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Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization

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Abstract. The strong conical hull intersection property and bounded linear regularity are properties of a collection of finitely many closed convex intersecting sets in Euclidean space. These fundamental notions occur in various branches of convex optimization (constrained approximation, convex feasibility problems, linear inequalities, for instance). It is shown that the standard constraint qualification from convex analysis implies bounded linear regularity, which in turn yields the strong conical hull intersection property. Jameson's duality for two cones, which relates bounded linear regularity to property (G), is re-derived and refined. For polyhedral cones, a statement dual to Hoffman's error bound result is obtained. A sharpening of a result on error bounds for convex inequalities by Auslender and Crouzeix is presented. Finally, for two subspaces, property (G) is quantified by the angle between the subspaces.

Key words. angle – asymptotic constraint qualification – basic constraint qualification – bounded linear regularity – CHIP – conical hull intersection property – convex feasibility problem – convex inequalities – constrained best approximation – error bound – Friedrichs angle – Hoffman's error bound – linear inequalities – linear regularity – orthogonal projection – property (G)

1. Introduction

An intriguing list of topics

We start with an informal discussion of several topics, which appear to be quite different from each other, drawn from convex optimization and bordering fields.

Constrained interpolation and optimization In a series of papers ([8], [9], [15], [14]), Deutsch *et al.* studied the problem of constrained interpolation from a convex subset or cone. One of the central ingredients in their analysis is the **(strong) conical hull intersection property ("(strong) CHIP")**, which captures a geometric property of the constraints. (See [16] and [11] for applications and further results.) Indeed, as

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shown recently by Deutsch *et al.* [13], [12], the property strong CHIP characterizes a strong relationship for a certain pair of optimization problems.

Projection algorithms The convex feasibility problem and the best approximation problem are often solved iteratively by projection algorithms. Result on (the rate of) convergence require **bounded linear regularity** of the constraints. (Details can be found, *e.g.*, in [4], [5], and [24].)

Duality theory for convex cones Jameson introduced in [22] a duality theory for two closed convex cones in a Banach space. He gave a generalization of the well-known fact (see, for instance, [23, Corollary 35.6]) that the sum of two closed subspaces in Banach space is closed if and only if the sum of their complements is. In our finite-dimensional setting, this linear result is trivial; for cones, however, several important questions remain unanswered. Central to Jameson's study is the **property (G)**.

Systems of linear inequalities In 1952, Hoffman [20] proved that the distance from an arbitrary point to the solution set of a system of linear inequalities is bounded above by a constant times the norm of the residual. This important result, which relies on the polyhedrality of the underlying system, is commonly referred to as **Hoffman's error bound result**. (A good starting point to generalizations and applications is Burke and Tseng's [7].)

Systems of convex inequalities Because of non-polyhedrality, different tools are required to tackle this natural generalization of the linear-inequality-case. Typically, **existence of error bounds is guaranteed under constraint qualifications**. (For additional information, see Lewis and Pang's recent [26].)

Geometry of two subspaces The rate of convergence of von Neumann's method of alternating projections can be formulated in terms of the **angle between the subspaces**. (See Deutsch's recent survey [10].)

The aim of this paper is to exhibit the (sometimes surprising) relationships among these topics, and to improve and unify results on the underlying common notions.

Before we describe our main result, let us fix some notation.

Notation

Throughout the paper,

X is some Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

The language employed follows Rockafellar's classical [32] and/or the more recent books [18, 19] by Hiriart-Urruty and Lemaréchal. These books contain all standard facts of convex analysis.

Notation is fairly standard: We write B_X for the *unit ball* $\{x \in X : \|x\| \leq 1\}$, S_X for the *unit sphere* $\{x \in X : \|x\| = 1\}$, and $B(z; r)$ for $z + rB_X$, where $z \in X$ and $r \geq 0$. Given convex functions f and g on X, the *domain* (*conjugate function* of f, *gradient* of f, *subdifferential* of f, *positive part* $\max\{f, 0\}$ of f, *infimal convolution* of f and g, resp.) is denoted by $\operatorname{dom} f$ (f^*, ∇f, ∂f, f^+, $f \Box g$, resp.). The infimal convolution $f \Box g$ is *exact* at a point x, if the infimum $(f \Box g)(x) = \inf_{u+v=x} f(u) + g(v)$ is attained.

By a *cone* we mean a *nonempty* set in X closed under nonnegative scalar multiplication; cones thus always contain the origin.

Suppose S is a set in X. The *interior* (resp. *relative interior*, *boundary*, *closure*, *convex hull*, *convex conical hull*, *affine span*, *linear span*, *(negative) polar*, *orthogonal complement*, *indicator function*) of S is denoted $\operatorname{int} S$, (resp. $\operatorname{ri} S$, $\operatorname{bd} S$, $\operatorname{cl} S$, $\operatorname{conv} S$, $\operatorname{cone} S$, $\operatorname{aff} S$, $\operatorname{span} S$, $S^{\ominus}$, $S^{\perp}$, ι_S). We abbreviate expressions like $\sup_{s\in S}\langle x, s\rangle$ by a more concise $\sup\langle x, S\rangle$. Assuming now that S is closed, convex, and nonempty, we remark that S induces a *distance function* $d(\cdot, S) = \|\cdot\| \Box \iota_S$ and the corresponding *projection* or *nearest point mapping* P_S. If $x \in S$, then the *tangent cone* (resp. *normal cone*) of S at x is denoted $T_S(x)$ (resp. $N_S(x)$). Thus $T_S(x) = \operatorname{cl}(\operatorname{cone}(S - x))$ and $N_S(x) = (S - x)^{\ominus}$. Finally, we use the acronyms "TFAE" (the following are equivalent) and "WLOG" (without loss of generality).

Main definitions and discussion of the results

For the remainder of this introduction, let us suppose that

$$C_1, \ldots, C_m \text{ are finitely many closed convex sets in } X \text{ with } C := \bigcap_i C_i \neq \emptyset.$$

We define the aforementioned concepts and recall the standard constraint qualification, ubiquitous in convex optimization.

(Strong) CHIP [8,15] The collection $\{C_1, \ldots, C_m\}$ has *strong CHIP* (resp. *CHIP*), if $N_C(x) = \sum_i N_{C_i}(x)$ (resp. $T_C(x) = \bigcap_i T_{C_i}(x)$), for every $x \in C$.

Bounded linear regularity [4] The collection $\{C_1, \ldots, C_m\}$ is *boundedly linearly regular*, if for every bounded subset S of X, there exists $\kappa_S > 0$ such that $d(x, C) \leq \kappa_S \max_i d(x, C_i)$, for every $x \in S$.

Property (G) [22] Provided that each C_i is a cone, the collection $\{C_1, \ldots, C_m\}$ *has property (G)*, if there exists $\alpha > 0$ such that $B_X \cap \sum_i C_i \subseteq \alpha \sum_i (C_i \cap B_X)$.

Standard constraint qualification The collection $\{C_1, \ldots, C_m\}$ satisfies the *standard constraint qualification*, if there exists $r \in \{0, \ldots, m\}$ such that $C_{r+1}, \ldots, C_m$ are polyhedral and $\bigcap_{i=1}^r \operatorname{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i \neq \emptyset$.

Our main results can be summarized as follows.

R1 The standard constraint qualification implies bounded linear regularity — a new proof via convex analysis.

R2 Bounded linear regularity implies strong CHIP.

R3 Property (G) holds for convex polyhedral cones.

R4 Bounded linear regularity and CHIP are the same properties when C is singleton or each C_i is smooth on C.

The proofs are mostly rooted in convex analysis. Result R1, which appears to be a "folk theorem" that can be proved using *metric regularity* (see also Remark 5), gives a very simple and natural criterion for bounded linear regularity. Consequently, several results on projection algorithms have now a broader range of applicability.

Similarly, in the area of constrained interpolation, the Result R2 recovers and sharpens many known instances of strong CHIP.

In the context of polyhedral cones (or homogeneous linear inequalities), Result R3 is dual to Hoffman's fundamental error bound result.

Result R4 suggests that examples distinguishing the properties bounded linear regularity, strong CHIP, and CHIP, are not very easy to come by. Indeed, the conical counter-examples appearing in [6] reside in $\mathbb{R}^4$ and are quite involved.

Two notable applications are the following.

A1 A sharpening of Auslender and Crouzeix's classical result on the existence of error bounds.
A2 An explicit formula relating the best possible constant in the definition of property (G) to the angle (for two subspaces).

Organization of the paper

In Section 2, we define a norm on the sum of a collection of finitely many closed convex cones; this "cone norm" is equivalent to the given norm precisely when the collection has property (G). Due to the well-known equivalence of norms on finite-dimensional vector spaces, this happens whenever the sum is linear. Convex calculus implies that the collection is boundedly linearly regular if and only if it has strong CHIP and the collection of polar cones has property (G). By Hoffman's famous error bound result, this holds for polyhedral cones. Result R3 follows. For later use, we derive technical extensions for collections of closed convex sets.

We show the following in Section 3: for a collection of finitely many closed convex intersecting sets, bounded linear regularity implies strong CHIP (Result R2) and hence CHIP. Since the converse implications are false in general, we exhibit conditions under which bounded linear regularity, strong CHIP, and CHIP do coincide (Result R4). Examples of collections that are boundedly linearly regular (and hence strong CHIP) are presented.

The fourth section contains an entirely self-contained proof that the standard constraint qualification implies bounded linear regularity (Result R1).

The foregoing results are applied to systems of convex inequalities $f_1(x) \leq 0, \dots, f_m(x) \leq 0$ in Section 5; each set C_i is given by the sublevel set $\{x \in X : f_i(x) \leq 0\}$. Linear regularity (the "global version" of bounded linear regularity) of $\{C_1, \dots, C_m\}$ follows when the weak Slater and the asymptotic constraint qualification both hold; this improves upon a result by Auslender and Crouzeix.

In the final Section 6, we focus on two closed convex cones. Jameson's result that the two cones are boundedly linearly regular exactly when their polars have property (G) is re-derived and refined. Finally, for two subspaces, property (G) is quantified by the angle.

2. Property (G) and Hoffman's error bound

Renorming the sum of finitely many closed convex cones

Definition 1. *Suppose $K_1, \dots, K_m$ are finitely many closed convex cones in X. Let $K := K_1 + \cdots + K_m$, $\mathbf{X} := X^m$, $\mathbf{K} := K_1 \times \cdots \times K_m$, and the* sum operator *S be given* *by*

$$\mathbf{X} \to X : \mathbf{x} = (x_1, \dots, x_m) \mapsto \textstyle\sum_i x_i.$$

Suppose that $|||\cdot|||$ *is a norm on* $\mathbf{X}$*. Then we define*

$$|||x|||_K := \min\big\{|||\mathbf{x}||| : \mathbf{x} \in \mathbf{K}, S\mathbf{x} = x\big\}, \quad \forall x \in K.$$

Remark 1. The notation $|||\cdot|||_K$ is very concise but not entirely unambiguous, because the sum K does not uniquely determine its terms. However, we hope that it will always be clear from the context which terms are meant. Note that the minimum really is a minimum, *i.e.*, the infimum is attained (since $\mathbf{K} \cap S^{-1}(x)$ is closed, $\forall x \in K$).

The following elementary proposition shows that $|||\cdot|||_K$ acts like a norm on K; hence we refer to $|||\cdot|||_K$ loosely as a "cone norm" on K.

Proposition 1. *Suppose* $K_1, \dots, K_m$ *are finitely many closed convex cones in* X*. Let* $K := \sum_i K_i$ *and* $x \in K$*. Then:*

(i) $|||x|||_K \geq 0$*; moreover,* $|||x|||_K = 0 \Leftrightarrow x = 0$.
(ii) For every $\lambda \in \mathbb{R}$ *such that* $\lambda x \in K$*:* $|||\lambda x|||_K = |\lambda|\ |||x|||_K$.
(iii) For every $y \in K$*:* $|||x + y|||_K \leq |||x|||_K + |||y|||_K$.

