The resolvent order: a unification of the orders by Zarantonello, by Loewner, and by Moreau

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Abstract

We introduce and investigate the resolvent order, which is a binary relation on the set of firmly nonexpansive mappings. It unifies well-known orders introduced by Loewner (for positive semidefinite matrices) and by Zarantonello (for projectors onto convex cones). A connection with Moreau's order of convex functions is also presented. We also construct partial orders on (quotient sets of) proximal mappings and convex functions. Various examples illustrate our results.

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1 Introduction

In this paper, we assume that

$$X \text{ is a real Hilbert space,} \tag{1}$$

with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We denote the set of all functions from X to $\mathbb{R}\cup\{+\infty\}$ that are *convex*[1], lower semicontinuous and proper by $\Gamma_0(X)$. Let $A\colon X \rightrightarrows X$ be a

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[1]We assume the reader is familiar with basic convex analysis; see, e.g., [13, 14, 18, 22, 19, 4].

set-valued operator, i.e., $(\forall x \in X)\ Ax \subseteq X$ and denote the graph of A by $\operatorname{gra} A$. Recall that A is monotone if

$$(\forall (x,x^*) \in \operatorname{gra} A)(\forall (y,y^*) \in \operatorname{gra} A) \quad \langle x-y, x^*-y^*\rangle \geq 0 \tag{2}$$

and that A is *maximally monotone* if it is monotone and cannot be extended without destroying monotonicity. The notion of maximal monotonicity has proven to be useful in modern optimization and nonlinear analysis; see, e.g., [4, 5, 7, 9, 19, 20, 21, 25, 26]. We denote the set of maximally monotone operators on X by $\mathcal{M}(X)$. This set includes subdifferential operators of functions in $\Gamma_0(X)$ as well as all square matrices with symmetric parts that are positive semidefinite. Furthermore, we denote by $\mathcal{F}(X)$ the set of all mappings $T\colon X \to X$ that are *firmly nonexpansive*[2], i.e.,

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 \leq \langle x-y, Tx-Ty\rangle\,. \tag{3}$$

Thanks to the work of Minty [16] (see also [10]), we can identify a maximally monotone operator A from $\mathcal{M}(X)$ with its *resolvent* in $\mathcal{F}(X)$ via

$$\mathrm{J}_A := (\mathrm{Id} + A)^{-1}. \tag{4}$$

Here $\mathrm{Id} = \nabla\, \mathrm{q} = \partial\, \mathrm{q}$ is the identity operator on X, where $\mathrm{q}\colon x \mapsto \frac{1}{2}\|x\|^2$. If we focus instead on the important subset of subdifferential operators in $\mathcal{M}(X)$, then we recover Moreau's [17] *proximal mapping* (or proximity operator)

$$\mathrm{P}_f := \mathrm{J}_{\partial f} = (\mathrm{Id} + \partial f)^{-1}, \tag{5}$$

where $f \in \Gamma_0(X)$ and $\partial f \in \mathcal{M}(X)$ is the subdifferential operator of f. If C is a nonempty closed convex subset of X, then we write P_C for the projector associated with C. The set of proximal mappings, which we write as $\mathcal{P}(X)$, can also be described as follows. Given $f \in \Gamma_0(X)$, let $\operatorname{env}(f) := \mathrm{q} \,\square\, f$ be the *(Moreau) envelope* of f, where $\square$ denotes infimal convolution. The set of all envelopes is written as $\mathrm{M}_0(X)$. Then, by e.g. [4, (14.7)],

$$\nabla \operatorname{env}(f^*) = \mathrm{P}_f = \mathrm{J}_{\partial f} = (\mathrm{Id} + \partial f)^{-1}, \tag{6}$$

where $f^* \in \Gamma_0(X)$ is the Fenchel conjugate of f. (Thus, we can loosely write $\mathrm{M}_0(X) = \Gamma_0(X) \,\square\, \mathrm{q}$ and $\nabla\, \mathrm{M}_0(X) = \mathcal{P}(X)$.)

Having set up the necessary notation, we can now describe the goal and the organization of this paper.

