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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 339
Date: Oct 8th, 2019 Duration: 50 minutes.
This exam has 5 questions for a total of 43 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise.
- No marks are given for answers without supporting work.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 50 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 30 minutes.

1. Consider the nonlinear ODE system below:

$$\dot{x} = x\cos(\pi y), \quad (1a)$$
$$\dot{y} = x^3 - y, \quad (1b)$$

where

$$\{(x,y) \in \mathbb{R}^2 \,|\, 0 \leqslant x \leqslant 2,\, 0 \leqslant y \leqslant 1\}.$$

$\boxed{8}$ (a) Find the nullclines and steady states (there are two), and determine their stability. For any steady state that is a saddle node, find the eigenvectors. Verify your work with Figure 1 (next page), which shows the nullclines and direction field arrows for (1).

Nullclines:

$\dot{x} = 0 \Leftrightarrow x\cos(\pi y) = 0 \Leftrightarrow x = 0$ or $y = \frac{2n+1}{2}$, $n \in \mathbb{Z}$

$\dot{y} = 0 \Leftrightarrow x^3 - y = 0 \Leftrightarrow y = x^3$

Steady states on $0 < x < 2$, $0 < y < 1$

$(0,0)$ or $y = \frac{1}{2}$, $x^3 - \frac{1}{2} = 0 \Leftrightarrow x = \frac{1}{\sqrt[3]{2}}$ so $\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{2}\right)$

Stability:

$$J = \begin{bmatrix} \cos(\pi y) & -x\pi\sin(\pi y) \\ 3x^2 & -1 \end{bmatrix}$$

At $(0,0)$: $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\therefore \lambda_1 = 1, \lambda_2 = -1 \Rightarrow$ saddle node

eigenvectors:

$\lambda_1 = 1$: $[J - \lambda_1 I]\vec{v}_1 = \vec{0} \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$: $[J - \lambda_2 I]\vec{v}_2 = \vec{0} \Leftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Choose $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

At $\left(\frac{1}{\sqrt[3]{2}}, \frac{1}{2}\right)$: $J = \begin{bmatrix} 0 & -\pi 2^{-1/3} \\ 3(2)^{-2/3} & -1 \end{bmatrix}$ $\therefore |J - \lambda I| = 0 \Leftrightarrow -\lambda(-1-\lambda) + \frac{3\pi}{2} = 0$

$\Leftrightarrow 2\lambda^2 + 2\lambda + 3\pi = 0$

$\Leftrightarrow \lambda = \frac{-1 \pm \sqrt{1-6\pi}}{2}$, complex w/ -ve real part

$\Rightarrow$ stable focus

$\boxed{6}$ (b) Use the information in Figure 1 and your results from part (a) to fill in solution curves throughout the phase plane. If there is a saddle, make sure the eigenvectors are also included.