Definition 2. *Suppose* $K_1, \dots, K_m$ *are finitely many closed convex cones in* X*. Let* $K := \sum_i K_i$*. Suppose further the product space* X^m *is equipped with the norm* $|||\cdot|||$*, defined as follows: for some fixed* $p \in [1, +\infty]$, $|||\mathbf{x}||| = \|(\|x_i\|)_i\|_p$, $\forall \mathbf{x} = (x_i)_i \in X^m$*. Then we write* $\|\cdot\|_{K,p}$ *for* $|||\cdot|||_K$*, and simply* $\|\cdot\|_K$ *when* $p = 2$*.*

We omit the proof of the following elementary result.

Proposition 2. *Suppose* $K_1, \dots, K_m$ *are finitely many closed convex cones in* X*. Let* $K := \sum_i K_i$*. Then the "cone norms" of Definition 2 are all equivalent in the following sense:* $\|x\|_{K,\infty} \leq \|x\|_{K,p} \leq \|x\|_{K,1} \leq m\|x\|_{K,\infty}$, $\forall x \in K, \forall p \in [1, +\infty]$*. Also,* $\|x\| \leq \|x\|_{K,1} \leq \sqrt{m}\|x\|_K$, $\forall x \in K$*.*

Jameson's property (G)

Remark 2. Jameson introduced property (G) for two closed convex cones in a locally convex vector space in the early 1970s. The results he obtained in [22] hold in a quite abstract setting, their proofs require more sophisticated tools from Functional Analysis. We will reprove some of his results in our setting; our proofs are much simpler.

Proposition 3. *(see also [22, Proposition 5]) Suppose* $K_1, \dots, K_m$ *are finitely many closed convex cones in* X*. If* $\{K_1, \dots, K_m\}$ *has property (G), then* $\sum_i K_i$ *is closed.*

Proof. Let $K := \sum_i K_i$ and $\alpha > 0$ such that $K \cap B_X \subseteq \alpha \sum_i (K_i \cap B_X)$. Fix an arbitrary $\bar{x} \in \mathrm{cl}\,(K)$. After scaling if necessary, we assume WLOG that $\|\bar{x}\| < 1$. Then $\bar{x} \in \mathrm{cl}\,(K \cap B_X) \subseteq \alpha \mathrm{cl}\,(\sum_i (K_i \cap B_X))$. But the terms in the last sum are all compact, hence so is the set $\sum_i (K_i \cap B_X)$. It follows that $\bar{x} \in \alpha \sum_i (K_i \cap B_X) \subseteq K$.

□

Proposition 4. *Suppose $K_1, \dots, K_m$ are finitely many closed convex cones in X. Let $K := \sum_i K_i$. Suppose further $\alpha > 0$. Then $K \cap B_X \subseteq \alpha\big[(K_1 \cap B_X) + \cdots + (K_m \cap B_X)\big]$ if and only if $\|x\|_{K,\infty} \le \alpha\|x\|$, $\forall x \in K$.*

Proof. "$\Rightarrow$": In view of Proposition 1.(ii), we can assume WLOG that $\|x\| = 1$. By assumption, there exist $y_i \in K_i \cap B_X$, $\forall i$, such that $x = \alpha \sum_i y_i$. Let $x_i := \alpha y_i \in K_i$, $\forall i$. Then $x = \sum_i x_i$ and hence $\|x\|_{K,\infty} \le \max_i \|x_i\| = \alpha \max_i \|y_i\| \le \alpha$, as desired.
"$\Leftarrow$": Suppose $x \in K \cap B_X$. Recall that the minimum in the definition of $\|x\|_{K,\infty}$ is attained; thus, there exist $x_i \in K_i$, $\forall i$, such that $x = \sum_i x_i$ and $\|x\|_{K,\infty} = \max_i \|x_i\| \le \alpha\|x\| \le \alpha$. Let $y_i := x_i/\alpha \in K_i \cap B_X$, $\forall i$. Then $x = \alpha \sum_i y_i$ and the proof is complete.
□

Corollary 1. *Suppose $K_1, \dots, K_m$ are finitely many closed convex cones in X. Let $K := \sum_i K_i$. Then the collection $\{K_1, \dots, K_m\}$ has property (G) if and only if there exists $\alpha > 0$ and $p \in [1, +\infty]$ such that $\|x\|_{K,p} \le \alpha\|x\|$, $\forall x \in K$.*

Proof. Combine Proposition 4 with Proposition 2.
□

Proposition 5. *(see also [22, Proposition 43]) Suppose $K_1, \dots, K_m$ are finitely many closed convex cones in X such that $\sum_i K_i$ is* linear. *Then $\{K_1, \dots, K_m\}$ has property (G).*

Proof. Let $K := \sum_i K_i$. Not only is K is linear but also $\|\cdot\|$ and $\|\cdot\|_K$ are norms on K (Proposition 1). Hence these two norms are equivalent ([35, Corollary 3]). By Corollary 1, $\{K_1, \dots, K_m\}$ has property (G).
□

Proposition 6. *(see also Jameson's [22, Theorem 2.1]) Suppose $C_1, \dots, C_m$ are finitely many closed convex cones in X and $\alpha > 0$. Let $C := \bigcap_i C_i$ and $K := \sum_i C_i^{\ominus}$. Then TFAE:*

(i) $\alpha\frac{1}{2}d^2(x, C) \le \sum_i \frac{1}{2}d^2(x, C_i)$, $\forall x \in X$.
(ii) $K = C^{\ominus}$ *and* $\min\big\{\sum_i \frac{1}{2}\|x_i^*\|^2 : \text{ each } x_i^* \in C_i^{\ominus}, \sum_i x_i^* = x^*\big\} \le \frac{1}{\alpha}\frac{1}{2}\|x^*\|^2$, $\forall x^* \in K$.
(iii) $K = C^{\ominus}$ *and* $\|x^*\|_K \le \frac{1}{\sqrt{\alpha}}\|x^*\|$, $\forall x^* \in K$.

Proof. Recall that $\mathrm{cl}\,(K) \subseteq C^{\ominus}$. Set $g := \alpha\frac{1}{2}d^2(\cdot, C) = \alpha(\frac{1}{2}\|\cdot\|^2 \Box \iota_C)$. Then $g^* := \frac{1}{\alpha}\frac{1}{2}\|\cdot\|^2 + \iota_{C^{\ominus}}$. Further set $f_i := \frac{1}{2}d^2(\cdot, C_i)$, $\forall i$, then $f_i^* := \frac{1}{2}\|\cdot\|^2 + \iota_{C_i^{\ominus}}$. The functions $g, f_1, \dots, f_m$ are closed convex proper; hence so is $f := \sum_i f_i$. Because each f_i is everywhere continuous, we learn that $f^* = f_1^* \Box \cdots \Box f_m^*$; moreover, this infimal convolution is *exact* on $\mathrm{dom}\, f^* = K$ (by [32, Theorem 16.4]). Using this, we obtain: (i) $\Leftrightarrow g \le f \Leftrightarrow f^* \le g^* \Leftrightarrow \inf\big\{\sum_i \frac{1}{2}\|x_i^*\|^2 : \text{ each } x_i^* \in C_i^{\ominus}, \sum_i x_i^* = x^*\big\} \le \frac{1}{\alpha}\frac{1}{2}\|x^*\|^2 + \iota_{C^{\ominus}}^*(x^*)$, $\forall x^* \in X \Leftrightarrow K = C^{\ominus}$ and $\min\big\{\sum_i \frac{1}{2}\|x_i^*\|^2 : \text{ each } x_i^* \in C_i^{\ominus}, \sum_i x_i^* = x^*\big\} \le \frac{1}{\alpha}\frac{1}{2}\|x^*\|^2$, $\forall x^* \in K \Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).
□

Remark 3. Proposition 6 says that "linear regularity (see Definition 3 below) of the original cones is equivalent to strong CHIP of the original cones and property (G) of the dual cones"; this will be refined in Theorem 10. Jameson's proof is very different from the present convex analytical approach.

Consequences of Hoffman's error bound

The classical Hoffman error bound result can be reformulated as follows.

Fact 1. *(Hoffman [20]; 1952) Suppose $C_1, \ldots, C_m$ are finitely many convex polyhedral subsets of X with $C := \bigcap_i C_i \neq \emptyset$. Then there exists $\alpha > 0$ such that $\alpha \frac{1}{2} d^2(x, C) \leq \sum_i \frac{1}{2} d^2(x, C_i)$, $\forall x \in X$.*

Proof. See [4, Corollary 5.26].

□

Our reformulation of Hoffman's result is somewhat unusual; however, it allows a direct application of Proposition 6. For instance, we deduce (using also Corollary 1) the following result, dual to Hoffman's result.

Corollary 2 (Dual of Hoffman's error bound). *Every finite collection of convex polyhedral cones has property (G).*

The next result is a useful extension of Corollary 2.

Theorem 1. *Suppose $C_1, \ldots, C_m$ are finitely many convex polyhedral subsets of X with $C := \bigcap_i C_i \neq \emptyset$. Then there exists $\beta > 0$ such that*

$$\min\Big\{\sum_i \|y_i\| : \text{ each } y_i \in N_{C_i}(c), \sum_i y_i = y\Big\} \leq \beta \|y\|, \quad \forall c \in C, \forall y \in N_C(c).$$

Proof. For brevity, let us write T_i for T_{C_i}, $\forall i$. Note that (by [32, Corollary 23.8.1]) $N_C(c) = \sum_i N_{C_i}(c)$ and thus (by taking polars and [32, Corollary 16.4.2]) $T_C(c) = \bigcap_i T_i(c)$, $\forall c \in C$. Now consider the collection of m-tuples of tangent cones $\mathcal{T} := \big\{(T_1(c), \ldots, T_m(c)) : c \in C\big\}$. Because each C_i is *polyhedral*, [18, Examples III.5.2.6.(b)] implies that the collection $\mathcal{T}$ is finite: there exist finitely many $c_1, \ldots, c_n \in C$ such that $\mathcal{T} = \bigcup_{j \in \{1, \ldots, n\}} \big\{(T_1(c_j), \ldots, T_m(c_j))\big\}$. Now every tangent cone $T_i(c_j)$ is polyhedral and $T_C(c_j) = \bigcap_i T_i(c_j)$, $\forall j$, so Hoffman's result (Fact 1) yields the existence of positive reals $\alpha_1, \ldots, \alpha_n$ such that $\alpha_j \frac{1}{2} d^2(x, T_C(c_j)) \leq \sum_i \frac{1}{2} d^2(x, T_i(c_j))$, $\forall x \in X, \forall j \in \{1, \ldots, n\}$. Hence after setting $\alpha := \min\{\alpha_1, \ldots, \alpha_n\} > 0$, we obtain $\alpha \frac{1}{2} d^2(x, T_C(c)) \leq \sum_i \frac{1}{2} d^2(x, T_i(c))$, $\forall x \in X$, $\forall c \in C$. Set $K(c) := \sum_i T_i^{\ominus}(c) = \sum_i N_{C_i}(c) = N_C(c) = T_C^{\ominus}(c)$, $\forall c \in C$. Then Proposition 6 results in $\|y\|_{K(c)} \leq \frac{1}{\sqrt{\alpha}} \|y\|$, $\forall c \in C$, $\forall y \in K(c)$. Use Proposition 2 to further get $\frac{1}{\sqrt{m}} \|y\|_{K(c),1} \leq \|y\|_{K(c)} \leq \frac{1}{\sqrt{\alpha}} \|y\|$, $\forall c \in C$, $\forall y \in K(c)$. Therefore, the desired inequality holds with $\beta := \sqrt{m/\alpha}$.