The goal of this paper is to introduce a new order[3] *on $\mathcal{F}(X)$ which we call the* resolvent order. *It induces orders on $\mathcal{P}(X)$, $\mathcal{M}(X)$, $\Gamma_0(X)$, and $\mathrm{M}_0(X)$ which will allow us to unify and connect to several well known orders from linear and nonlinear analysis, namely to the orders by Zarantonello, by Loewner, and by Moreau. These orders are powerful: indeed, positive semidefinite optimization hinges upon the Loewner order, Moreau introduced his order in his seminal paper on proximal mappings [17], and Zarantonello*

[2] Note that by the Cauchy–Schwarz inequality, every firmly nonexpansive mapping is *nonexpansive*, i.e., Lipschitz continuous with constant 1.

[3] To keep the language in this paper from being overly technical, we will refer to an "order" as a binary relation that is at least reflexive.

used his order for spectral resolution with respect to projectors onto convex cones [23, 24]. We tie these important orders together within one generalized framework which is the subject matter of this paper. We provide several examples and also present a partial order on (a quotient set of) the set of proximal mappings $\mathcal{P}(X)$ and on (a quotient set of) the set of convex functions $\Gamma_0(X)$.

The remainder of the paper is organized as follows. In Section 2, we present various results that make the proofs of the main results more structured. The resolvent order on $\mathcal{F}(X)$ is defined in Section 3, where we also provide basic properties and characterizations for $\mathcal{P}(X)$. In fact, transitivity of the order is established for $\mathcal{P}(X)$. In Section 4, we discuss partitions of the identity and show that transitivity fails for $\mathcal{F}(X)$. In Sections 5, 6, and 7, we connect the resolvent order to the orders by Zarantonello, by Loewner, and by Moreau, respectively. New orders on $\mathcal{M}(X)$ and $\Gamma_0(X)$ are introduced in Section 8. These orders are not *partial orders*. A quotient construction is presented in Section 9 which results in partial orders on $\mathcal{P}(X)$ and on $\Gamma_0(X)$.

The notation we employ is standard and follows, e.g., [4].

2 Auxiliary results

In this section, we collect various results that will be useful later.

Fact 2.1. *Let $T\colon X \to X$. Then the following are equivalent:*

(i) *T is* firmly nonexpansive, *i.e.,* $(\forall x \in X)(y \in X)\ \langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2$.
(ii) $(\forall x \in X)(y \in X)\ \|Tx - Ty\|^2 + \|(x - Tx) - (y - Ty)\|^2 \leq \|x - y\|^2$.
(iii) $\operatorname{Id} - T$ *is firmly nonexpansive.*
(iv) $2T - \operatorname{Id}$ *is nonexpansive.*

Proof. See, e.g., [4], [11], or [12]. ■

Corollary 2.2. *The sets $\mathcal{F}(X)$ and $\mathcal{P}(X)$ are convex. If $\lambda \in [0, 1]$, then $\lambda\,\mathcal{F}(X) \subseteq \mathcal{F}(X)$ and $\lambda\,\mathcal{P}(X) \subseteq \mathcal{P}(X)$.*

Proof. For $\mathcal{F}(X)$, the convexity follows using the last item from Fact 2.1 (see also [2, Corollary 1.8]). For the convexity of $\mathcal{P}(X)$, see [17]. To obtain the inclusions, it suffices to note that $0 \in \mathcal{P}(X) \subseteq \mathcal{F}(X)$. ■

Lemma 2.3. *Let T_1 and T_2 be firmly nonexpansive on X. Then $T_2 - T_1$ is nonexpansive.*

Proof. By Fact 2.1, we can write each $T_i = (\operatorname{Id} + N_i)/2$, where N_i is nonexpansive. It follows that $T_2 - T_1 = (N_2 - N_1)/2$ is nonexpansive. ■

Fact 2.4. *Let $f \in \Gamma_0(X)$ and set $h := f^* - \mathrm{q}$. Then the following are equivalent:*

(i) *f is Fréchet differentiable on X and ∇f is nonexpansive.*

(ii) f *is Fréchet differentiable on* X *and* ∇f *is firmly nonexpansive.*
(iii) $\mathrm{q} - f$ *is convex*
(iv) $f^* - \mathrm{q}$ *is convex.*
(v) $h \in \Gamma_0(X)$ *and* $f = \operatorname{env}(h^*)$.
(vi) $h \in \Gamma_0(X)$ *and* $\nabla f = \mathrm{P}_h$.