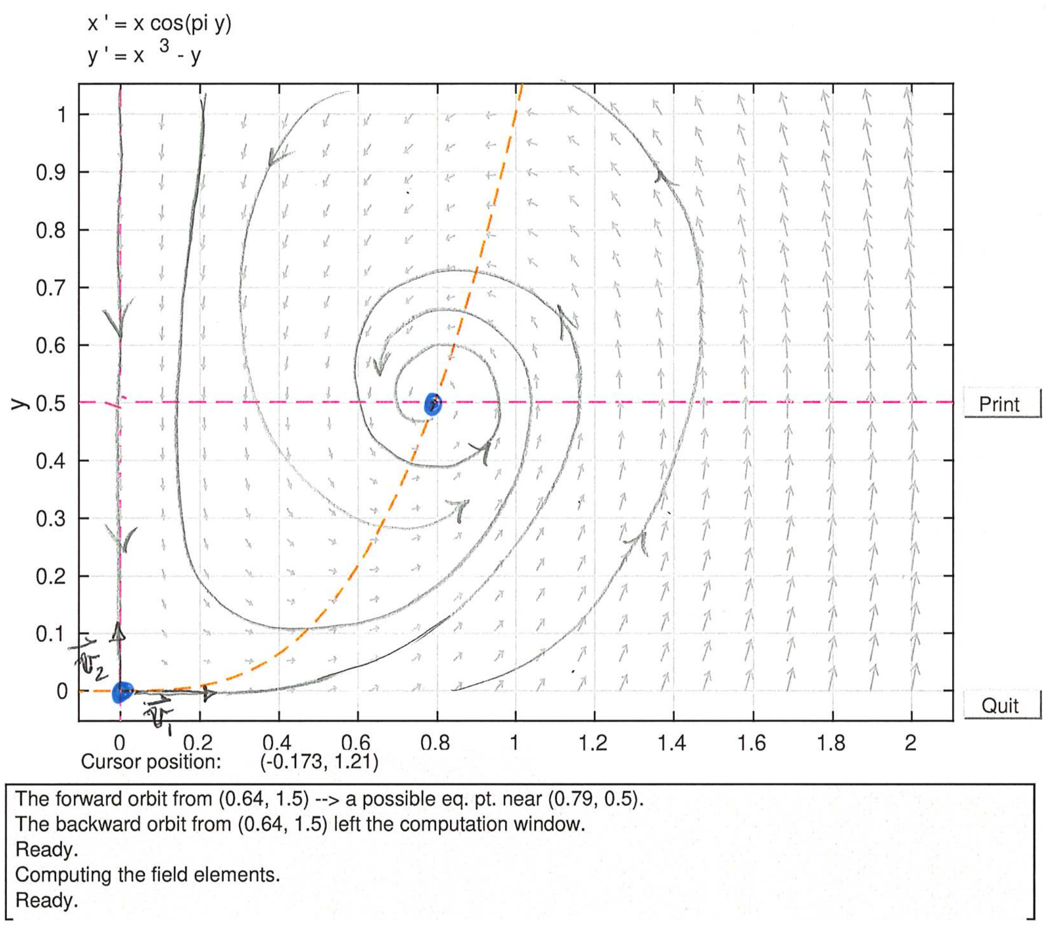


Figure 1: Some components of the phase plane plot of (1).

$\vec{v}_1$ is associated w/ $\lambda_1 = 1 > 0$ so flow is away from $(0,0)$
$\vec{v}_2$ " " " $\lambda_2 = -1 < 0$ " " " toward $(0,0)$

$\boxed{6}$ 2. Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y, && \text{(2a)} \\ \dot{y} &= -(x+x^2) - y(3y^2+3x^2+2x^3-1). && \text{(2b)} \end{aligned}$$

$\boxed{4}$ (a) Show that the set $L(x,y) = 0$ where

$$L(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{6}$$

is an invariant set for (2). Make sure you write your proof carefully.

$$\dot{L} = 0 \iff \frac{1}{2}2y\dot{y} + \frac{1}{2}2x\dot{x} + \frac{1}{3}3x^2\dot{x} = 0$$

$$\iff y\dot{y} + x\dot{x} + x^2\dot{x} = 0$$

$$\iff y\left[\cancel{-(x+x^2)} - y(3y^2+3x^2+2x^3-1)\right] + \cancel{(x+x^2)[y]} = 0$$

$$\iff -y^2(3y^2+3x^2+2x^3-1) = 0$$

$\therefore$ $\dot{L} = 0$ if $y = 0$ (trivial case) or if

$3y^2+3x^2+2x^3-1 = 0 \iff \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{6} = 0$

$\iff L = 0$

$\therefore$ $\dot{L} = 0$ if $L = 0$ & so $L = 0$ is an invariant set for the system (2).

$\boxed{1}$ (b) Show that the $L = 0$ invariant set includes the steady state at (-1,0).

$$L(-1,0) = 0 + \frac{1}{2}(-1)^2 + \frac{1}{3}(-1)^3 - \frac{1}{6} = 0$$

$\therefore$ $L=0$ includes the point $(-1,0)$.