□

We conclude this section with an extension of Theorem 1 to the case when nonpolyhedral sets are involved; this will turn out to be very useful in the later development.

Theorem 2. *Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X, where, for some $r \in \{0, \ldots, m\}$, the sets $C_{r+1}, \ldots, C_m$ are polyhedral. Suppose further $z \in \bigcap_{i=1}^r \mathrm{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i$ and let $C := \bigcap_{i=1}^m C_i$. Then $N_C(x) = \sum_i N_{C_i}(x)$,*

$\forall x \in C$. *Let* $A_i := \operatorname{aff}(C_i)$, $\forall i \in \{1, \dots, r\}$ *and* $\delta > 0$ *such that* $A_i \cap B(z;\delta) \subseteq C_i$, $\forall i \in \{1, \dots, r\}$. *Then there exists* $\beta > 0$, *independent of* z *and* δ, *such that*

$$\min\left\{\sum_{i=1}^m \|y_i\| : \text{ each } y_i \in N_{C_i}(x), \sum_{i=1}^m y_i = y\right\} \le \tfrac{\|x-z\|(1+\beta)+\delta\beta}{\delta}\|y\|,$$

$\forall x \in C, \forall y \in N_C(x)$.

Proof. The statement concerning the normal cones follows from [32, Corollary 23.8.1]. Let $L_i := \operatorname{span}(C_i - z)$ so that $A_i = z + \operatorname{span}(C_i - z) = z + L_i = C_i + L_i$, $\forall i \in \{1, \dots, r\}$. Define further $D_0 := \bigcap_{i=1}^r A_i \cap \bigcap_{i=r+1}^m C_i$ and $D_i := C_i + L_i^\perp$, $\forall i \in \{1, \dots, r\}$. After verification of $(C_i + L_i) \cap (C_i + L_i^\perp) = C_i$, $\forall i \in \{1, \dots, r\}$, it is easy to see that $C = \bigcap_{i=0}^r D_i$. Let us fix momentarily $i \in \{1, \dots, r\}$ and $b \in B_X$. Write $b = l_i + l_i^\perp$, where $l_i \in L_i$ and $l_i^\perp \in L_i^\perp$. Then $l_i, l_i^\perp \in B_X$. Hence $z + \delta b = (z + \delta l_i) + \delta l_i^\perp \in (B(z;\delta) \cap A_i) + L_i^\perp \subseteq C_i + L_i^\perp$. Thus $B(z;\delta) \subseteq D_i$, $\forall i \in \{1, \dots, r\}$ and so $z \in D_0 \cap \bigcap_{i=1}^r \operatorname{int}(D_i)$, which in turn implies (by [32, Corollary 23.8.1] again) $N_C(x) = \sum_{i=0}^r N_{D_i}(x)$, $\forall x \in C$. For the remainder of the proof, fix $x \in C$ and $y \in N_C(x) = \sum_{i=1}^m N_{C_i}(x) = \sum_{i=0}^r N_{D_i}(x)$. Obtain $\bar{y}_i \in N_{D_i}(x)$, $\forall i \in \{0, \dots, r\}$ such that $y = \sum_{i=0}^r \bar{y}_i$. Then $\langle x, \bar{y}_i\rangle = \sup\langle D_i, \bar{y}_i\rangle \ge \sup\langle B(z;\delta), \bar{y}_i\rangle = \langle z, \bar{y}_i\rangle + \delta\|\bar{y}_i\|$, $\forall i \in \{1, \dots, r\}$ and $\langle x, \bar{y}_0\rangle = \sup\langle D_0, \bar{y}_0\rangle \ge \langle z, \bar{y}_0\rangle$. Hence $\langle x, y\rangle = \sum_{i=0}^r \langle x, \bar{y}_i\rangle \ge \sum_{i=0}^r \langle z, \bar{y}_i\rangle + \delta \sum_{i=1}^r \|\bar{y}_i\| = \langle z, y\rangle + \delta \sum_{i=1}^r \|\bar{y}_i\|$, which implies

$$\sum_{i=1}^r \|\bar{y}_i\| \le \frac{1}{\delta}\langle x - z, y\rangle \le \frac{\|x - z\|}{\delta}\|y\| \tag{1}$$

and further

$$\|\bar{y}_0\| = \left\| y - \sum_{i=1}^r \bar{y}_i \right\| \le \|y\| + \sum_{i=1}^r \|\bar{y}_i\| \le \frac{\|x - z\| + \delta}{\delta}\|y\|. \tag{2}$$

Now $A_1, \dots, A_r, C_{r+1}, \dots, C_m$ are polyhedral and their intersection equals D_0. By Theorem 1, there exist $\beta > 0$ (depending only on $A_1, \dots, A_r, C_{r+1}, \dots, C_m$) and $\tilde{y}_i \in N_{A_i}(x) = L_i^\perp$, $\forall i \in \{1, \dots, r\}$, $y_i \in N_{C_i}(x)$, $\forall i \in \{r+1, \dots, m\}$ such that

$$\sum_{i=1}^r \tilde{y}_i + \sum_{i=r+1}^m y_i = \bar{y}_0 \quad \text{and} \quad \sum_{i=1}^r \|\tilde{y}_i\| + \sum_{i=r+1}^m \|y_i\| \le \beta\|\bar{y}_0\|. \tag{3}$$

Define $y_i := \bar{y}_i + \tilde{y}_i$, $\forall i \in \{1, \dots, r\}$. On the one hand, $C_i \subseteq D_i$, and so $\bar{y}_i \in N_{D_i}(x) \subseteq N_{C_i}(x)$, $\forall i \in \{1, \dots, r\}$. On the other hand, $C_i \subseteq A_i$, and so $\tilde{y}_i \in N_{A_i}(x) \subseteq N_{C_i}(x)$, $\forall i \in \{1, \dots, r\}$. Altogether, $y_i \in N_{C_i}(x)$, $\forall i \in \{1, \dots, m\}$. Also, using (3),

$$\sum_{i=1}^m y_i = \sum_{i=1}^r (\bar{y}_i + \tilde{y}_i) + \sum_{i=r+1}^m y_i = \bar{y}_0 + \sum_{i=1}^r \bar{y}_i = y.$$

Therefore, invoking (1), (2), and (3),

$$\begin{aligned}\sum_{i=1}^m \|y_i\| &\le \sum_{i=1}^r (\|\bar{y}_i\| + \|\tilde{y}_i\|) + \sum_{i=r+1}^m \|y_i\| \le \frac{\|x - z\|}{\delta}\|y\| + \beta\|\bar{y}_0\| \\ &\le \left(\frac{\|x - z\|}{\delta} + \beta\frac{\|x - z\| + \delta}{\delta}\right)\|y\|.\end{aligned}$$

□

3. Strong CHIP and bounded linear regularity

(Strong) CHIP . . .

It is clear (take polars) that strong CHIP implies CHIP. The relationship between these two concepts is made more precise in the next proposition.

Proposition 7. *(see also [15, Deutsch* et al.*'s Lemma 2.4]) Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Then TFAE:*

(i) $\{C_1, \ldots, C_m\}$ *has strong CHIP.*
(ii) $\partial(\sum_i \iota_{C_i}) = \sum_i \partial\iota_{C_i}$.
(iii) $\{C_1, \ldots, C_m\}$ *has CHIP and* $\sum_i N_{C_i}(x)$ *is closed,* $\forall x \in C$.

Proof. "(i) ⇔ (ii)": obvious. "(i) ⇔ (iii)": use [32, Corollary 16.4.2].

□

Remark 4. The properties strong CHIP and CHIP are pivotal in the study of constrained approximation problems; see [8], [9], [11], [15], and [16]. Recall that the following inclusions always hold: $\mathrm{cl}\,(\sum_i N_{C_i}(x)) \subseteq N_C(x)$ and $T_C(x) \subseteq \bigcap_i T_{C_i}(x)$, $\forall x \in C$.

In $\mathbb{R}^2$, CHIP and strong CHIP are the same: indeed, all cones in $\mathbb{R}^2$ are polyhedral and so the sum of polyhedral cones is (polyhedral and hence) closed.

. . . and (bounded) linear regularity

Definition 3. *Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$. Then $\{C_1, \ldots, C_m\}$ is* linearly regular, *if there exists $\kappa > 0$ such that $d(x, C) \leq \kappa \max_i d(x, C_i)$, $\forall x \in X$.*

It is clear that linear regularity implies bounded linear regularity.

Remark 5 (bounded linear and metric regularity). Bounded linear regularity of the collection $\{C_1, \ldots, C_m\}$ is a quantitative version of a very intuitive idea: "closeness to all sets C_i implies closeness to their intersection". This concept is of fundamental importance in the study of projection methods; see [3], [4, Section 5], and [2, Chapters 4 and 5].

For clarity, we now consider two closed convex sets C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$. Define a set-valued map $\Omega : X \to 2^X$ by $\Omega(x) = x - C_2$, if $x \in C_1$; $\Omega(x) = \emptyset$, otherwise. Then the range of Ω is $C_1 - C_2$ and the Slater-type condition $0 \in \mathrm{int}\,(C_1 - C_2)$ yields *metric regularity* (in the sense of Robinson [31]) of Ω and thus the existence of κ and $\delta > 0$ depending on an arbitrary but fixed $\bar{x} \in C_1 \cap C_2 = \Omega^{-1}(0)$ such that $d(x, C_1 \cap C_2) \leq \kappa d(x, C_2)$, $\forall x \in C_1 \cap B(\bar{x}; \delta)$. (The literature on metric regularity is vast. We refer the interested reader to [33, Section 9.G and the commentary to Chapter 9] as a starting point.) This last condition in turn guarantees the formally stronger bounded linear regularity of $\{C_1, C_2\}$; see [3, Lemma 4.1].

Theorem 3 (bounded linear regularity implies strong CHIP). *Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$. Consider the following three conditions.*

(i) $\{C_1, \ldots, C_m\}$ *is boundedly linearly regular.*
(ii) $\{C_1, \ldots, C_m\}$ *has strong CHIP.*
(iii) $\{C_1, \ldots, C_m\}$ *has CHIP.*

Then: (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii).*

Proof. "(i)⇒(ii)": Fix $x \in C$ and $x^* \in \partial \iota_C(x) = N_C(x)$. In view of Proposition 7 and Remark 4, it suffices to show that $x^* \in \sum_i \partial \iota_{C_i}(x) = \sum_i N_{C_i}(x)$. We may assume WLOG $x^* \neq 0$ and then let $\hat{x}^* := x^*/\|x^*\|$. Then $\hat{x}^* \in N_C(x) \cap B_X = \partial d(\cdot, C)(x)$; see [18, Example VI.3.3]. Thus $\langle \hat{x}^*, y - x \rangle \leq d(y, C) - d(x, C) = d(y, C)$, $\forall y \in X$. On the other hand, $\{C_1, \ldots, C_m\}$ is boundedly linearly regular; we thus obtain $\kappa > 0$ such that $d(y, C) \leq \kappa \sum_i d(y, C_i)$, $\forall y \in B(x; 1)$. Altogether $\langle \hat{x}^*, y - x \rangle \leq \kappa \sum_i d(y, C_i)$, $\forall y \in B(x; 1)$. It follows that $\hat{x}^*/\kappa \in \partial(\sum_i d(\cdot, C_i) + \iota_{B(x;1)})(x) = \sum_i \big(N_{C_i}(x) \cap B_X\big)$; consequently, $x^* \in \sum_i N_{C_i}(x)$. "(ii)⇒(iii)": Proposition 7.

□

Remark 6. Theorem 3 was proved independently by Pang [29, Proposition 6] resting on Lewis and Pang's [26, Proposition 2].

Converse implications

In general, the properties bounded linear regularity, strong CHIP, and CHIP are all different [6]. In this subsection, we focus on conditions that make the three properties coincide.

