The Douglas–Rachford algorithm in the affine-convex case

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Abstract

The Douglas–Rachford algorithm is a simple yet effective method for solving convex feasibility problems. However, if the underlying constraints are inconsistent, then the convergence theory is incomplete. We provide convergence results when one constraint is an affine subspace. As a consequence, we extend a result by Spingarn from halfspaces to general closed convex sets admitting least-squares solutions.

Keywords:
Affine subspace, convex feasibility problem, Douglas–Rachford splitting operator, halfspace, least-squares solution, normal cone operator, projection, Spingarn's method.

1. Introduction

We shall assume throughout this paper that X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and that

$$A \text{ and } B \text{ are nonempty closed convex (not necessarily intersecting) subsets of } X. \tag{1}$$

Consider the problem of finding a best approximation pair relative to A and B (see [3], [12]), that is to

$$\text{find } (a,b) \in A \times B \text{ such that } \|a-b\| = \inf \|A-B\|. \tag{2}$$

Recall that the Douglas–Rachford splitting operator (see, e.g., [11] or [3, Proposition 3.3(i)]) for the ordered pair of sets (A,B) is defined by

$$T = T_{(A,B)} := \tfrac{1}{2}(\mathrm{Id} + R_B R_A) = \mathrm{Id} - P_A + P_B R_A, \tag{3}$$

where P_A is the projector onto A and $R_A := 2P_A - \mathrm{Id}$ is the reflector onto A. Let $x \in X$. Consider momentarily the consistent case, i.e., when $A \cap B \neq \varnothing$. In this case the *"governing sequence"* $(T^n x)_{n\in\mathbb{N}}$ generated by iterating the Douglas–Rachford operator converges weakly to a fixed point[1] of T (see [11]), and the *"shadow sequence"* $(P_A T^n x)_{n\in\mathbb{N}}$ converges weakly to a point in $A \cap B$ (see [17] or [2, Theorem 25.6]). For further information on the Douglas–Rachford algorithm (DRA), see also [11] and [9]. In fact, DRA is a splitting method to find a point in the set of minimizers of $\iota_A + \iota_B$, i.e., in $A \cap B$. Other methods that can be used for the same set are e.g., the *forward-backward* method (see, e.g., [2, Section 25.3]) or *FISTA* (see, e.g.,

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[1]Fix $T = \{x \in X \mid x = Tx\}$ is the set of fixed points of T.

[7] and [6]) which view $A \cap B$ as the set of minimizers[2] of $\iota_A + \frac{1}{2}d_B^2$, and the *subgradient projection* method (see, e.g., [14] and [8]) applied to the function $\frac{1}{2}d_A^2 + \frac{1}{2}d_B^2$.

In [3], the authors showed that in the inconsistent case, when $A \cap B = \varnothing$, $(P_A T^n x)_{n\in\mathbb{N}}$ remains bounded with the weak cluster points of $(P_A T^n x, P_B P_A T^n x)_{n\in\mathbb{N}}$ being best approximation pairs relative to A and B whenever $g := P_{\overline{B-A}}0 \in B - A$. *The goal of this paper is to study the case when $A \cap B$ is possibly empty in the setting that one of the sets A and B is a closed affine subspace of X.* Our results show that the shadow sequence will always converge to a best approximation solution in $A \cap (B - g)$. As a consequence we obtain a far-reaching refinement of Spingarn's splitting method introduced in [16].

2. Main results

We start with the following key lemma, which is well known when $A = X$ (see, e.g., [2, Theorem 5.5]).

