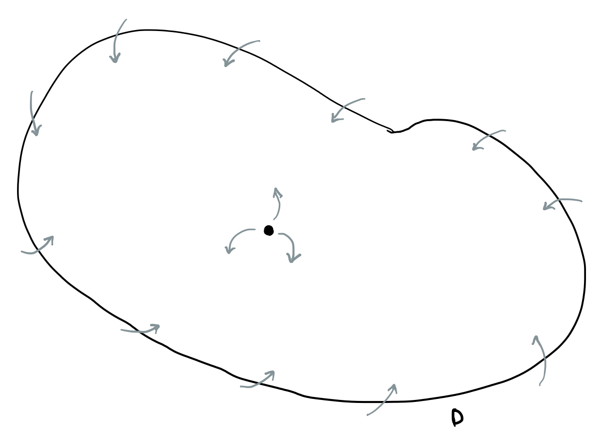
For the —/--- case, the limit cycle is stable
For the —/--- case, the limit cycle is unstable

To prove that a limit cycle exists, we can use the

Poincaré - Beudixson Theorem: $n=2$

If, for $t > t_0$, a trajectory is bounded & does not approach any singular point, then it is either a closed periodic orbit or approaches a closed periodic orbit as $t \to \infty$

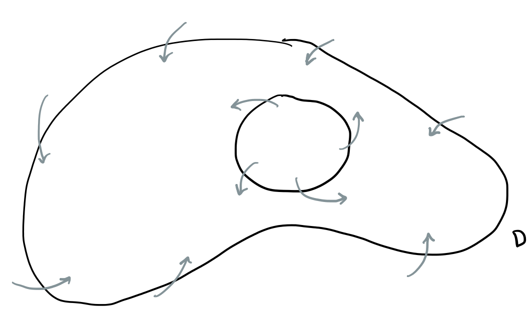
a)



• = unstable node
D = bounded region where no flow exits.

In this scenerio, the P-B Theorem says there must be a limit cycle

b)



P-B Theorem says there must be a limit cycle in D (no singular point in D)

In order to rule out the presence of a limit cycle, we can use the following two criteria:

1. Bendixson Criteria (Negative Criteria)

Suppose D is simply connected (i.e. no holes)
If

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is not identically 0 (i.e not $0 \; \forall (x,y) \in D$) & does not change sign in D, then there are no closed orbits in D.

2. Dulec's Criterion:

Suppose D is a simply-connected region in the plane, & suppose $\exists B(x,y)$, continuously differentiable in D, such that:

$$\frac{\partial}{\partial x}(B \cdot f) + \frac{\partial}{\partial y}(B \cdot g)$$

is not identically zero in D & does not change sign in D, then there are no closed orbits in D.

Example 1,

(Bendixson's Criterion)
Use B's criterion to indicate whether or not limit cycles can be ruled out in

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = dxy - cy \end{cases} \quad (a, b, c, d \text{ are all positive})$$

in $D = \{(x,y) \mid x > 0, \; y > 0\}$.

Ans:

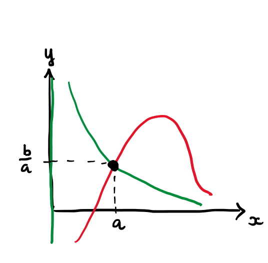
$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = a - by + dx - c$$
$$= (a + dx) - (c + by)$$

← can we guarantee that this eq. is greater than 0 in D?