Proposition 8. *Suppose* $C_1, \ldots, C_m$ *are finitely many closed convex sets in* X *with* $C := \bigcap_i C_i \neq \emptyset$. *Suppose further that* $\{C_1, \ldots, C_m\}$ *is not boundedly linearly regular. Then there exist: (i) a sequence* (c_n) *in* C *converging to some* $\bar{c} \in C$, *(ii) a point* $q \in S_X \cap N_C(\bar{c})$, *(iii) sequences* $(y_{i,n})$ *such that* $y_{i,n} \in T_{C_i}(c_n)$, $\forall n$ *and* $y_{i,n} \to q$, $\forall i$. *Moreover, if each* C_i *is a cone, then we can additionally arrange that (iv)* $\bar{c} \in S_X$.

Proof. Because $\{C_1, \ldots, C_m\}$ is not boundedly linearly regular, there exists a bounded sequence (x_n) in C such that

$$d(x_n, C) > n \max_i d(x_n, C_i), \qquad \forall n. \tag{4}$$

(*Digression.* Suppose momentarily that each C_i is a cone. Then (4) shows that $x_n \neq 0$, $\forall n$. Because projections onto cones are positively homogeneous, we can replace each x_n by $x_n/\|x_n\|$ to get a new sequence lying entirely in S_X and still satisfying (4). This will prove the "Moreover" part. *End of Digression.*) Now (x_n) is bounded, hence so is $(d(x_n, C))$. By (4), $\max_i d(x_n, C_i) \to 0$. This implies that $d(x_n, C) \to 0$ (by either a straight-forward proof by contradiction or [4, Proposition 5.4.(iii)]). Hence all cluster points of (x_n) lie in C. For brevity, set $c_n := P_C(x_n)$ and $c_{i,n} := P_{C_i}(x_n)$, $\forall n, \forall i$. Then $0 < d(x_n, C) = \|x_n - c_n\|$ and $d(x_n, C_i) = \|x_n - c_{i,n}\|$, $\forall n, \forall i$. After passing to a subsequence if necessary, we assume WLOG that (x_n) converges to some $\bar{c} \in C$ and that $((x_n - c_n)/\|x_n - c_n\|)$ converges to some $q \in S_X$. Hence (c_n) and each $(c_{i,n})$

converge to $\bar{c}$, too. Set $t_n := \|x_n - c_n\| > 0$, $\forall n$. Then $t_n \to 0$. We have, using $d(x_n, C_i)/d(x_n, C) \le 1/n \to 0$ from (4),

$$y_{i,n} := \frac{c_{i,n} - c_n}{\|x_n - c_n\|} = \frac{c_{i,n} - x_n}{\|x_n - c_n\|} + \frac{x_n - c_n}{\|x_n - c_n\|} \to q, \quad \forall i.$$

Since $c_n + t_n y_{i,n} = c_{i,n} \in C_i$, we obtain $y_{i,n} \in T_{C_i}(c_n)$, $\forall n, \forall i$. Also, $\langle c - c_n, x_n - c_n\rangle / \|x_n - c_n\| \le 0$, $\forall n, \forall c \in C$ (using $c_n = P_C(x_n)$ and a well-known property of projections; see, for instance, Proposition 12.(i) below). Taking limits yields $q \in N_C(\bar{c})$. Therefore, we have constructed objects that possess all announced properties.

□

Definition 4. *Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$ and $\bar{x} \in C$. We say that the collection $\{C_1, \ldots, C_m\}$ is* intersection-closed at x, *if whenever (c_n) is a sequence in C converging to $\bar{c}$ and $(y_{i,n})$ are sequences converging to y_i with $y_{i,n} \in T_{C_i}(c_n)$, $\forall n, \forall i$, then necessarily $y_i \in T_{C_i}(\bar{c})$, $\forall i$. If D is a closed nonempty subset of C and $\{C_1, \ldots, C_m\}$ is intersection-closed at every point in D, then we say that $\{C_1, \ldots, C_m\}$* is intersection-closed on D.

Definition 5. *(see also [21, Section 7.D] or [34, Definition 7.5]) Suppose C is a closed convex set in X with* $\operatorname{int}(C) \neq \emptyset$ *and* $\bar{x} \in \operatorname{bd}(C)$. *We say that C is* smooth at $\bar{x}$, *if $T_C(\bar{x})$ is a halfspace; equivalently, if $N_C(\bar{x})$ is a ray. If C is smooth at $\bar{x}$, then $N_C(\bar{x}) = [0, +\infty[\cdot \hat{d}_C(\bar{x})$, for some unique vector $\hat{d}_C(\bar{x}) \in S_X$, called* the normal direction of C at $\bar{x}$. *Suppose D is a closed nonempty subset of* $\operatorname{bd}(C)$. *If C is smooth at every $x \in D$, then we say that C* is smooth on D. *If there exists some $\epsilon > 0$ such that C is smooth on $C \cap B(\bar{x}; \epsilon)$, then we say that C* is locally smooth at $\bar{x}$. *Finally, if C is locally smooth at every $x \in D$, then we say that C* is locally smooth on D.

Proposition 9. *Suppose C is a closed convex set in X with* $\operatorname{int}(C) \neq \emptyset$ *and D is a closed subset of* $\operatorname{bd}(C)$ *on which C is smooth. Then the mapping $\hat{d}_C|_D : D \to S_X : x \mapsto \hat{d}_C(x)$ is continuous.*

Proof. Suppose (x_n) is a sequence in D converging to some point $\bar{x} \in D$. Let $\hat{d}_n := \hat{d}_C(x_n) \in S_X$, $\forall n$. We have to show that $(\hat{d}_n)$ converges to $\hat{d}_C(\bar{x})$. Let d' be an arbitrary cluster point of $(\hat{d}_n)$. Then $d' \in S_X$ and there exists a subsequence $(\hat{d}_{n'})$ of $(\hat{d}_n)$ that converges to d'. Now $\hat{d}_n \in N_C(x_n)$, hence $\langle c - x_n, \hat{d}_n\rangle \le 0$, $\forall n, \forall c \in C$. Taking limits along (n') yields $\langle c - \bar{x}, d'\rangle \le 0$, $\forall c \in C$. Thus $d' \in N_C(\bar{x}) \cap S_X = \{\hat{d}_C(\bar{x})\}$. Therefore, the entire sequence $(\hat{d}_n)$ converges to $\hat{d}_C(\bar{x})$ and the proof is complete.

□

Proposition 10. *Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$ and $\bar{c} \in C$. If C_i is locally smooth at $\bar{c}$ or $T_{C_i}|_C$ does not change on a neighborhood of $\bar{c}$, $\forall i$, then $\{C_1, \ldots, C_m\}$ is intersection-closed at $\bar{c}$.*

Proof. Let (c_n) be a sequence in C converging to $\bar{c}$ and $(y_{i,n})$ be a sequence converging to some y_i such that $y_{i,n} \in T_{C_i}(c_n)$, $\forall i, \forall n$. Fix $i \in \{1, \ldots, m\}$. If $T_{C_i}|_C$ is constant on a neighborhood of $\bar{c}$, then $y_i \in T_{C_i}(\bar{c})$ and we are done (tangent cones are closed). Otherwise, C_i is locally smooth at $\bar{c}$. Let $d_{i,n} := \hat{d}_{C_i}(c_n)$ be the normal direction of C_i

at c_n, $\forall n$. Then $y_{n,i} \in T_{C_i}(c_n) = N_{C_i}^{\ominus}(c_n) = ([0,+\infty[\cdot d_{i,n})^{\ominus} \Rightarrow \langle y_{n,i}, d_{i,n}\rangle \le 0$, $\forall n$. Let $d_i := \hat{d}_{C_i}(\bar{c})$. Then, by Proposition 9, $d_i = \lim_n d_{i,n}$. $\forall i$. Thus $\langle y_i, d_i\rangle \le 0$, *i.e.*, $y_i \in N_{C_i}^{\ominus}(\bar{c}) = T_{C_i}(\bar{c})$. Altogether, $\{C_1, \dots, C_m\}$ is intersection-closed at $\bar{c}$.

□

Proposition 11. *Suppose $C_1, \dots, C_m$ are finitely many closed convex cones in X such that $C := \bigcap_i C_i$ is a* ray*: $C = [0,+\infty[\cdot\bar{c}$, for some $\bar{c} \in C \setminus \{0\}$. Then $\{C_1, \dots, C_m\}$ is intersection-closed at $\bar{c}$.*

Proof. Suppose (c_n) is a sequence in C converging to $\bar{c}$. WLOG $c_n = p_n\bar{c}$, $\forall n$, where (p_n) is a sequence of positive reals. By Proposition 18, $T_{C_i}(c_n) = T_{C_i}(p_n\bar{c}) = \mathrm{cl}\,(C_i + \mathbb{R}p_n\bar{c}) = \mathrm{cl}\,(C_i + \mathbb{R}\bar{c}) = T_{C_i}(\bar{c})$, $\forall n$. Hence $\{C_1, \dots, C_m\}$ is intersection-closed at $\bar{c}$.

□

Theorem 4. *Suppose $C_1, \dots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$. Suppose further that $\{C_1, \dots, C_m\}$ is intersection-closed on C. In particular, this holds when (i) C is singleton; or (ii) each C_i is smooth on C. Then $\{C_1, \dots, C_m\}$ is boundedly linearly regular if and only if it has CHIP.*

Proof. By Theorem 3, bounded linear regularity implies CHIP, so we have to show that CHIP implies bounded linear regularity. We do so by contradiction: assume $\{C_1, \dots, C_m\}$ has CHIP but is not boundedly linearly regular. Then, by Proposition 8, there exist

a sequence (c_n) in C converging to some $\bar{c} \in C$, a point $q \in S_X \cap N_C(\bar{c})$, sequences $(y_{i,n})$ such that $y_{i,n} \in T_{C_i}(c_n)$, $\forall n$ and $y_{i,n} \to q$, $\forall i$.

Now $\{C_1, \dots, C_m\}$ is intersection-closed, whence $q \in T_{C_i}(\bar{c})$, $\forall i$. Since $\{C_1, \dots, C_m\}$ has also CHIP, we conclude $q \in T_C(\bar{c})$. On the other hand, $q \in N_C(\bar{c}) \cap S_X$. Altogether, we contradict $N_C(\bar{c}) \cap T_C(\bar{c}) = \{0\}$. The "In particular" case follows from Proposition 10.

□

Theorem 5. *Suppose $C_1, \dots, C_m$ are finitely many closed convex cones in X and let $C := \bigcap_i C_i$. Suppose further that $\{C_1, \dots, C_m\}$ is intersection-closed on $C \cap S_X$. In particular, this holds when (i) each C_i is a subspace or locally smooth on $C \cap S_X$; or (ii) C is a ray or linear. Then $\{C_1, \dots, C_m\}$ is boundedly linearly regular if and only if it has CHIP.*

Proof. The main statement follows similarly to the proof of Theorem 4 (use the "cone" version of Proposition 8). The "In particular" part follows from Proposition 10, Proposition 11, Corollary 9, and Remark 15 below.

□

Examples

Several conditions sufficient for bounded linear regularity can be found in [3, Section 4], [4, Section 5], and [2, Chapter 4 and 5]. In view of Theorem 3, we thus immediately obtain criteria for strong CHIP.

It is most satisfying that the standard constraint qualification yields bounded linear regularity. (For a self-contained proof, see Corollary 5 below.)

Corollary 3. *Suppose $C_1, \dots, C_m$ are finitely many closed convex sets in X, where $C_{r+1}, \dots, C_m$ are polyhedral, for some $r \in \{0, \dots, m\}$. Suppose further $\bigcap_{i=1}^{r} \operatorname{ri}(C_i) \cap \bigcap_{i=r+1}^{m} C_i \neq \emptyset$. Then $\{C_1, \dots, C_m\}$ is boundedly linearly regular and hence has strong CHIP.*

Proof. The collection $\{C_1, \dots, C_m\}$ is boundedly linearly regular ([2, Theorem 5.6.2]) and hence has strong CHIP (Theorem 3).