Lemma 2.1. *Let A be a closed linear subspace of X, let C be a nonempty closed convex subset of A, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Suppose that $(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to C, i.e., $(\forall n \in \mathbb{N})$ $(\forall c \in C)$ $\|x_{n+1} - c\| \le \|x_n - c\|$, and that all weak cluster points of $(P_A x_n)_{n\in\mathbb{N}}$ lie in C. Then $(P_A x_n)_{n\in\mathbb{N}}$ converges weakly to some point in C.*

Proof. The proof follows largely along the lines of [2, Lemma 2.39 and Theorem 5.5]; however, we point out below where the linearity of P_A is used. Since $(x_n)_{n\in\mathbb{N}}$ is bounded (by e.g., [2, Proposition 5.4(i)]) and P_A is (firmly) nonexpansive we learn that $(P_A x_n)_{n\in\mathbb{N}}$ is bounded and by assumption, its weak cluster points lie in $C \subseteq A$. Now let c_1 and c_2 be in C. On the one hand the Fejér monotonicity of $(x_n)_{n\in\mathbb{N}}$ implies the convergence of the sequences $(\|x_n - c_1\|^2)_{n\in\mathbb{N}}$ and $(\|x_n - c_2\|^2)_{n\in\mathbb{N}}$ by e.g., [2, Proposition 5.4(ii)]. On the other hand, expanding and simplifying yield $\|x_n - c_1\|^2 - \|x_n - c_2\|^2 = \|x_n\|^2 + \|c_1\|^2 - 2\langle x_n, c_1\rangle - \|x_n\|^2 - \|c_2\|^2 + 2\langle x_n, c_2\rangle = \|c_1\|^2 - 2\langle x_n, c_1 - c_2\rangle - \|c_2\|^2$, which in turn implies that $(\langle x_n, c_1 - c_2\rangle)_{n\in\mathbb{N}}$ converges. Since $c_1 \in A$ and $c_2 \in A$, using the linearity of P_A, we have

$$\langle x_n, c_1 - c_2\rangle = \langle x_n, P_A c_1 - P_A c_2\rangle = \langle x_n, P_A(c_1 - c_2)\rangle = \langle P_A x_n, c_1 - c_2\rangle. \tag{4}$$

Now assume that $(P_A x_{k_n})_{n\in\mathbb{N}}$ and $(P_A x_{l_n})_{n\in\mathbb{N}}$ are subsequences of $(P_A x_n)_{n\in\mathbb{N}}$ such that $P_A x_{k_n} \rightharpoonup c_1$ and $P_A x_{l_n} \rightharpoonup c_2$. By the uniqueness of the limit in (4) we conclude that $\langle c_1, c_1 - c_2\rangle = \langle c_2, c_1 - c_2\rangle$ or equivalently $\|c_1 - c_2\|^2 = 0$, hence $(P_A x_n)_{n\in\mathbb{N}}$ has a unique weak cluster point which completes the proof. ■

From now on we work under the assumption that

$$g = g_{(A,B)} := P_{\overline{B-A}}0 \in B - A. \tag{5}$$

In view of (5) we have

$$E = E_{(A,B)} := A \cap (B - g) \neq \varnothing \quad \text{and} \quad F = F_{(A,B)} := (A + g) \cap B \neq \varnothing. \tag{6}$$

For sufficient conditions on when $g \in B - A$ (or equivalently the sets E and F are nonempty) we refer the reader to [1, Facts 5.1].

Lemma 2.2. *Let $x \in X$. Then the following hold:*

(i) *If $C \in \{A, B\}$ is a closed affine subspace of X, then $g \in (C - C)^\perp$.*

(ii) *The sequence $(T^n x - ng)_{n\in\mathbb{N}}$ is Fejér monotone with respect to E.*

[2]Let C be a nonempty closed convex subset of X. We use ι_C to denote the *indicator function* associated with C, defined by $\iota_C(x) = 0$ for all $x \in C$ and $\iota_C(x) = +\infty$; otherwise, and d_C to denote the *distance* function from the set C defined by $d_C : X \to [0, +\infty[\, : x \mapsto \min_{c\in C}\|x - c\| = \|x - P_C x\|$.

(iii) *The sequence* $(P_A T^n x)_{n\in\mathbb{N}}$ *is bounded and all weak cluster points lie in* E.

(iv) *If* B *is a closed affine subspace, then* $P_B T^n x - P_A T^n x \to g$*, the sequence* $(P_B T^n x)_{n\in\mathbb{N}}$ *is bounded and all weak cluster points lie in* F.