Proof. See [4, Theorem 18.15], and also [1], [3], [17]. ■

Corollary 2.5. *Any linear combination of proximal mappings that is monotone and nonexpansive is actually a proximal mapping.*

Proof. Let $h_1, \ldots, h_n$ be in $\Gamma_0(X)$ such that $P := \sum_{i=1}^n \alpha_i \mathrm{P}_{h_i}$ is nonexpansive and monotone. Set $(\forall i \in \{1, \ldots, n\})\ f_i := \operatorname{env}(h_i^*) = h_i^* \,\square\, \mathrm{q} = (h_i + \mathrm{q})^*$, $f := \sum_{i=1}^n \alpha_i f_i$ and $h := f^* - \mathrm{q}$. Then $P = \nabla f$ and f is thus convex and Fréchet differentiable. By Fact 2.4, $h \in \Gamma_0(X)$ and $P = \nabla f = \mathrm{P}_h$. ■

Corollary 2.6. *Let* P_1 *and* P_2 *be proximal mappings. Then* $P_1 - P_2$ *is a proximal mapping if and only if* $P_1 - P_2$ *is monotone.*

Proof. "$\Rightarrow$": Clear. "$\Leftarrow$": By Lemma 2.3, $P_1 - P_2$ is nonexpansive. Now apply Corollary 2.5. ■

Fact 2.7. *Let* $T\colon X \to X$ *be linear and self-adjoint. Then* T *is firmly nonexpansive if and only if* T *is monotone and nonexpansive, in which case* T *is a proximal mapping.*

Proof. See [4, Corollary 18.15] and also [1]. ■

3 The resolvent order

From the point of view of monotone operator theory, the set of firmly nonexpansive mappings is the same as the set of resolvents. This motivates the language in the following definition.

Definition 3.1. (resolvent order) *We define on* $\mathcal{F}(X)$ *a binary relation via*

$$T_1 \preceq T_2 \;:\Leftrightarrow\; T_2 - T_1 \in \mathcal{F}(X). \tag{7}$$

Let us collect some basic properties.

Lemma 3.2. *Let* T, T_0, T_1 *be in* $\mathcal{F}(X)$. *The binary relation* $\preceq$ *satisfies the following:*

(i) **(reflexivity)** $T \preceq T$.
(ii) **(existence of least and greatest element)** $0 \preceq T \preceq \mathrm{Id}$.
(iii) **(order reversal)** $T_0 \preceq T_1 \Leftrightarrow \mathrm{Id} - T_1 \preceq \mathrm{Id} - T_0$.
(iv) $T_0 \preceq T_1 \Leftrightarrow (\forall \lambda \in [0,1])\ T_0 \preceq (1-\lambda)T_0 + \lambda T_1 \preceq T_1$.

Proof. (i): $T - T = 0$ is firmly nonexpansive. (ii): $T - 0 = T$ is firmly nonexpansive as is $\mathrm{Id} - T$ by Fact 2.1. (iii): Indeed, $T_1 - T_0 = (\mathrm{Id} - T_0) - (\mathrm{Id} - T_1)$. (iv): Suppose that $T_0 \preceq T_1$, i.e., $T_1 - T_0 \in \mathcal{F}(X)$. Let $\lambda \in [0,1]$. Then $T_1 - ((1-\lambda)T_0 + \lambda T_1) = (1-\lambda)(T_1 - T_0)$ and $(1-\lambda)T_0 + \lambda T_1 - T_0 = \lambda(T_1 - T_0)$ both of which are firmly nonexpansive by Corollary 2.2. The converse implication is trivial. ■

The following two observations are easily verified.