□

It is worth noting two special cases.

Example 1. (see also [15, Theorems 3.6 and 3.12]) Suppose A is a linear operator from X to some Euclidean space Y, and C_1 is a closed convex subset in X. Let $C_2 := A^{-1}(b)$ and suppose $C_1 \cap C_2 \neq \emptyset$. Then $\{C_1, C_2\}$ is boundedly linearly regular and hence has strong CHIP, whenever (i) C_1 is polyhedral; or (ii) $b \in \operatorname{ri} A(C_1)$.

Example 2. A collection of finitely many subspaces of X is boundedly linearly regular and has strong CHIP.

Remark 7. With care, some of the results of this subsection generalize to the case when X is an infinite-dimensional Hilbert space: Example 1 remains true without change (use [4, Corollary 5.26] and [2, Proposition 4.7.1]). Example 2 is no longer true in infinite-dimensional Hilbert space: in fact, a collection $\{C_1, \dots, C_m\}$ of finitely many closed subspaces is boundedly linearly regular if and only if it has strong CHIP if and only if $\sum_i C_i^{\perp}$ is closed; see [4, Theorem 5.19].

4. Bounded linear regularity via convex analysis

Work in this section culminates in a self-contained convex-analytical proof of Corollary 3.

Proposition 12. *Suppose S is a closed convex nonempty set in X and let $x, x^* \in X$. Then:*

(i) $x - P_S(x) \in N_S(P_S(x))$.
(ii) $\frac{1}{2}d^2(x, S) + \frac{1}{2}\|x^*\|^2 + \iota_S^*(x^*) \geq \langle x^*, x\rangle$.
(iii) $x^* \in N_S(x)$ *if and only if* $x \in S$ *and* $\langle x^*, x\rangle = \iota_S^*(x^*)$.

Proof. (i) is a well-known property of the projection; see [18, Theorem III.3.1.1]. (ii): Note that $\frac{1}{2}d^2(\cdot, S) = \frac{1}{2}\|\cdot\|^2 \Box \iota_S$, hence $(\frac{1}{2}d^2(\cdot, S))^*(x^*) = \frac{1}{2}\|x^*\|^2 + \iota_S^*(x^*)$.The statements follow from the Fenchel-Young inequality; see [32, Theorem 23.5]. (iii): Follows from the characterization of equality in the Fenchel-Young inequality applied to the function ι_S.

□

Theorem 6. *Suppose $C_1, \dots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$. Suppose further $x \in X$ with $T_C(P_C(x)) = \bigcap_i T_{C_i}(P_C(x))$ and there exists $\lambda_x > 0$ such that for every $y \in \sum_i N_{C_i}(P_C(x))$,*

$$\min\Big\{\sum_i \|y_i\|^2 : \textit{ each } \ y_i \in N_{C_i}(P_C(x)), \sum_i y_i = y\Big\} \le \lambda_x^2 \|y\|^2.$$

Then $\sum_i N_{C_i}(P_C(x)) = N_C(P_C(x))$ and $d(x, C) \le \lambda_x \sum_i d(x, C_i)$.

Proof. "$\sum_i N_{C_i}(P_C(x)) = N_C(P_C(x))$": On the one hand, the assumption on the tangent cones yields (take polars), $N_C(P_C(x)) = \mathrm{cl}\,(\sum_i N_{C_i}(P_C(x)))$. On the other hand, by the assumption on the minimum and Corollary 1, the collection $\{N_{C_1}(P_C(x)), \dots, N_{C_m}(P_C(x))\}$ has property (G). Thus, by Proposition 3, $\sum_i N_{C_i}(P_C(x))$ is closed.
"$d(x, C) \le \lambda_x \sum_i d(x, C_i)$": By Proposition 12.(i), $x - P_C(x) \in N_C(P_C(x))$. By the preceding statement on the normal cones, there exists $y_i \in N_{C_i}(P_C(x))$, $\forall i$, such that $x - P_C(x) = \sum_i y_i$ and $\sum_i \|y_i\|^2 \le \lambda_x^2 \|x - P_C(x)\|^2$. Also, by Proposition 12.(iii), $\langle y_i, P_C(x)\rangle = \iota^*_{C_i}(y_i)$. Using the last displayed inequality and Proposition 12.(ii), we obtain the inequalities in the following chain of equalities and inequalities.

$$\begin{aligned}
\tfrac{1}{2} d^2(x, C) &= \tfrac{1}{2}\|x - P_C(x)\|^2 \\
&= \langle x - P_C(x), x - P_C(x)\rangle - \tfrac{1}{2}\|x - P_C(x)\|^2 \\
&= \langle x - P_C(x), \textstyle\sum_i y_i\rangle - \tfrac{1}{2}\|x - P_C(x)\|^2 \\
&= \textstyle\sum_i \langle x, y_i\rangle - \sum_i \langle P_C(x), y_i\rangle - \tfrac{1}{2}\|x - P_C(x)\|^2 \\
&= \textstyle\sum_i \langle x, y_i\rangle - \sum_i \iota^*_{C_i}(y_i) - \tfrac{1}{2}\|x - P_C(x)\|^2 \\
&\le \textstyle\sum_i \langle x, y_i\rangle - \sum_i \iota^*_{C_i}(y_i) - \tfrac{1}{2}\tfrac{1}{\lambda_x^2}\sum_i \|y_i\|^2 \\
&= \tfrac{1}{\lambda_x^2}\textstyle\sum_i \big(\langle \lambda_x^2 x, y_i\rangle - \iota^*_{\lambda_x^2 C_i}(y_i) - \tfrac{1}{2}\|y_i\|^2\big) \\
&\le \tfrac{1}{\lambda_x^2}\textstyle\sum_i \tfrac{1}{2} d^2(\lambda_x^2 x, \lambda_x^2 C_i) \\
&= \tfrac{\lambda_x^2}{2}\textstyle\sum_i d^2(x, C_i).
\end{aligned}$$

□

The next example shows that the assumption on the intersection of the tangent cones in Theorem 6 is important.

Example 3. Let $X := \mathbb{R}^2$, $C_1 := B((1, 0); 1)$ and $C_2 := B((-1, 0); 1)$. Then $C := C_1 \cap C_2 := \{0\}$. Hence $P_C(x) = 0$, $\forall x \in X$. Now $T_{C_1}(0) = [0, +\infty[\times\mathbb{R}$, $T_{C_2}(0) =]-\infty, 0] \times \mathbb{R}$, and $T_C(0) = \{0\}$. Hence $T_{C_1}(0) \cap T_{C_2}(0) = \{0\} \times \mathbb{R} \supsetneqq \{0\} = T_C(0)$, *i.e.*, the condition on the intersection of the tangent cones in Theorem 6 is not satisfied and $\{C_1, C_2\}$ does not have CHIP. On the other hand, the condition on the minimum is satisfied with $\lambda_x = 1$, $\forall x \in X$, but the first conclusion of Theorem 6 fails. Now consider $x := (0, r)$, where $r > 0$. Then $d(x, C_1) = d(x, C_2) = \sqrt{1 + r^2} - 1$ and $d(x, C) = r$. Since $d(x, C)/(d(x, C_1) + d(x, C_2)) = r/(2\sqrt{1 + r^2} - 2) \to +\infty$, the second conclusion of Theorem 6 is not true either.

Applications

Corollary 4. *Suppose $C_1, \dots, C_m$ are finitely many closed convex sets in X with $C := \bigcap_i C_i \neq \emptyset$. Suppose further $T_C(c) = \bigcap_i T_{C_i}(c)$, $\forall c \in C$ and there exists $\lambda > 0$ such that*

$$\min\Big\{\sum_i \|y_i\|^2 : \text{ each } y_i \in N_{C_i}(c), \sum_i y_i = y\Big\} \leq \lambda^2 \|y\|^2, \quad \forall c \in C, \forall y \in N_C(c).$$

Then $d(x, C) \leq \lambda \sum_i d(x, C_i)$, $\forall x \in X$.

Proof. Theorem 6.

□

Remark 8. Corollary 4 says that "if the tangent cones of $\{C_1, \dots, C_m\}$ have CHIP and the collection of normal cones has property (G) (uniformly on C), then the collection $\{C_1, \dots, C_m\}$ is linearly regular."

Theorem 7. *Suppose $C_1, \dots, C_m$ are finitely many closed convex sets in X, where, for some $r \in \{0, \dots, m\}$, the sets $C_{r+1}, \dots, C_m$ are polyhedral. Suppose further there exists $z \in \bigcap_{i=1}^r \operatorname{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i$ and let $C := \bigcap_{i=1}^m C_i$. Then there exists $\gamma > 0$, depending on z, such that*

$$d(x, C) \leq \gamma(\|P_C(x) - z\| + 1) \sum_i d(x, C_i), \quad \forall x \in X.$$

Proof. Let $\delta > 0$ such that $\operatorname{aff}(C_i) \cap B(z; \delta) \subseteq C_i$, $\forall i \in \{1, \dots, r\}$. Then Theorem 2 yields $\beta > 0$, independent of z and δ, such that for every $c \in C$ and $y \in N_C(c)$,

$$\min\Big\{\sum_{i=1}^m \|y_i\| : \text{ each } y_i \in N_{C_i}(c), \sum_{i=1}^m y_i = y\Big\} \leq \tfrac{\|c-z\|(1+\beta)+\delta\beta}{\delta} \|y\|.$$

For an arbitrary but fixed $x \in X$, define $\lambda_x := \big(\|P_C(x) - z\|(1+\beta) + \delta\beta\big)/\delta$. Then for every $y \in N_C(P_C(x)) = \sum_i N_{C_i}(P_C(x))$,

$$\min\Big\{\sum_{i=1}^m \|y_i\|^2 : \text{ each } y_i \in N_{C_i}(P_C(x)), \sum_{i=1}^m y_i = y\Big\} \leq \lambda_x^2 \|y\|^2.$$

Theorem 6 implies $d(x, C) \leq \frac{\|P_C(x)-z\|(1+\beta)+\delta\beta}{\delta} \sum_i d(x, C_i)$, $\forall x \in X$; therefore, $\gamma := \max\{(1+\beta)/\delta, \beta\}$ does the job.

□

Corollary 5 (standard constraint qualification implies bounded linear regularity). *Suppose $C_1, \dots, C_m$ are finitely many closed convex sets in X, where, for some $r \in \{0, \dots, m\}$, the sets are polyhedral. Suppose further $\bigcap_{i=1}^r \operatorname{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i \neq \emptyset$. Then for every $\rho > 0$, there exists $\lambda_\rho > 0$ such that $d(x, C) \leq \lambda_\rho \sum_i d(x, C_i)$, $\forall x \in \rho B_X$.*

Proof. Pick an arbitrary $z \in \bigcap_{i=1}^r \operatorname{ri}(C_i) \cap \bigcap_{i=r+1}^m C_i$. By Theorem 7, there exists $\gamma > 0$ such that $d(x, C) \leq \gamma(\|P_C(x) - z\| + 1) \sum_i d(x, C_i)$, $\forall x \in X$. Therefore, $\lambda_\rho := \gamma(\rho + \|z\| + 1)$ does the job.

□

Remark 9. If $C_1, \ldots, C_m$ are closed convex sets in X with $\bigcap_i \operatorname{int}(C_i) \neq \emptyset$, then the Robinson-Ursescu theorem [33, Theorem 9.48] can be used to prove Corollary 5. The general statement of Corollary 5 can be pieced together from this and Fact 1, but this metric regularity route appears to be more involved and much less self-contained.

