Real-valued Lipschitz functions and metric properties of functions

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Abstract

The purpose of this expository article is to explore the very general phenomenon that a function betweeen metric spaces has a particular metric property if and only if whenever it is followed in a composition by an arbitrary real-valued Lipschitz function, the composition has this property. Continuity and boundedness are easily shown to be such properties. The Efremovič lemma plays the key role in showing that uniform continuity is such a property. It is a 2004 theorem of Garrido and Jaramillo that a function between metric spaces is Lipschitz if and only if whenever it is followed by a Lipschitz real-valued function in a composition, the composition is Lipschitz. We present a streamlined proof of the Garrido-Jaramillo result itself, but one that still relies on their natural continuous linear operator from the Lipschitz space for the target space to the Lipschitz space for the domain.