(v) *If* $E = \{\bar{x}\}$ *and hence* $F = \{\bar{x} + g\}$*, then* $P_A T^n x \rightharpoonup \bar{x}$ *and* $P_B T^n x \rightharpoonup \bar{x} + g$.

Proof. (i): See [3, Corollary 2.7 and Remark 2.8(ii)]. (ii): It follows[3] from [3, Theorem 3.5] that $E + N_{\overline{A-B}}(-g) \subseteq \operatorname{Fix}(-g + T) := \{x \in X \mid x = -g + Tx\}$. Consequently, $E \subseteq \operatorname{Fix}(-g + T)$. Moreover, [3, Remark 3.15] implies that the sequence $(T^n x - ng)_{n\in\mathbb{N}}$ is Fejér monotone with respect to $\operatorname{Fix}(-g + T)$. (iii): See [3, Theorem 3.13(iii)(b)]. (iv): See [3, Theorem 3.17]. (v): This follows from (iii) and (iv). ■

We are now ready for our main results.

Theorem 2.3 (convergence of DRA when A is a closed affine subspace)**.** *Suppose that* A *is a closed affine subspace of* X*, and let* $x \in X$*. Then the following hold:*

(i) *The* shadow sequence $(P_A T^n x)_{n\in\mathbb{N}}$ *converges weakly to some point in* $E = A \cap (B - g)$.

(ii) *No general conclusion can be drawn about the boundedness of the sequence* $(P_B T^n x)_{n\in\mathbb{N}}$.

Proof. (i): After translating the sets A and B by a vector, if necessary, we can and do assume that A is a closed linear subspace of X. Using Lemma 2.2(i) we learn that $(\forall n \in \mathbb{N})\ P_A T^n x = P_A(T^n x - ng)$. Note that $E = A \cap (B - g) \subseteq A$. Now combine Lemma 2.2(ii)–(iii) and Lemma 2.1 with $C = E$, and $(x_n)_{n\in\mathbb{N}}$ replaced by $(T^n x - ng)_{n\in\mathbb{N}}$. (ii): In fact, $(P_B T^n x)_{n\in\mathbb{N}}$ can be unbounded (see Example 2.4) or bounded (e.g., when $A = B = X$). ■

Example 2.4. Suppose that $X = \mathbb{R}^2$, that $A = \mathbb{R} \times \{0\}$ and that $B = \operatorname{epi}(|\cdot| + 1)$. Then $A \cap B = \varnothing$ and for the starting point $x \in [-1, 1] \times \{0\}$ we have $(\forall n \in \{1, 2, \ldots\})\ T^n x = (0, n) \in B$ and therefore $\|P_B T^n x\| = \|T^n x\| = n \to \infty$.

Proof. Let $x = (\alpha, 0)$ with $\alpha \in [-1, 1]$. We proceed by induction. When $n = 1$ we have $T(\alpha, 0) = P_{A^\perp}(\alpha, 0) + P_B R_A(\alpha, 0) = P_B(\alpha, 0) = (0, 1)$. Now suppose that for some $(n \in \{1, 2, \ldots\})\ T^n x = (0, n)$. Then $T^{n+1} x = T(0, n) = P_{A^\perp}(0, n) + P_B R_A(0, n) = (0, n) + P_B(0, -n) = (0, n + 1) \in B$. ■

When B is an affine subspace, the convergence theory is even more satisfying:

Theorem 2.5 (convergence of DRA when B is a closed affine subspace)**.** *Suppose that* B *is a closed affine subspace of* X*, and let* $x \in X$*. Then the following hold:*

(i) *The* shadow sequence $(P_A T^n x)_{n\in\mathbb{N}}$ *converges weakly to some point in* $E = A \cap (B - g)$.

(ii) *The sequence* $(P_B T^n x)_{n\in\mathbb{N}}$ *converges weakly to some point in* $F = (A + g) \cap B$.