Corollary 6. *Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in X, where, for some $r \in \{0, \ldots, m\}$, the sets $C_{r+1}, \ldots, C_m$ are polyhedral. Suppose further $\bigcap_{i=1}^{r} \operatorname{ri}(C_i) \cap \bigcap_{i=r+1}^{m} C_i \neq \emptyset$ and $C := \bigcap_{i=1}^{m} C_i$ is* bounded. *Then there exists $\lambda > 0$ such that $d(x, C) \leq \lambda \sum_i d(x, C_i)$, $\forall x \in X$.*

Proof. Similarly to the proof of Corollary 5.

□

Remark 10. Corollary 6 can be proved differently by using Corollary 5 and Lewis's observation that bounded linear regularity plus a bounded intersection yields linear regularity (see [2, Theorems 4.2.6 and 5.2.3]).

5. Applications to convex inequalities

Throughout this section, we assume that $f_1, \ldots, f_m$ are finitely many functions on X that are convex and finite everywhere on X.

Let $C_i := \{x \in X : f_i(x) \leq 0\}$, $\forall i$. We assume $C := \bigcap_i C_i$, *i.e.*, the system of convex inequalities

$$f_1(x) \leq 0,\ f_2(x) \leq 0, \ldots,\ f_m(x) \leq 0 \tag{5}$$

has at least one solution.

Systems of convex inequalities have been receiving much attention lately; for more information, the reader is referred to Li's [27], Klatte and Li's [25], Lewis and Pang's [26], and the many references therein.

To demonstrate the usefulness of the results of the previous sections, we present some selected applications to systems of convex inequalities.

Basic constraint qualification and strong CHIP

Definition 6. *([18, Section VII.2.2]) Suppose $x \in C$. Then the system of convex inequalities (5) satisfies the* basic constraint qualification at x, *if* $N_C(x) = \operatorname{cone}\{\partial f_i(x) : f_i(x) = 0\}$.

Remark 11. • The basic constraint qualification is formulated in terms of the normal cone of C. There exists an equivalent "dual" condition, called *Abadie's constraint qualification*, which is formulated in terms of the tangent cone of C; see Li's [27] for more. • If $x \in C$ and $f_i(x) = 0$, then $\operatorname{cone}(\partial f_i(x)) \subseteq N_{C_i}(x)$ (with equality if x is not a minimizer of f_i; see [32, Corollary 23.7.1]).

Definition 6 and Remark 4 thus immediately imply the following result.

Proposition 13. *Suppose $x \in C$ and the system of convex inequalities (5) satisfies the basic constraint qualification at x. Then $\{C_1, \dots, C_m\}$ has strong CHIP at x.*

Note that strong CHIP is a purely geometric property, while the basic constraint qualification depends on the analytic representation of each C_i. The next example shows that strong CHIP is genuinely less restrictive than the basic constraint qualification.

Example 4. Let $X := \mathbb{R}^2$, $f_1(x_1, x_2) := x_1^2$, and $f_2(x_1, x_2) := x_2^2$. Then $C_1 := \{x \in X : f_1(x) \le 0\} = \{0\} \times \mathbb{R}$ and $C_2 := \{x \in X : f_2(x) \le 0\} = \mathbb{R} \times \{0\}$, which implies $N_{C_1}(0) = \mathbb{R} \times \{0\}$ and $N_{C_2}(0) = \{0\} \times \mathbb{R}$. Let $C := C_1 \cap C_2 = \{0\}$, then $N_C(0) = X$. On the one hand, $N_{C_1}(0) + N_{C_2}(0) = N_C(0)$, so $\{C_1, C_2\}$ has strong CHIP. On the other hand, $\nabla f_1(0) = 0 = \nabla f_2(0)$, so the system of convex inequalities $f_1(x) \le 0$, $f_2(x) \le 0$ does not satisfy the basic constraint qualification at 0.

The next proposition shows that strong CHIP and the basic constraint qualification do coincide, provided the subdifferentials are well-behaved.

Proposition 14. *Suppose $x \in C$ and $N_{C_i}(x) = \operatorname{cone}(\partial f_i(x))$, for each i with $f_i(x) = 0$. Then the system of convex inequalities (5) satisfies the basic constraint qualification if and only if the collection $\{C_1, \dots, C_m\}$ has strong CHIP at x.*

Proof. "$\Rightarrow$": Proposition 13. "$\Leftarrow$": if $f_i(x) < 0$, then $x \in \operatorname{int} C_i$ and so $N_{C_i}(x) = \{0\}$. Let $I := \{i : f_i(x) = 0\}$. Then $N_C(x) = \sum_i N_{C_i}(x) = \sum_{i \in I} N_{C_i}(x) = \sum_{i \in I} \operatorname{cone}(\partial f_i(x)) = \operatorname{cone}\{\partial f_i(x) : i \in I\}$, and therefore (5) satisfies the basic constraint qualification at x.

□

Definition 7. *([18, Definition VII.2.2.3]) The system of convex inequalities (5) satisfies the* weak Slater assumption, *if there exists some $\hat{x} \in C$, called a* weak Slater point, *such that for every i, f_i is affine or $f_i(\hat{x}) < 0$.*

Corollary 7. *Suppose the system of convex inequalities (5) satisfies the weak Slater assumption. Then it satisfies the basic constraint qualification at every point in C and $\{C_1, \dots, C_m\}$ has strong CHIP.*

Proof. By [18, Section VII.2.2], (5) satisfies the basic constraint qualification at every point in C. Apply Proposition 13.

□

Asymptotic constraint qualification and error bounds

Definition 8. *(Auslender and Crouzeix [1]) We say that the system of convex inequalities (5) satisfies the* asymptotic constraint qualification, *if the following is satisfied: "Suppose (x_n) is a sequence in C with $\|x_n\| \to +\infty$. Let I be a subset of $\{1, \dots, m\}$ such that $f_i(x_n) = 0$ eventually, $\forall i \in I$. Let $g_{i,n} \in \partial f_i(x_n)$, $\forall n, \forall i \in I$. Define $I_{\text{bounded}} := \{i \in I : (g_{i,n})$ is bounded$\}$ and $I_{\text{unbounded}} := I \setminus I_{\text{bounded}}$. Suppose (k_n) is a subsequence of (n) such that for every $i \in I$: (g_{i,k_n}) converges, if $i \in I_{\text{bounded}}$; $(g_{i,k_n}/\|g_{i,k_n}\|)$ converges, if $i \in I_{\text{unbounded}}$. Denote the limit by g_i, $\forall i \in I$. Then necessarily* $0 \notin \operatorname{conv}\{g_i : i \in I\}$."

Remark 12. • Note that if C is bounded, then the asymptotic constraint qualification holds trivially. • Auslender and Crouzeix's asymptotic constraint qualification is an extension of Mangasarian's asymptotic constraint qualification [28] introduced for differentiable convex inequalities (5). When each f_i is differentiable, then these two definitions are equivalent; see [1, Remark (ii) on page 250]. More on the asymptotic constraint qualification can be found in Auslender and Crouzeix's [1] as well as in Klatte and Li's recent study [25].

The distinction between bounded subgradient sequences and unbounded subgradient sequences in Definition 8 can be removed as follows.

Proposition 15. *The system of convex inequalities (5) satisfies the asymptotic constraint qualification if and only if the following is satisfied: "Suppose* (x_n) *is a sequence in* C *with* $\|x_n\| \to +\infty$*. Let* I *be a subset of* $\{1, \ldots, m\}$ *such that* $f_i(x_n) = 0$ *eventually,* $\forall i \in I$*. Let* $g_{i,n} \in \partial f_i(x_n)$, $\forall n, \forall i \in I$*. Suppose* (k_n) *is a subsequence of* (n) *such that* $(g_{i,k_n}/(1 + \|g_{i,k_n}\|)$ *converges to* $\hat{g}_i$, $\forall i \in I$*. Then necessarily* $0 \notin \operatorname{conv}\{\hat{g}_i : i \in I\}$*."*

Proof. This follows from the (easy) fact that whenever $z_1, \ldots, z_m \in X$ and $t_1, \ldots, t_m > 0$, then: $0 \in \operatorname{conv}\{z_1, \ldots, z_m\}$ if and only if $0 \in \operatorname{conv}\{t_1 z_1, \ldots, t_m z_m\}$. □

Theorem 8 (sharpening of Auslender and Crouzeix's result). *Suppose the system of convex inequalities (5) satisfies the weak Slater assumption and the asymptotic constraint qualification. Then the collection* $\{C_1, \ldots, C_m\}$ *is linearly regular: there exists* $\gamma > 0$ *such that*

$$d(x, C) \le \gamma \sum_i d(x, C_i), \qquad \forall x \in X.$$

Proof. Step 1. The collection $\{C_1, \ldots, C_m\}$ has strong CHIP (by Corollary 7).

Step 2. $N_{C_i}(x) = \begin{cases} \operatorname{cone}(\partial f_i(x)), & \text{if } f_i(x) = 0; \\ \{0\}, & \text{otherwise,} \end{cases} \quad \forall x \in C, \forall i \in \{1, \ldots, m\}.$

If $f_i(x) = 0$, then the result follows from the weak Slater assumption combined with [32, Corollary 23.7.1]. Otherwise, $f_i(x) < 0$ in a neighborhood of x (by continuity of f_i on X). Step 2 thus holds.

Step 3. There exists $\gamma > 0$ such that $\forall x \in C, \forall y \in N_C(x) = \sum_i N_{C_i}(x)$:

$$\min\big\{\textstyle\sum_i \|y_i\|^2 : \text{ each } y_i \in N_{C_i}(x), \sum_i y_i = y\big\} \le \gamma^2 \|y\|^2.$$

Suppose to the contrary that Step 3 is not true. Then there exist sequences (x_n) in C and (y_n) with $y_n \in N_C(x_n)$, $\forall n$, such that

$$\min\big\{\textstyle\sum_i \|y_i'\|^2 : \text{ each } y_i' \in N_{C_i}(x_n), \sum_i y_i' = y_n\big\} > n^2 \|y_n\|^2 > 0. \tag{6}$$

We claim that $\|x_n\| \to +\infty$: indeed, if (6) held on a bounded subsequence of (x_n), then we would contradict Theorem 2 (use the weak Slater point for the system of convex inequalities (5) and recall Proposition 2). After normalizing (the N_{C_i} are cones), we assume WLOG that $y_n \in S_X$, $\forall n$. Let $y_{i,n} \in N_{C_i}(x_n)$, $\forall i$, such that $\sum_i y_{i,n} = y_n$, $\forall n$. Then (6) yields

$$\textstyle\sum_i \|y_{i,n}\| > n \to +\infty. \tag{7}$$

After passing to a subsequence if necessary, we assume WLOG that the sets $\{i : y_{i,n} \neq 0\}$ are all the same, say I. Then, using Step 2, we see that $f_i(x_n) = 0$ and obtain $\mu_{i,n} > 0$ such that $y_{i,n} = \mu_{i,n} g_{i,n}$, for some $g_{i,n} \in \partial f_i(x_n)$, $\forall n, \forall i \in I$. Then

$$\|y_{i,n}\| \leq \mu_{i,n}(1 + \|g_{i,n}\|), \quad \forall n, \forall i \in I. \tag{8}$$

Define

$$\lambda_{i,n} := \frac{\mu_{i,n}(1 + \|g_{i,n}\|)}{\sum_{j\in I} \mu_{j,n}(1 + \|g_{j,n}\|)}, \quad \forall i \in I, \forall n.$$

Set $\hat{g}_{i,n} := g_{i,n}/(1 + \|g_{i,n}\|)$, $\forall n, \forall i \in I$. After passing to another subsequence if necessary, we assume WLOG that $\lambda_{i,n} \to \lambda_i$ and that $\hat{g}_{i,n} \to \hat{g}_i$, $\forall i \in I$. Note that $\sum_{i\in I} \lambda_i = 1$ and $\lambda_i \geq 0$, $\forall i \in I$. Using (7), (8), and $\|y_n\| = 1$, $\forall n$, we deduce

$$\sum_{i\in I} \lambda_i \hat{g}_i \leftarrow \sum_{i\in I} \lambda_{i,n} \hat{g}_{i,n} = \frac{1}{\sum_{j\in I} \mu_{j,n}(1 + \|g_{j,n}\|)} y_n \to 0.$$

Therefore, $0 = \sum_{i\in I} \lambda_i \hat{g}_i$, which contradicts the characterization of the asymptotic constraint qualification in Proposition 15. Step 3 thus holds true.

