JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 102, No. 3, pp. 697-703, SEPTEMBER 1999

TECHNICAL NOTE

Proof of a Conjecture by Deutsch, Li, and Swetits on Duality of Optimization Problems[1]

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Communicated by F. Giannessi

Abstract. A conjecture of Deutsch, Li, and Swetits on duality relationships among three optimization problems is shown to hold true. The proof relies on a reformulation of one of the problems in a suitable product space, to which then a version of the classical Fenchel duality theorem applies.

Key Words. Convex optimization, Fenchel duality.

1. Introduction

The main result of a very recent paper by Deutsch, Li, and Swetits (Ref. 1, Theorem 4.4) rests crucially on the following theorem.

Theorem 1.1. See Ref. 1, Theorem 3.1. Suppose that X is a real Hilbert space and $f: X \to \mathfrak{R}$ is a strictly convex, lower semicontinuous, Gâteaux differentiable function with the property

$$\lim_{\|x\| \to +\infty} [f(x) - \langle y, x \rangle] = +\infty, \qquad \text{for every } y \in X.$$

[1]This research was completed while the author visited the Department of Mathematics at Pennsylvania State University in Spring 1998. The author thanks Frank Deutsch for his hospitality, support, and helpful comments; Wu Li for careful reading of an earlier version of this manuscript; and the Department of Combinatorics and Optimization, University of Waterloo, where this manuscript was prepared while the author was an NSERC Postdoctoral Fellow.

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Suppose further that $C_1, \dots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_{i=1}^m C_i \neq \varnothing$. Consider the following three optimization problems[3]:

$$\text{(OP1)} \quad \alpha := \inf_{x \in C} f(x),$$

$$\text{(OP2)} \quad \beta := \max_{y \in X} \left[\inf_{x \in X} (\langle x, y \rangle + f(x)) - \iota_C^*(y) \right],$$

$$\text{(OP3)} \quad \gamma := \sup_{(y_1, \dots, y_m) \in X^m} \left[\inf_{x \in X} \left(\left\langle x, \sum_{i=1}^m y_i \right\rangle + f(x) \right) - \sum_{i=1}^m \iota_{C_i}^*(y_i) \right].$$

Let $\bar{x}$ be the unique solution of (OP1). Then, (OP3) has a solution if and only if[4]

$$-\nabla f(\bar{x}) \in \sum_{i=1}^m N_{C_i}(\bar{x}). \tag{1}$$

Moreover, if $(\bar{y}_1, \dots, \bar{y}_m)$ is a solution of (OP3), then $\sum_{i=1}^m \bar{y}_i$ is the unique solution of (OP2), each $\bar{y}_i$ belongs to $N_{C_i}(\bar{x})$, and $\alpha = \gamma$.

The Deutsch-Li-Swetits conjecture (see Ref. 1, Remark 3.1 following Theorem 3.1) states that, if the assumption on Gâteaux differentiability of f is dropped, then the conclusion of Theorem 1.1 still holds with (1) replaced by

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^m N_{C_i}(\bar{x}). \tag{2}$$

2. New Results

We will now prove[5] (a generalization of) this conjecture in the following theorem.

Theorem 2.1. Suppose that X is a real Hilbert space and that $C_1, \dots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_{i=1}^m C_i \neq \varnothing$. Suppose

[3]Here and elsewhere, ι_S stands for the indicator function of a closed convex nonempty set S in X; the Fenchel conjugate of a function g is denoted g^*; in particular, ι_S^* is the support function of S.

[4]If S is a closed convex nonempty set in X and $x \in S$, then $N_S(x) = \{y \in X : \langle y, S - x \rangle \leq 0\} = \partial \iota_S(x)$ denotes the normal cone of S at x.

[5]Shortly after the author discussed his solution with Frank Deutsch, Ivan Singer circulated independently his manuscript "Duality for Optimization and Best Approximation over Finite Intersection," which contains related results.

further that $f: X \to \Re \cup \{+\infty\}$ is a convex, lower semicontinuous function which satisfies[6]

$$\operatorname{int}(\operatorname{dom} f) \cap C \neq \varnothing \quad \text{and} \quad 0 \in \operatorname{int}(\operatorname{dom} f^*).$$

Define (OP1), (OP2), (OP3), and α, β, γ as in Theorem 1.1. Then, $\alpha = \gamma$ and (OP1) has solutions. Let $\bar{x}$ be an abitrary solution for (OP1). Then, (OP3) has a solution if and only if

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} N_{C_i}(\bar{x}). \tag{3}$$

Moreover, if $(\bar{y}_1, \ldots, \bar{y}_m)$ is an arbitrary solution of (OP3), which happens precisely when γ is attained, then $\sum_{i=1}^{m} \bar{y}_i$ solves (OP2), and each $\bar{y}_i \in N_{C_i}(\bar{x})$.