$\boxed{2}$ (c) The eigenvalues and eigenvectors at the (-1,0) steady state are (rounded to the nearest second decimal point):

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The invariant set $L = 0$ is shown in Figure 2. Use the eigenvalues and eigenvectors to determine which branch of the invariant set is stable with respect to the steady state, and which is unstable. Sketch the initial portions of the invariant sets passing through (-1,0) in the region $x > -1$.

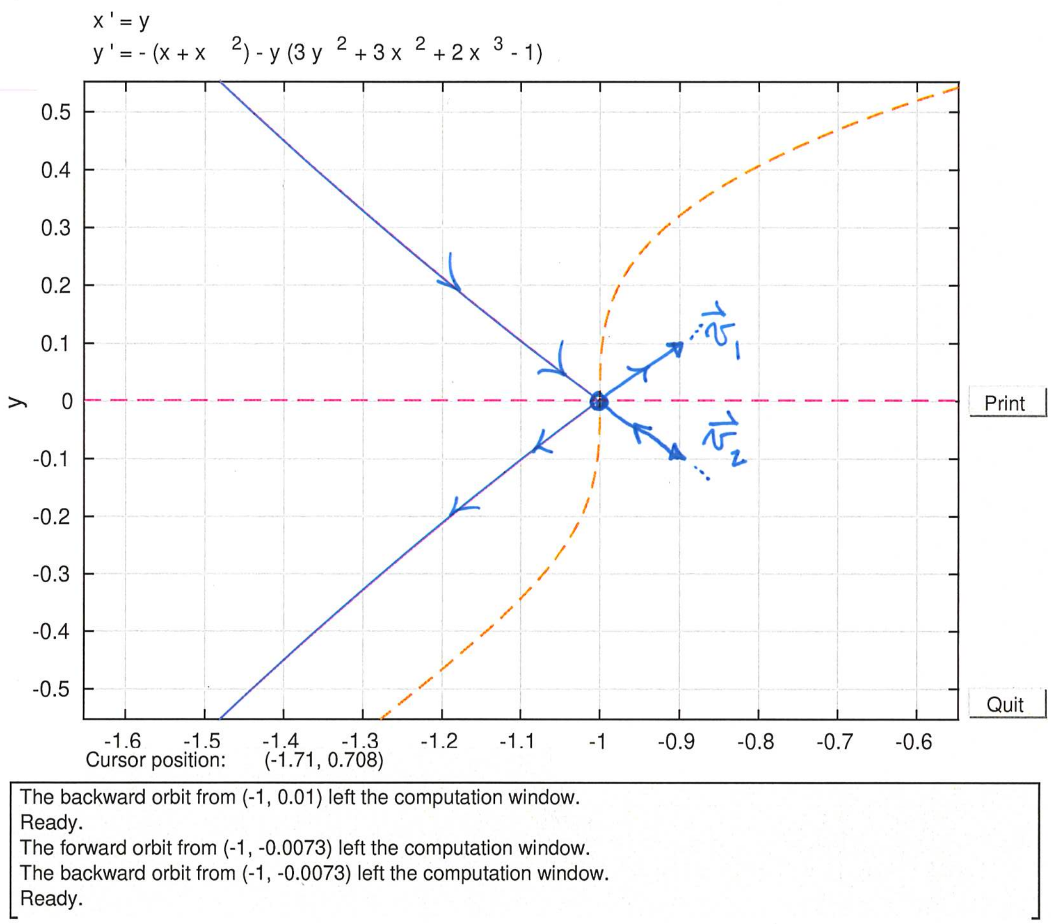


Figure 2: Plane plot of (2) showing the invariant set $L = 0$ and the nullclines at the (-1,0) steady state.