Final Step. The result now follows from Steps 1, 3, and Corollary 4.

□

To see how Theorem 8 implies Auslender and Crouzeix's main result, we need the following simple result on error bounds, which was implicitly proved in [25, Proof of Theorem 8, (C1)⇒(C3)] and [26, Proof of Corollary 1, (b)⇒(a)], for instance.

Proposition 16. *Suppose f is a convex function that is everywhere finite on X. Let $S := \{x \in X : f(x) \leq 0\}$ be nonempty. Suppose further f is affine or there exists a Slater point $\hat{x} \in X$ with $f(\hat{x}) < 0$. If the convex inequality $f(x) \leq 0$ satisfies the asymptotic constraint qualification, then there exists $\alpha > 0$ such that $d(x, S) \leq \alpha f^+(x)$, $\forall x \in X$.*

Corollary 8. *(see also Auslender and Crouzeix's [1, Theorem 2]) Suppose the system of convex inequalities (5) satisfies the weak Slater assumption and the asymptotic constraint qualification. Then there exists $\beta > 0$ such that $d(x, C) \leq \beta \sum_i f_i^+(x)$, $\forall x \in X$.*

Proof. On the one hand, by Theorem 8, there exists $\gamma > 0$ such that $d(x, C) \leq \gamma \sum_i d(x, C_i)$, $\forall x \in X$. On the other hand, each f_i *by itself* satisfies the weak Slater condition and the asymptotic constraint qualification; in particular, by Proposition 16, there exists $\alpha_i > 0$ such that $d(x, C_i) \leq \alpha_i f_i^+(x)$, $\forall x \in X, \forall i$. So $\beta := \max_i \gamma\alpha_i$ does the job.

□

Remark 13. Corollary 8 is not exactly the same as Auslender and Crouzeix's [1, Theorem 2]. They considered a system of convex and linear inequalites

$$g_i(x) \leq 0 \quad (\text{for } 1 \leq i \leq r), \qquad \langle a_i, x\rangle \leq b_i \quad (\text{for } r + 1 \leq i \leq m), \tag{9}$$

where each $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, and g_i is closed convex proper on $\mathbb{R}^n$. They used the following Slater condition: there exists $\bar{x} \in \mathbb{R}^n$ such that

$$g_i(\bar{x}) < 0 \quad (\text{for } 1 \le i \le r), \qquad \langle a_i, \bar{x}\rangle \le b_i \quad (\text{for } r+1 \le i \le m). \tag{10}$$

If we define $f_i(x) = g_i(x)$ for $1 \le i \le r$ and $f_i(x) = \langle a_i, x\rangle - b_i$ for $r+1 \le i \le m$, then (10) implies the weak Slater condition and so [1, Theorem 2] follows from Corollary 8. Note that our proof works even if each f_i is only a closed proper convex function. Corollary 8 gives a Hoffman-type error bound for convex inequalities (an algebraic property for (5)), while Theorem 8 establishes the linear regularity of the underlying convex sets defined by (5) (a geometric property). It is easy to construct examples, such as $C_i := \{x \in \mathbb{R}^n : \|x\|^2 \le 1\}$ with $f_i(x) = \|x\|^2$, where one has $\lim_{\|x\|\to\infty} f_i^+(x)/d(x, C_i) = \infty$. In this sense, Theorem 8 is indeed a sharpening of Auslender and Crouzeix's [1, Theorem 2].

The constrained case

As a last application to error bounds, we briefly discuss the constrained case:

As before, we assume that $f_1, \ldots, f_m$ are finitely many functions on X that are convex and finite everywhere, and we let $C_i := \{x \in X : f_i(x) \le 0\}$, $\forall i$. In addition, let C_0 be a closed convex set in X with $C := \bigcap_{i=0}^m C_i \ne \emptyset$. That is, *the system of constrained convex inequalities*

$$f_1(x) \le 0, \ldots, f_m(x), \le 0, x \in C_0 \tag{11}$$

has at least one solution.

We require a special case of a result due to Robinson:

Proposition 17. *([30, Section 3]) Suppose f is a closed convex function on X and there exists a Slater point $\hat{x} \in X$ with $f(\hat{x}) < 0$. Let $S := \{x \in X : f(x) \le 0\}$. Then $d(x, S) \le -(\|x - \hat{x}\|/f(\hat{x}))f^+(x)$, $\forall x \in X$.*

Theorem 9. *Consider the constrained system of convex inequalities (11). Suppose there exists a point $\hat{x} \in C_0$ such that $f_i(\hat{x}) < 0$, $\forall i \in \{1, \ldots, m\}$. Then for every $\rho > 0$, there exists $\beta_\rho > 0$ such that $d(x, C) \le \beta_\rho(d(x, C_0) + \sum_{i=1}^m f_i^+(x))$, $\forall x \in \rho B_X$.*

Proof. By [32, Corollary 6.3.2], there exists some z (close to $\hat{x}$) such that $z \in \mathrm{ri}(C_0) \cap \bigcap_{i=1}^m C_i$. On the one hand, by Theorem 7, there exists $\gamma > 0$ (depending on z) such that

$$d(x, C) \le \gamma(\|P_C(x) - z\| + 1)(d(x, C_0) + \textstyle\sum_{i=1}^m d(x, C_i)), \quad \forall x \in X.$$

On the other hand, by Proposition 17, $d(x, C_i) \le -(\|x - \hat{x}\|/f_i(\hat{x}))f_i^+(x)$, $\forall i \in \{1, \ldots, m\}$, $\forall x \in X$. Altogether, using $\|P_C(x) - z\| \le \|x - z\| \le \rho + \|z\|$, we are done by letting

$$\beta_\rho := \gamma(\rho + \|z\| + 1)\max\{1, (\rho + \|\hat{x}\|)/\min_{i\in\{1,\ldots,m\}}(-f_i(\hat{x}))\}$$

□

Remark 14. Suppose that $m = 1$. Then the conclusion of Theorem 9 also follows from Lewis and Pang's [26, Proposition 3]. Note that Lewis and Pang pose a certain boundedness assumption on ∂f_1, which is trivially satisfied if f_1 is everywhere finite (use [4, Corollary 7.9]).

6. More on cones and subspaces

Two closed convex cones

Fact 2. *([32, Corollary 16.4.2]) Suppose A and B are two closed convex cones in X. Then* $\mathrm{cl}\,(A^{\ominus} + B^{\ominus}) = (A \cap B)^{\ominus}$.

Proposition 18. *Suppose K is a closed convex cone in X. Then $N_K(x) = K^{\ominus} \cap \{x\}^{\perp}$ and $T_K(x) = \mathrm{cl}\,(K + \mathbb{R}x)$, $\forall x \in K$.*

Proof. Pick $x^* \in N_K(x)$, *i.e.*, $\langle x^*, k - x\rangle \le 0$, $\forall k \in K$. If we let $k := k' + x$, where $k' \in K$ and hence $k \in K$, we learn that $\langle x^*, k'\rangle \le 0$, $\forall k' \in K$ and thus $x^* \in K^{\ominus}$. Furthermore, if we let $k := 2x \in K$, then $\langle x^*, x\rangle \le 0$; and if $k := 0$, then $\langle x^*, -x\rangle \le 0$. Altogether, $N_K(x) \subseteq K^{\ominus} \cap \{x\}^{\perp}$. The reverse inclusion is even simpler and the normal cone formula thus follows. The tangent cone formula follows by taking polars and invoking Fact 2.

□

Proposition 19. *Suppose A and B are two closed convex cones in X. Then*

$$(A^{\ominus} + B^{\ominus}) \cap \{x\}^{\perp} = [A^{\ominus} \cap \{x\}^{\perp}] + [B^{\ominus} \cap \{x\}^{\perp}], \qquad \forall x \in A \cap B.$$

Proof. Clearly, the right-hand side is contained in the left-hand side. Let z be a member of the left-hand side. Then $z = a^{\ominus} + b^{\ominus}$, where $a^{\ominus} \in A^{\ominus}$, $b^{\ominus} \in B^{\ominus}$, and $\langle z, x\rangle = 0$. Now $x \in A \cap B$; hence $\langle x, a^{\ominus}\rangle \le 0$ and $\langle x, b^{\ominus}\rangle \le 0$. Altogether, $0 = \langle x, z\rangle = \langle x, a^{\ominus}\rangle + \langle x, b^{\ominus}\rangle \le 0 + 0 = 0$. Hence $a^{\ominus}$ and $b^{\ominus}$ belong to $\{x\}^{\perp}$, which yields $z = a^{\ominus} + b^{\ominus} \in [A^{\ominus} \cap \{x\}^{\perp}] + [B^{\ominus} \cap \{x\}^{\perp}]$.

□

Proposition 20. *Suppose A and B are two closed convex cones in X. Then TFAE:*

(i) $\{A, B\}$ *has strong CHIP.*
(ii) $\{A, B\}$ *has strong CHIP at* 0*:* $(A \cap B)^{\ominus} = A^{\ominus} + B^{\ominus}$.
(iii) $A^{\ominus} + B^{\ominus}$ *is closed.*

Proof. "(i)⇒(ii)": $(A\cap B)^{\ominus} = N_{(A\cap B)}(0) = N_A(0) + N_B(0) = A^{\ominus} + B^{\ominus}$. "(ii)⇔(iii)": Fact 2. "(i)⇐(iii)": Fix an arbitrary $x \in A \cap B$. We have to show that $N_A(x) + N_B(x) = N_{A\cap B}(x)$; equivalently, by Proposition 18, that $[A^{\ominus} \cap \{x\}^{\perp}] + [B^{\ominus} \cap \{x\}^{\perp}] = (A \cap B)^{\ominus} \cap \{x\}^{\perp}$; equivalently, by Fact 2, that $[A^{\ominus} \cap \{x\}^{\perp}] + [B^{\ominus} \cap \{x\}^{\perp}] = \mathrm{cl}\,(A^{\ominus} + B^{\ominus}) \cap \{x\}^{\perp}$. The last condition is satisfied by assumption (iii) and Proposition 19.

□

We now re-derive and refine Jameson's main duality result.

Theorem 10. *(see also [22, Theorem 2.1]) Suppose A and B are two closed convex cones in X. Then TFAE:*

(i) $\{A, B\}$ *is linearly regular.*
(ii) $\{A, B\}$ *is boundedly linearly regular.*
(iii) $\{A, B\}$ *has strong CHIP and* $\{A^\ominus, B^\ominus\}$ *has property (G).*
(iv) $A^\ominus + B^\ominus$ *is closed and* $\{A^\ominus, B^\ominus\}$ *has property (G).*
(v) $\{A^\ominus, B^\ominus\}$ *has property (G).*

Proof. "(i)⇒(ii)": obvious. "(i)⇐(ii)": easy to verify (use homogeneity or [3, Theorem 3.17]). "(i)⇔(iii)": Proposition 6 (see also Remark 3). "(iii)⇔(iv)": Proposition 20. "(iv)⇒(v)": obvious. "(iv)⇐(v)": Proposition 3.