Proof. (ii): Combine Theorem 2.3(i) and [5, Corollary 2.8(i)]. (i): Combine (ii) and Lemma 2.2(iv). ■

It is tempting to conjecture that Theorem 2.3(i) remains true when A is just convex and not necessarily a subspace. While this *statement* may be true[4], the *proof* of Theorem 2.3(i) does *not* admit such an extension:

Example 2.6. Suppose that $X = \mathbb{R}$, that $A = [1, 2]$ and that $B = \{0\}$. Then $g = -1$ and $E = \{1\}$. Let $x = 4$. We have $(T^n x)_{n\in\mathbb{N}} = (4, 2, 0, -1, -2, -3, \ldots)$, $P_A T^n x \to 1 \in E$ and $(\forall n \in \{2, 3, 4, \ldots\})$ $T^n x - ng = -(n - 2) - n(-1) = 2 \in A$ and $P_A(T^n x - ng) = 2 \in A \smallsetminus E$. In the proof of Theorem 2.3(i), we had $(P_A T^n x)_{n\in\mathbb{N}} = (P_A(T^n x - ng))_{n\in\mathbb{N}}$ which is strikingly false here. Figure 1 provides an illustration of the first few iterates of $(T^n x)_{n\in\mathbb{N}}$.

[3] We use N_C to denote the *normal cone* operator associated with a nonempty closed convex subset C of X.

[4] In [3, Remark 3.14(ii)], the authors claim otherwise but forgot to list the assumption that $A \cap B \neq \varnothing$.

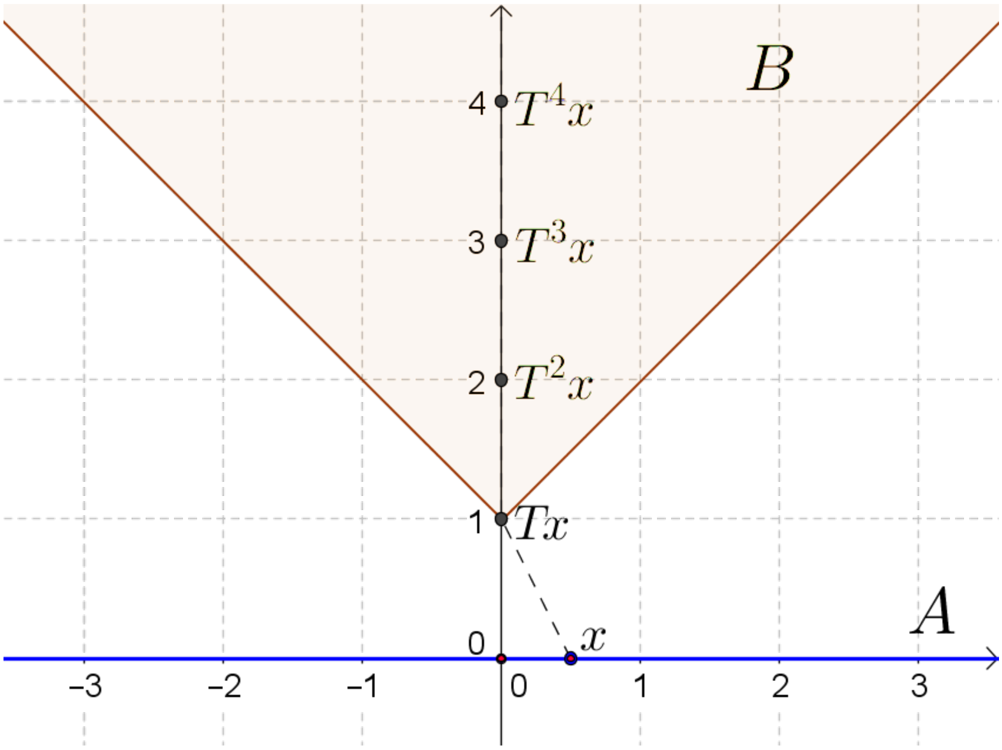

Figure 1: A `GeoGebra` [10] snapshot that illustrates Example 2.4 with the starting point $x = (0.5, 0)$.