Example 3.3. (lack of symmetry) *Suppose that $X \neq \{0\}$. Then $0 \preceq \mathrm{Id}$ but $\mathrm{Id} \not\preceq 0$.*

Example 3.4. (lack of antisymmetry) *Suppose that x_1 and x_2 are two distinct vectors in X, and set $(\forall x \in X)\ T_1(x) := x_1$ and $T_2(x) := x_2$. Then $T_1 \preceq T_2$ and $T_2 \preceq T_1$ yet $T_1 \neq T_2$.*

In Section 9, we will present a quotient construction that makes the binary relation antisymmetric.

We now turn to proximal mappings which allows us to obtain stronger conclusions.

Theorem 3.5. *Let f and g be in $\Gamma_0(X)$. Then the following are equivalent:*

(i) $\mathrm{P}_f \preceq \mathrm{P}_g$, *i.e.*, $\mathrm{P}_g - \mathrm{P}_f \in \mathcal{F}(X)$.
(ii) $\mathrm{P}_g - \mathrm{P}_f \in \mathcal{P}(X)$.
(iii) $\mathrm{env}(g^*) - \mathrm{env}(f^*) \in \mathrm{M}_0(X)$.
(iv) $\mathrm{env}(f) - \mathrm{env}(g) \in \mathrm{M}_0(X)$.

Proof. "(i)⇐(ii)": Clear. "(i)⇒(ii)": Since $\mathrm{P}_g - \mathrm{P}_f$ is firmly nonexpansive it is also monotone. Now apply Corollary 2.6. "(ii)⇒(iii)": Integrate. "(ii)⇐(iii)": Differentiate. "(iii)⇐(iv)": This is clear since $\mathrm{env}(g^*) - \mathrm{env}(f^*) = (\mathrm{q} - \mathrm{env}(g)) - (\mathrm{q} - \mathrm{env}(f)) = \mathrm{env}(f) - \mathrm{env}(g)$. ■

Theorem 3.6. (transitivity for proximal mappings) *Let f, g, h be in $\Gamma_0(X)$ such that $\mathrm{P}_f \preceq \mathrm{P}_g$ and $\mathrm{P}_g \preceq \mathrm{P}_h$. Then $\mathrm{P}_f \preceq \mathrm{P}_h$.*

Proof. By the hypothesis and Theorem 3.5, there exist a and b in $\Gamma_0(X)$ such that

$$\mathrm{P}_g - \mathrm{P}_f = \mathrm{P}_a \quad \text{and} \quad \mathrm{P}_h - \mathrm{P}_g = \mathrm{P}_b\,. \tag{8}$$

Adding yields $\mathrm{P}_h - \mathrm{P}_f = \mathrm{P}_a + \mathrm{P}_b$. On the other hand, $\mathrm{P}_a + \mathrm{P}_b$ is monotone because P_a and P_b are monotone. Altogether, we deduce from Corollary 2.6, that $\mathrm{P}_h - \mathrm{P}_f$ is a proximal mapping. ■

Corollary 3.7. (proximal mappings are directed) *$(\mathcal{P}(X), \preceq)$ is a directed set.*

Proof. The reflexivity of $\preceq$ was observed in Lemma 3.2(i) while the transitivity of $\preceq$ is a consequence of Theorem 3.6. Finally, if P_1 and P_2 are in $\mathcal{P}(X)$, then $P_1 \preceq \mathrm{Id}$ and $P_2 \preceq \mathrm{Id}$ by Lemma 3.2(ii). ■

We conclude this section with an example.

Example 3.8. *Denote the unit ball centered at* 0 *of radius* 1 *in* X *by* C, *and set* $T := \mathrm{Id} - \mathrm{P}_C$. *Then* $(\forall n \in \mathbb{N})$ $T^n = \mathrm{Id} - \mathrm{P}_{nC} \in \mathcal{P}(X)$ *and* $T^n - T^{n+1} = \mathrm{P}_{(n+1)C} - \mathrm{P}_{nC} \in \mathcal{P}(X)$. *Consequently,*

$$(\forall n \in \mathbb{N}) \quad 0 \preceq T^{n+1} \preceq T^n \preceq \cdots \preceq T \preceq T^0 = \mathrm{Id}\,. \tag{9}$$