Proof. We split the proof in three parts.

Part 1. Relating (OP1) to (OP3). Let $Y := X^m$, with

$$\|y\|^2 := \sum_{i=1}^{m} \|y_i\|^2, \qquad \text{for every } y = (y_1, \ldots, y_m) \in Y.$$

Then, Y is a real Hilbert space with

$$\langle y, z \rangle = \sum_i \langle y_i, z_i \rangle, \qquad \text{for all } y, z \in Y.$$

Set $D := C_1 \times \cdots \times C_m$. Define

$$F: Y \to \Re \cup \{+\infty\}: (y_1, \ldots, y_m) \mapsto \iota_D^*(y_1, \ldots, y_m),$$

$$G: X \to \Re: x \mapsto f^*(x).$$

Also, let

$$A: Y \to X: (y_1, \ldots, y_m) \mapsto -\sum_{i=1}^{m} y_i.$$

Then, A is a continuous linear operator from Y onto X with conjugate

$$A^*: X \to Y: x \mapsto (-x, \ldots, -x).$$

Consider the following optimization problem:

$$\text{(P)} \quad p := \inf_{y \in Y} [F(y) + G(Ay)].$$

[6]In particular, these assumptions hold when f is as in Theorem 1.1.

Note that $p<+\infty$, since $y=0$ is a feasible point for (P). Up to a minus sign, (P) is actually equal to (OP3),

$$
\begin{aligned}
-\gamma &= -\sup_{(y_1,\ldots,y_m)\in X^m}\left[\inf_{x\in X}\left(\left\langle x, \sum_{i=1}^m y_i\right\rangle + f(x)\right) - \sum_{i=1}^m \iota_{C_i}^*(y_i)\right] \\
&= -\sup_{y\in Y}\left[-\iota_D^*(y) + \inf_{x\in X}(\langle x, -Ay\rangle + f(x))\right] \\
&= \inf_{y\in Y}\left[\iota_D^*(y) + \sup_{x\in X}(\langle x, Ay\rangle - f(x))\right] \\
&= \inf_{y\in Y}[\iota_D^*(y) + f^*(Ay)] \\
&= \inf_{y\in Y}[F(y) + G(Ay)] \\
&= p.
\end{aligned}
$$

The classical Fenchel dual of (P) is defined by

$$\text{(D)}\quad d := -\inf_{x\in X}[F^*(-A^*x) + G^*(x)].$$

Moreover, (P) and (D) are related as follows (see, for instance, Ref. 2, Theorem 3.2.4 or Ref. 3, Theorem 52.B):

(i) Weak Duality. $p \geq d$.
(ii) Karush–Kuhn–Tucker Conditions. Suppose that $\bar{y}\in Y$ and $\bar{x}\in X$. Then, $\bar{y}$ solves (P), $\bar{x}$ solves (D), and $p=d$, if and only if $A\bar{y}\in\partial G^*(\bar{x})$ and $-A^*\bar{x}\in\partial F(\bar{y})$.
(iii) Strong Duality via Constraint Qualification. If $A(\operatorname{dom} F)\cap \operatorname{int}(\operatorname{dom} G)\neq\varnothing$ and p is finite, then $p=d$ and d is attained.

First, up to a minus sign again, (D) is (OP1),

$$
\begin{aligned}
-d &= \inf_{x\in X}[F^*(-A^*x) + G^*(x)] \\
&= \inf_{x\in X}[\iota_D^{**}(-A^*x) + f^{**}(x)] \\
&= \inf_{x\in X}[\iota_D(-A^*x) + f(x) \\
&= \inf_{x\in X:\, -A^*x\in D} f(x) \\
&= \inf_{x\in C} f(x) \\
&= \alpha.
\end{aligned}
$$

Now, $\operatorname{dom} f \cap C \neq \varnothing$; hence, $\alpha < +\infty$ and so $d > -\infty$. On the other hand, we saw that $p < +\infty$. Altogether, using (i), we conclude that p is finite, and so is γ.