$\therefore$ x & y are independent, we cannot guarantee that it will be greater than 0.
⇒ cannot rule out limit cycle

Example 2. Brusselator

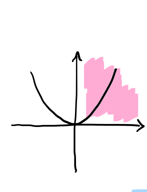
$$\dot{x} = a - bx + x^2y - x$$
$$\dot{y} = bx - x^2y$$



requires $a > 0$
$b > 1 + a^2$
$\therefore$ coexistence SS exists.

$$J\big|_{ss} = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}$$

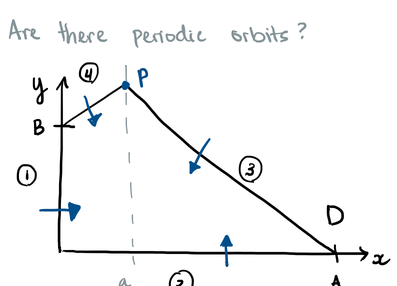
$$\text{Det}(J) = -a^2(b-1) + ba^2$$
$$= a^2 \quad > 0$$



$$\text{Tr}(A) = b - 1 - a^2$$
$$= b - (1 + a^2) \quad > 0 \quad \text{since } b > 1 + a^2$$

$\therefore$ unstable.

Are there periodic orbits?



Boundary
① $x = 0$
② $y = 0$
③ $y = A - x$
④ $y = B + x$

assume slope is 1 to make our lives easier.

Along ①, only need x-component of flow

$\dot{x}$ when $x = 0$; $\dot{x} = a > 0$ flow is into region D

Along ②, only need y-component of flow

$\dot{y}$ when $y = 0$; $\dot{y} = bx \geq 0$ flow is into D

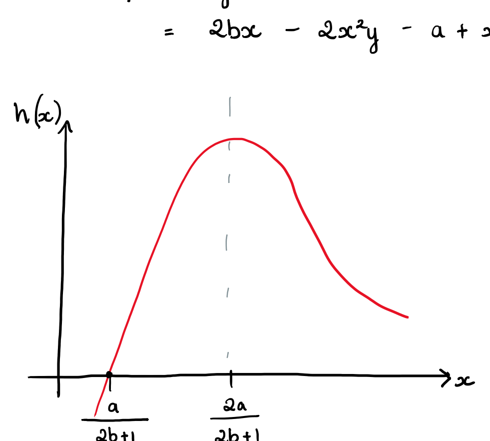
Along ③
Consider $L_3 = y + x - A$, then $L_3 > 0$ for ouside/above of region D
$L_3 < 0$ for inside/below of region D.

$$\dot{L}_3 = \dot{y} + \dot{x} = a - x < 0 \text{ if } x > a$$

$\therefore$ so flow is into D if the point P has x-coordinate a

Along ④
Consider $L_4 = y - x - B$, then $L_4 > 0$ above $L_4 = 0$ (ouside D)
$L_4 < 0$ below $L_4 = 0$ (inside D)

$$\dot{L}_4 = \dot{y} - \dot{x} - B$$
$$= 2bx - 2x^2y - a + x < 0 \text{ if } y > \frac{(2b+1)x - a}{2x^2} = h(x)$$



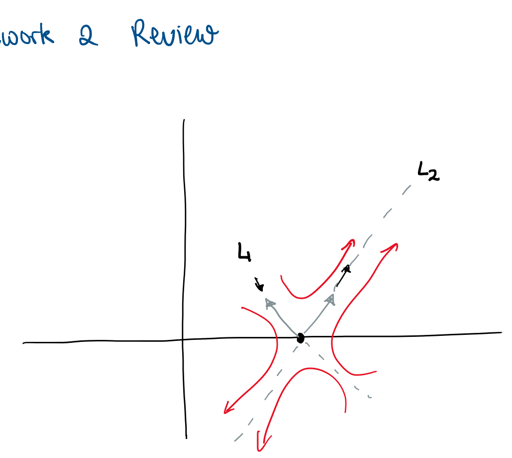
If we let $B = \frac{(2b+1)^2}{8a}$, then

along ④
$$y = B + x = \frac{(2b+1)^2}{8a} + x > h(x)$$

So $\dot{L}_4 < 0$ ⇒ flow is into D

⇒ D contains a limit cycle

Homework 2 Review



both vectors are invariant sets, but one is stable (heading to SS), & one is unstable