Proof. The identity for T^n is easily verified by mathematical induction and discussing cases. To verify that $T^n - T^{n+1}$ belongs to $\mathcal{P}(X)$, observe that by Corollary 2.6 it suffices to show that $T^n - T^{n+1}$ is monotone. In turn, this is achieved by discussing cases and invoking the Cauchy–Schwarz inequality. ■

4 Partitions of the identity and the partial sum property

In this section, we discuss partial sums of firmly nonexpansive mappings arising in partitions of the identity. Somewhat surprisingly, we also show that the transitivity result for proximal mappings (Theorem 3.6) *fails* for firmly nonexpansive mappings (see Example 4.5 below).

We start with a positive result.

Lemma 4.1. *Let* T_1, T_2, T_3 *be in* $\mathcal{F}(X)$ *such that such that* $T_1 + T_2 + T_3 = \mathrm{Id}$. *Then* $T_1 + T_2$ *is firmly nonexpansive.*

Proof. Since $T_1 + T_2 = \mathrm{Id} - T_3$, this follows from Fact 2.1. ■

For proximal mappings we are able to extend Lemma 4.1 from 3 to any number of operators:

Theorem 4.2. (partial sum property for proximal mappings) *Let* $n \in \{1, 2, \ldots\}$, *let* $P_1, \ldots, P_n$ *be in* $\mathcal{P}(X)$ *such that* $P_1 + P_2 + \cdots + P_n = \mathrm{Id}$, *and let* $m \in \{1, \ldots, n\}$. *Then* $P_1 + \cdots + P_m \in \mathcal{P}(X)$.

Proof. There exist functions $f_1, \ldots, f_n$ in $\Gamma_0(X)$ such that for each i, $\nabla \operatorname{env}(f_i^*) = P_i$, and $\operatorname{env}(f_1^*) + \cdots + \operatorname{env}(f_n^*) = \mathrm{q}$. It follows that

$$\mathrm{q} - \big(\operatorname{env}(f_1^*) + \cdots + \operatorname{env}(f_m^*)\big) = \operatorname{env}(f_{m+1}^*) + \cdots + \operatorname{env}(f_n^*) \tag{10}$$

is convex. By Fact 2.4, $\nabla(\operatorname{env}(f_1^*) + \cdots + \operatorname{env}(f_m^*)) = P_1 + \cdots + P_m$ is a proximal mapping. ■

Surprisingly, the counterpart of Theorem 4.2 for firmly nonexpansive mappings is false as the next two results show.

Lemma 4.3. *In* $X = \mathbb{R}^2$, *let* $n \in \{2, 3, \ldots\}$, *let* $\theta \in \big]\arccos(1/\sqrt{2}), \arccos(1/\sqrt{2n})\big]$, *set* $\alpha := 1/(2n\cos(\theta))$, *and denote by* R_θ *be the counterclockwise rotator by* θ. *Then the following hold:*

(i) αR_θ *and* $\alpha R_{-\theta}$ *are firmly nonexpansive.*
(ii) $n\alpha R_\theta$ *and* $n\alpha R_{-\theta}$ *are not firmly nonexpansive.*
(iii) $n\alpha R_\theta + n\alpha R_{-\theta} = \mathrm{Id}$.

Proof. Observe that $(\forall x \in X)\ \langle x, R_\theta x\rangle = \cos(\theta)\|x\|^2 = \cos(\theta)\|R_\theta x\|^2$. Because R_θ is a linear isometry, we obtain for every $\alpha \in \mathbb{R}_+$ the equivalences

$$\alpha R_\theta \in \mathcal{F}(X) \Leftrightarrow (\forall x \in X) \quad \langle x, \alpha R_\theta x\rangle \geq \|\alpha R_\theta x\|^2 \tag{11a}$$

$$\Leftrightarrow (\forall x \in X) \quad \alpha(\cos\theta)\|x\|^2 \geq \alpha^2\|x\|^2 \tag{11b}$$

$$\Leftrightarrow \alpha \in [0, \cos(\theta)]. \tag{11c}$$

Let $\alpha \in \mathbb{R}_{++}$ satisfy

$$n\alpha R_\theta + n\alpha R_{-\theta} = n\alpha(R_\theta + R_{-\theta}) = n\alpha \begin{pmatrix} 2\cos(\theta) & 0 \\ 0 & 2\cos(\theta) \end{pmatrix} = \mathrm{Id}; \tag{12}$$

equivalently,

$$\alpha := \frac{1}{2n\cos(\theta)}. \tag{13}$$

Combining (11) and (13) yields

$$\alpha R_\theta \in \mathcal{F}(X) \quad \Leftrightarrow \quad \frac{1}{2n} \leq \cos^2(\theta). \tag{14}$$