□

Corollary 9. *Suppose A and B are two closed convex cones in X such that $A \cap B$ is* linear. *Then $\{A, B\}$ is linearly regular and has strong CHIP, $A^\ominus + B^\ominus$ is closed, and $\{A^\ominus, B^\ominus\}$ has property (G).*

Proof. $A \cap B$ linear $\Rightarrow$ $\operatorname{cl}(A^\ominus + B^\ominus)$ is linear (Fact 2) $\Rightarrow$ $A^\ominus + B^\ominus$ is linear ([32, Theorem 6.3]) $\Rightarrow$ $A^\ominus + B^\ominus$ is closed $\Rightarrow$ $\{A, B\}$ has strong CHIP (Proposition 20). Since $A^\ominus + B^\ominus$ is closed and *linear*, Proposition 5 implies $\{A^\ominus, B^\ominus\}$ has property (G). The result now follows from Theorem 10.

□

The next example shows the importance of the assumption on linearity in Corollary 9.

Example 5. Let $X := \mathbb{R}^3$, $A := \{(x_1, x_2, x_3) \in X : x_1^2 + x_2^2 \le x_3^2, x_3 \ge 0\}$, and $B := \{(x_1, x_2, x_3) : x_1 = x_3\}$. Then $\{A, B\}$ fails to have CHIP at every nonzero point in the intersection of $A \cap B$.

Corollary 10. *Suppose A and B are two convex* polyhedral *cones in X. Then $\{A, B\}$ is linearly regular and has strong CHIP, $A^\ominus + B^\ominus$ is closed, and $\{A^\ominus, B^\ominus\}$ has property (G).*

Proof. Theorem 10 and Fact 1.

□

Remark 15. All results in this subsection remain valid for finitely many closed convex cones; some will generalize to infinite-dimensional Hilbert space (see Jameson's [22]).

Two subspaces and their angle

Suppose A and B are two subspaces of X. Then, by Proposition 5, the collection $\{A, B\}$ has property (G). By Corollary 1, there exists $\alpha > 0$ such that $\|x\|_{A+B} \le \alpha\|x\|$, $\forall x \in A + B$. In this subsection, we identify the "best" possible α, the square of which we define now.

Definition 9. *Suppose A and B are two subspaces of X. Define*

$$\beta(A, B) := \sup_{x \in (A \cap B)^{\perp} \cap B_X} \min_{u \in A^{\perp}, v \in B^{\perp}, u+v=x} \|u\|^2 + \|v\|^2.$$

The following notion of the angle between two subspaces goes back to 1937.

Definition 10. *(Friedrichs [17]) Suppose A and B are two subspaces of X. The* angle *between A and B is the angle in $[0, \pi/2]$ whose cosine is defined by*

$$c(A, B) := \sup\big\{\langle a, b\rangle : a \in A \cap (A \cap B)^{\perp} \cap B_X, b \in B \cap (A \cap B)^{\perp} \cap B_X\big\}.$$

Remark 16. Because X is *finite-dimensional*, we conclude: • the supremum in Definition 10 is attained and hence a maximum; • thus the angle is always bigger than 0; equivalently, the cosine is always less than 1. For more information on the angle, the reader is referred to Deutsch's comprehensive survey [10].

Proposition 21. *Suppose A and B are two subspaces of X. Then:*

(i) $A = B$ *if and only if* $A^{\perp} \cap (A + B) = \{0\}$ *and* $B^{\perp} \cap (A + B) = \{0\}$.
(ii) *if* $A = B \subsetneqq X$, *then* $\beta(A, B) = \frac{1}{2}$.

Proof. (i): "⇒" is trivial. We prove the contrapositive of "⇐": assume $A \neq B$. Then $A + B \supsetneqq A$ or $A + B \supsetneqq B$. WLOG $A + B \supsetneqq A$. Pick $s \in (A + B) \setminus A$ and write $s = P_A s + P_{A^{\perp}} s$. Then $P_{A^{\perp}} s$ is *nonzero* and contained in $A^{\perp}$. Also, $P_{A^{\perp}} s = s - P_A s \in (A + B) - A = A + B$. Altogether, $P_{A^{\perp}} s \in A^{\perp} \cap (A + B) \setminus \{0\}$ and (i) is proven. (ii): pick an arbitrary $x \in A^{\perp} = B^{\perp} = (A \cap B)^{\perp}$. Then $\min_{y \in A^{\perp}} \|x - y\|^2 + \|y\|^2$ is attained at $y = \frac{1}{2}x$ (use Convex Calculus) with value $\frac{1}{2}\|x\|^2$. Because $A^{\perp}$ contains nonzero vectors, we deduce $\beta(A, B) = \frac{1}{2}$.

□

Theorem 11 (property (G) and the angle). *Suppose A and B are two subspaces of X. Then $\beta(A, B) \le 1/(1 - c(A, B))$; more precisely:*

$$\beta(A, B) = \begin{cases} 0, & \text{if } A = B = X; \\ \frac{1}{2}, & \text{if } A = B \subsetneqq X; \\ \frac{1}{1-c(A,B)}, & \text{otherwise.} \end{cases}$$

Proof. Recall that $(A \cap B)^{\perp} = A^{\perp} + B^{\perp}$. Fix an arbitrary $x \in (A \cap B)^{\perp} \cap B_X$. Then there exist $u \in A^{\perp}$ and $v \in B^{\perp}$ such that $x = u + v$. For brevity, let $S := A + B$ and $S^{\perp} = (A + B)^{\perp} = A^{\perp} \cap B^{\perp}$. Write $u = P_{S^{\perp}}(u) + P_S(u)$ and $v = P_{S^{\perp}}(v) + P_S(v)$. Then $P_S(u) = u - P_{S^{\perp}}(u) \in A^{\perp} - (A^{\perp} \cap B^{\perp}) \subseteq A^{\perp}$ and similarly $P_S(v) \in B^{\perp}$. Thus

$$P_S(u) \in A^{\perp} \cap (A + B) \text{ and } P_S(v) \in B^{\perp} \cap (A + B).$$

By [10, Lemma 2.10.2.(b)], $\langle -P_S(u), P_S(v)\rangle \le c(A^{\perp}, B^{\perp})\|P_S(u)\|\|P_S(v)\|$. Since $c(A, B) = c(A^{\perp}, B^{\perp})$ ([10, Theorem 2.16]), we get $\langle -P_S(u), P_S(v)\rangle \le \frac{1}{2}c(A, B)(\|P_S(u)\|^2 + \|P_S(v)\|^2)$, which easily leads to $\|P_S(u) + P_S(v)\|^2 \ge (1 -$

$c(A, B))(\|P_S(u)\|^2 + \|P_S(v)\|^2)$. Using this, $0 \le c(A, B) < 1$, and Pythagoras (twice), we obtain the following chain of inequalities:

$$\begin{aligned}\|x\|^2 &= \|u+v\|^2 = \|P_S(u+v) + P_{S^\perp}(u+v)\|^2\\ &= \|P_S(u) + P_S(v)\|^2 + \|P_{S^\perp}(u+v)\|^2\\ &\ge (1-c(A,B))(\|P_S(u)\|^2 + \|P_S(v)\|^2) + \|P_{S^\perp}(u+v)\|^2\\ &\ge (1-c(A,B))(\|P_S(u)\|^2 + \|P_S(v)\|^2 + \|P_{S^\perp}(u+v)\|^2)\\ &= (1-c(A,B))(\|P_S(u)\|^2 + \|P_S(v) + P_{S^\perp}(u+v)\|^2).\end{aligned}$$

Now $P_S(u) \in A^\perp$ and $P_S(v) + P_{S^\perp}(u+v) \in B^\perp$, hence

$$\begin{aligned}\min_{u'\in A^\perp, v'\in B^\perp, u'+v'=x} \|u'\|^2 + \|v'\|^2 &\le \|P_S(u)\|^2 + \|P_S(v) + P_{S^\perp}(u+v)\|^2\\ &\le \frac{1}{1-c(A,B)}\|x\|^2 \le \frac{1}{1-c(A,B)}.\end{aligned}$$

Since we chose $x \in (A\cap B)^\perp \cap B_X$ arbitrarily, we deduce $\beta(A, B) \le 1/(1 - c(A, B))$. "more precisely": *Case 1:* $A = B = X$. Then $A^\perp = B^\perp = (A \cap B)^\perp = \{0\}$, which yields $\beta(A, B) = 0$. *Case 2:* Proposition 21.(ii). *Case 3:* $A \ne B$. Note that $(A^\perp \cap B^\perp)^\perp = A + B$. Since $c(A, B) = c(A^\perp, B^\perp)$ (again [10, Theorem 2.16]) and since $(A^\perp \cap (A + B) \cap B_X) \times (B^\perp \cap (A + B) \cap B_X)$ is compact, there exist $\bar u \in A^\perp \cap (A+B)\cap B_X$ and $\bar v \in B^\perp \cap (A+B)\cap B_X$ such that $\langle -\bar u, \bar v\rangle = c(A^\perp, B^\perp) = c(A, B)$. Now $A \ne B$. Hence, in view of Proposition 21.(i), we assume WLOG that $\|\bar u\|^2 + \|\bar v\|^2 > 0$. Suppose $u \in A^\perp$ and $v \in B^\perp$ are such that $u + v = \bar u + \bar v$. Then $u - \bar u = \bar v - v \in A^\perp \cap B^\perp = (A+B)^\perp$, hence $\langle \bar u, u - \bar u\rangle = 0$ and $\langle \bar v, v - \bar v\rangle = 0$. This implies $\|u\| \ge \|\bar u\|$ and $\|v\| \ge \|\bar v\|$ (expand $\|u\|^2 = \|(u - \bar u) + \bar u\|^2$). Thus

$$\min_{u\in A^\perp, v\in B^\perp: u+v=\bar u+\bar v} \|u\|^2 + \|v\|^2 = \|\bar u\|^2 + \|\bar v\|^2.$$

It follows that $\|\bar u\|^2 + \|\bar v\|^2 \le \beta(A, B)\|\bar u + \bar v\|^2$; equivalently, $-2\beta(A, B)\langle \bar u, \bar v\rangle \le (\beta(A, B) - 1)(\|\bar u\|^2 + \|\bar v\|^2)$. By definition of $\bar u$ and $\bar v$, we obtain

$$2\beta(A, B)c(A, B) \le (\beta(A, B) - 1)(\|\bar u\|^2 + \|\bar v\|^2).$$

In the last inequality, the left-hand side is nonnegative and the factor $(\|\bar u\|^2 + \|\bar v\|^2)$ on the right-hand side is strictly positive, hence $\beta(A, B) \ge 1 > 0$. After dividing by $2\beta(A, B)$ and recalling that both $\bar u$ and $\bar v$ belong to B_X, we obtain

$$c(A, B) \le \frac{\beta(A, B) - 1}{2\beta(A, B)}(\|\bar u\|^2 + \|\bar v\|^2\|) \le \frac{\beta(A, B) - 1}{\beta(A, B)};$$

equivalently, $\beta(A, B) \ge 1/(1 - c(A, B))$ (since $c(A, B) < 1$).

□

Theorem 11 allows us to write concisely: $c(A, B) = (1 - 1/\beta(A, B))^+$.

We can now use the angle between the two subspaces to estimate the quantity in the definition of bounded linear regularity.

Corollary 11. *Suppose A and B are two subspaces of X. Then:*

$$d(x, A \cap B) \leq \sqrt{\beta(A, B)}(d(x, A) + d(x, B)) \leq \frac{d(x, A) + d(x, B)}{\sqrt{1 - c(A, B)}}, \quad \forall x \in X.$$

Proof. Theorem 11 and Proposition 6.

□

Remark 17. We do not know how Theorem 11 generalizes to more than two subspaces. Definition 9 has an obvious analog. As for the angle between more than two subspaces, one could employ the definition of the angle suggested in [5, Definition 3.7.5]. Finally, Theorem 11 generalizes to infinite-dimensional Hilbert space.

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