$\boxed{11}$ 3. Let x be the state variable and $\mu \in \mathbb{R}$ a bifurcation parameter. Sketch the bifurcation diagram for the differential equation

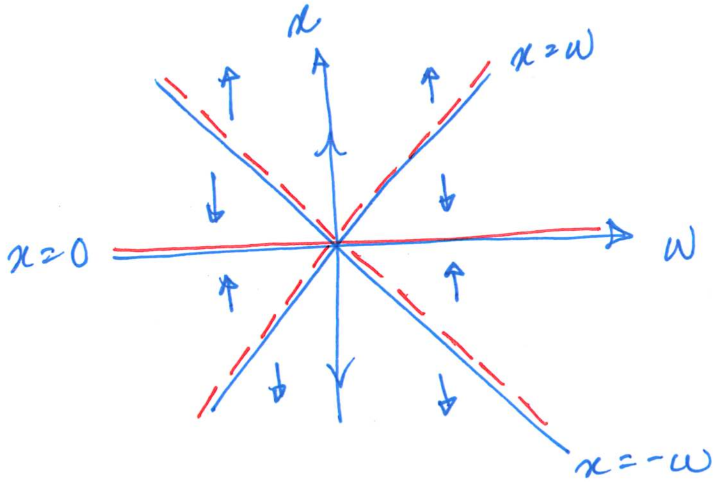
$$\dot{x} = x^3 - \omega^2 x,$$

where $\omega \in \mathbb{R}$. Show all of your work!

Steady states:

$\dot{x} = 0 \Leftrightarrow x^3 - \omega^2 x = 0 \Leftrightarrow x(x^2 - \omega^2) = 0 \Leftrightarrow x(x+\omega)(x-\omega) = 0$

$\therefore\ x^* = 0, \omega, -\omega$



Stability calculations:

case 1: $x > |\omega| > 0$, $\underset{>0}{x}\,\underbrace{(x^2 - \omega^2)}_{>0} = \dot{x} > 0$

case 2: $x < -|\omega| < 0$, $\underset{<0}{x}\,\underbrace{(x^2 - \omega^2)}_{>0} = \dot{x} < 0$

case 3: $0 < x < |\omega|$, $\underset{>0}{x}\,\underbrace{(x^2 - \omega^2)}_{<0} = \dot{x} < 0$

case 4: $-|\omega| < x < 0$, $\underset{<0}{x}\,\underbrace{(x^2 - \omega^2)}_{<0} = \dot{x} > 0$

$\boxed{5}$ 4. Consider the system of equations

$$\dot{x} = y, \tag{3a}$$
$$\dot{y} = -y^3 + \gamma y - x, \tag{3b}$$

where $\gamma \in \mathbb{R}$ is a parameter. The only steady state for this system is at $(x, y) = (0, 0)$. The eigenvalues of the Jacobian at the steady state are given by

$$\lambda_{\pm} = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2}.$$

Assume $|\gamma| < 2$. Explain how the stability of the (0,0) steady state varies as γ increases from values below 0 to values above 0. What is the bifurcation that occurs at $\gamma = 0$? What structure might emerge as γ passes through the value 0?

$-2 \qquad 0 \qquad 2 \qquad \gamma$

Here, $Re(\lambda_\pm) < 0$ & $Im(\lambda_\pm) \neq 0$, so (0,0) is a stable focus.

Here, $Re(\lambda_\pm) > 0$ and $Im(\lambda_\pm) \neq 0$, so (0,0) is an unstable focus.

Here, $\lambda_\pm = \pm i$, pure imaginary, so (0,0) is a centre.

The bifurcation at $\gamma = 0$ is a Hopf bifurcation and we expect a Limit Cycle to emerge.

5. [2] **BONUS POINTS:** Name the bifurcation in question 3! *(Hint: It's a compound bifurcation, that is, one bifurcation that you should recognise immediately followed by another bifurcation that you should also recognise.)*

The bifurcation is a subcritical pitchfork bifurcation immediatly followed by a supercritical pitchfork bifurcation.