3. Spingarn's method

In this section we discuss the problem to find a *least-squares solution* of $\bigcap_{i=1}^{M} C_i$, i.e., to

$$\text{find a minimizer of } \sum_{i=1}^{M} d_{C_i}^2, \tag{7}$$

where $C_1, \ldots, C_M$ are nonempty closed convex (possibly nonintersecting) subsets of X with corresponding distance functions $d_{C_1}, \ldots, d_{C_M}$. Following Pierra [13], we now consider the product Hilbert space $\mathbf{X} := X^M$, with the inner product $((x_1, \ldots, x_M), (y_1, \ldots, y_M)) \mapsto \sum_{i=1}^{M} \langle x_i, y_i \rangle$. We set

$$\mathbf{A} = \{(x, \ldots, x) \in \mathbf{X} \mid x \in X\} \quad \text{and} \quad \mathbf{B} = C_1 \times \cdots \times C_M. \tag{8}$$

Then the projections of $\mathbf{x} = (x_1, \ldots, x_M) \in \mathbf{X}$ onto $\mathbf{A}$ and $\mathbf{B}$ are given by, respectively, $P_{\mathbf{A}}\mathbf{x} = \left(\frac{1}{M}\sum_{i=1}^{M} x_i, \ldots, \frac{1}{M}\sum_{i=1}^{M} x_i\right)$ and $P_{\mathbf{B}}\mathbf{x} = \left(P_{C_1}x_1, \ldots, P_{C_M}x_M\right)$. Now assume that

$$\mathbf{g} = (g_1, \ldots, g_M) := P_{\overline{\mathbf{B}-\mathbf{A}}} 0 \in \mathbf{B} - \mathbf{A}. \tag{9}$$

Then we have

$$\mathbf{E} := \mathbf{A} \cap (\mathbf{B} - \mathbf{g}) \neq \varnothing, \quad \text{and} \quad (x, \ldots, x) \in \mathbf{A} \cap (\mathbf{B} - \mathbf{g}) \;\Leftrightarrow\; x \in \bigcap_{i=1}^{M} (C_i - g_i). \tag{10}$$

Using [1, Section 6], we see that the M-set problem (7) is equivalent to the *two-set* problem

$$\text{find a least-squares solution of } \mathbf{A} \cap \mathbf{B}. \tag{11}$$

It follows from (9) and (10) that $\mathbf{g}$ is the unique vector in $\overline{\mathbf{B}-\mathbf{A}}$ that satisfies

$$\left.\begin{array}{c}(w_1, w_2, \dots, w_M) \neq (g_1, g_2, \dots, g_M), \\ \text{and } \bigcap_{i=1}^{M} (C_i - w_i) \neq \varnothing\end{array}\right\} \Rightarrow \sum_{i=1}^{M} \|w_i\|^2 > \sum_{i=1}^{M} \|g_i\|^2. \tag{12}$$

We have the following result for the problem of finding a least-squares solution for the intersection of a finite family of sets.

Corollary 3.1. *Suppose that $C_1, \dots, C_M$ are closed convex subsets of X. Let $\mathbf{T} = \operatorname{Id} - P_{\mathbf{A}} + P_{\mathbf{B}} R_{\mathbf{A}}$, let $\mathbf{x} \in \mathbf{X}$ and recall assumption* (9). *Then the* shadow sequence $(P_{\mathbf{A}}\mathbf{T}^n\mathbf{x})_{n\in\mathbb{N}}$ *converges to* $\bar{\mathbf{x}} = (\bar{x}, \dots, \bar{x}) \in \mathbf{A} \cap (\mathbf{B} - \mathbf{g})$, *where $\bar{x} \in \bigcap_{i=1}^{M}(C_i - g_i)$ and $\bar{x}$ is a least-squares solution of* (7).

Proof. Combine Theorem 2.3 with (12) and (10). ■

In Figure 2 we visualize Corollary 3.1 in the case when $X = \mathbb{R}^2$, and $M = 3$.