Next,

$$0 \in \operatorname{int}(\operatorname{dom} f^*) = \operatorname{int}(\operatorname{dom} G),$$

by assumption and clearly,

$$0 \in \operatorname{dom} \iota_D^* = \operatorname{dom} F.$$

Hence,

$$0 \in A(\operatorname{dom} F) \cap \operatorname{int}(\operatorname{dom} G).$$

By (iii), $p = d$ and d is attained. Hence, $\alpha = \gamma$ and α is attained.

Let $\bar{x}$ be an arbitrary solution of (OP1), that is, of (D). Further, let $y = (y_1, \ldots, y_m)$ be an abitrary point in Y. Then, using $p = d$, (ii), and convex calculus, we obtain the equivalences

$$\begin{aligned} y \text{ solves (OP3)} &\Leftrightarrow y \text{ solves (P)} \\ &\Leftrightarrow Ay \in \partial G^*(\bar{x}) \text{ and } -A^*\bar{x} \in \partial F(y) \\ &\Leftrightarrow -\sum_{i=1}^{m} y_i \in \partial f(\bar{x}) \text{ and } y \in \partial F^*(-A^*\bar{x}) \\ &\Leftrightarrow -\sum_{i=1}^{m} y_i \in \partial f(\bar{x}) \text{ and } y \in \partial \iota_D(-A^*\bar{x}) \\ &\Leftrightarrow -\sum_{i=1}^{m} y_i \in \partial f(\bar{x}) \text{ and each } y_i \in N_{C_i}(\bar{x}). \end{aligned}$$

It follows that (OP3) has a solution if and only if (3) holds.

Part 2. Statement on $(\bar{y}_1, \ldots, \bar{y}_m)$. Now, assume that (OP3) has solutions, and let $\bar{y}$ be any one of them. Then, each $\bar{y}_i \in N_{C_i}$ and $-\sum_{i=1}^{m} \bar{y}_i \in \partial f(\bar{x})$ by Part 1. On the other hand, it is easy to check that (OP2) is the classical Fenchel dual of (OP1). Moreover, our assumption on f guarantees, in view of (iii) applied to (OP1) as the primal problem, that $\alpha = \beta$ and β is attained. The Karush–Kuhn–Tucker conditions (ii), once again applied to (OP1) as the primal problem, show that a vector $z \in X$ solves (OP2) if and only if

$$z \in N_C(\bar{x}) \quad \text{and} \quad -z \in \partial f(\bar{x}).$$

As an aside, this explains why (OP2) in the context of Theorem 1.1 must have a unique solution: indeed, by strict convexity of f, we see first that $\bar{x}$

is unique. Then, the solution of (OP2) is unique, because it must equal $-\nabla f(\bar{x})$ by differentiability of f. Set

$$\bar{z} := \sum_{i=1}^{m} \bar{y}_i.$$

Then,

$$-\bar{z} \in \partial f(\bar{x}) \quad \text{and} \quad \bar{z} \in \sum_{i=1}^{m} N_{C_i}(\bar{x}).$$

Since $\sum_{i=1}^{m} N_{C_i}(\bar{x})$ is always a subset of $N_C(\bar{x})$, we conclude that $\bar{z}$ indeed solves (OP2).

Part 3. Statement in Footnote 6. It remains to show that a function that satisfies the assumptions in Theorem 1.1 must necessarily satisfy the assumptions in Theorem 2.1. So, let g have the same properties as f in Theorem 1.1. Fix an arbitrary point $x^* \in X$ and choose a real number r such that

$$S := \{x \in X : g(x) - \langle x^*, x \rangle \leq r\} \neq \emptyset.$$

By assumption, S is weakly compact and convex. If $x \in S$, then

$$\langle x^*, x \rangle - g(x) \geq -r;$$

otherwise,

$$\langle x^*, x \rangle - g(x) < -r.$$

Hence,

$$\begin{aligned} g^*(x^*) &= \sup_{x \in X} [\langle x^*, x \rangle - g(x)] \\ &= \sup_{x \in S} [\langle x^*, x \rangle - g(x)] \\ &= -\inf_{x \in S} [g(x) - \langle x^*, x \rangle]. \end{aligned}$$

Now, $g(\cdot) - \langle x^*, \cdot \rangle$ is weakly lower semicontinuous and S is weakly compact; consequently,

$$g^*(x^*) = \langle x^*, \hat{x} \rangle - g(\hat{x}), \qquad \text{for some } \hat{x} \in S.$$

Since $g(\hat{x}) \in \Re$ by assumption, it follows that $x^* \in \operatorname{dom} g^*$. Now, x^* was chosen arbitrarily; hence, $\operatorname{dom} g^* = X$. Therefore, g satisfies the present assumptions and the entire theorem is proved. □

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