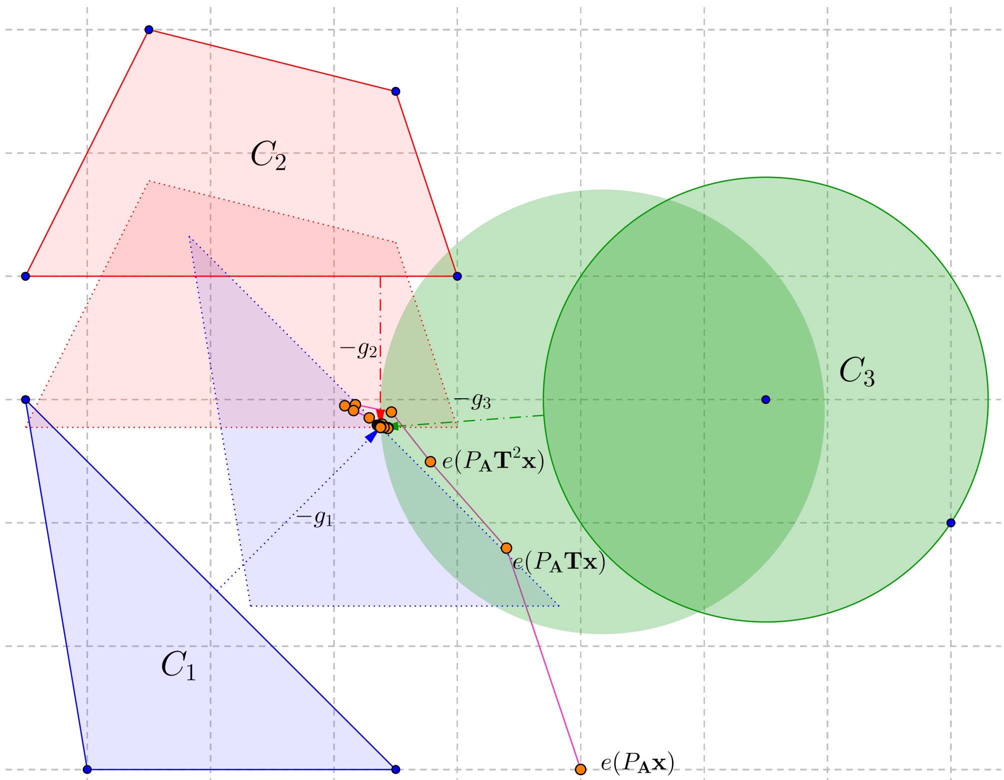

Figure 2: A `GeoGebra` [10] snapshot that illustrates Corollary 3.1. Three nonintersecting closed convex sets, C_1 (the blue triangle), C_2 (the red polygon) and C_3 (the green circle), are shown along with their translations forming the generalized intersection. The first few terms of the sequence $(e(P_{\mathbf{A}}\mathbf{T}^n\mathbf{x}))_{n\in\mathbb{N}}$ (yellow points) are also depicted. Here $e : \mathbf{A} \to \mathbb{R}^2 : (x, x, x) \mapsto x$.

Remark 3.2. When we particularize Corollary 3.1 from convex sets to *halfspaces* and X is *finite-dimensional*, we recover Spingarn's [16, Theorem 1]. Note that in this case, in view of [1, Facts 5.1(ii)] we have $\mathbf{g} \in \mathbf{B} - \mathbf{A}$. Recall that Spingarn used the following version of his *method of partial inverses* from [15]:

$$(\mathbf{a}_0, \mathbf{b}_0) \in \mathbf{A} \times \mathbf{A}^{\perp} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} \mathbf{a}_n' = P_{\mathbf{B}}(\mathbf{a}_n + \mathbf{b}_n), & \mathbf{b}_n' = \mathbf{a}_n + \mathbf{b}_n - \mathbf{a}_n', \\ \mathbf{a}_{n+1} = P_{\mathbf{A}}\mathbf{a}_n', & \mathbf{b}_{n+1} = \mathbf{b}_n' - P_{\mathbf{A}}\mathbf{b}_n'. \end{cases} \tag{13}$$

This method is the DRA in $\mathbf{X}$, applied to $\mathbf{A}$ and $\mathbf{B}$ with starting point $(\mathbf{a}_0 - \mathbf{b}_0)$ (see, e.g., [4, Lemma 2.17]).

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