On the other hand, $(n\alpha)R_\theta \notin \mathcal{F}(X) \Leftrightarrow n\alpha > \cos(\theta) \Leftrightarrow (2\cos(\theta))^{-1} > \cos(\theta)$. Altogether,

$$\left[\alpha R_\theta \in \mathcal{F}(X) \text{ and } n\alpha R_\theta \notin \mathcal{F}(X)\right] \Leftrightarrow \cos(\theta) < \frac{1}{2\cos(\theta)} \leq n\cos(\theta) \tag{15a}$$

$$\Leftrightarrow \cos^2(\theta) < \frac{1}{2} \leq n\cos^2(\theta) \tag{15b}$$

$$\Leftrightarrow \cos(\theta) < \frac{1}{\sqrt{2}} \leq \sqrt{n}\cos(\theta) \tag{15c}$$

$$\Leftrightarrow \frac{1}{\sqrt{2n}} \leq \cos(\theta) < \frac{1}{\sqrt{2}}. \tag{15d}$$

Note that (15) has no solution for $n = 1$; however, (15) has solutions for every $n \geq 2$. Because $R_{-\theta} = R_\theta^*$, the result follows with [4, Corollary 4.3] or by arguing along the same lines as above for $R_{-\theta}$. ■

We now obtain the following direct consequence of Lemma 4.3:

Example 4.4. (partial sum property fails for general firmly nonexpansive mappings) *Let $n \in \{2, 3, \ldots\}$, and let θ, α, and $R_{\pm\theta}$ be as in Lemma 4.3. Furthermore, set $T_1 := \cdots = T_n := \alpha R_\theta$ and $T_{n+1} := \cdots = T_{2n} := \alpha R_{-\theta}$. Then each T_i is firmly nonexpansive, $T_1 + \cdots + T_{2n} = \mathrm{Id}$, yet $T_1 + \cdots + T_n$ is* not *firmly nonexpansive.*

We conclude this section with another negative result.

Example 4.5. (lack of transitivity for firmly nonexpansive mappings) *Suppose that $X = \mathbb{R}^2$, and set $R := \alpha R_\theta$ and $S := \alpha R_{-\theta}$, where θ and α are as in Example 4.4 for $n = 2$. Then*

$$\textit{R and S are firmly nonexpansive,} \tag{16a}$$

$$2R \text{ and } 2S \text{ are not firmly nonexpansive,} \tag{16b}$$

$$2R + 2S = \mathrm{Id}. \tag{16c}$$

Now set

$$T_1 := S, \quad T_2 := R + S, \quad T_3 := 2R + S. \tag{17}$$

Then $T_1 \in \mathcal{F}(X)$ *by* (16a). *Next,* (16a) *and* (16c) *imply that* $T_3 = \mathrm{Id} - S \in \mathcal{F}(X)$. *Since* $\mathcal{F}(X)$ *is convex (Corollary 2.2), it follows that* $T_2 = (T_1 + T_3)/2 \in \mathcal{F}(X)$. *Because* $T_2 - T_1 = T_3 - T_2 = R \in \mathcal{F}(X)$ *by* (16a), *we have*

$$T_1 \preceq T_2 \text{ and } T_2 \preceq T_3. \tag{18}$$

On the other hand, $T_3 - T_1 = 2R \notin \mathcal{F}(X)$ *by* (16b). *Thus,*

$$T_1 \not\preceq T_3. \tag{19}$$

Altogether, we deduce that

$$(\mathcal{F}(X), \preceq) \text{ is not transitive.} \tag{20}$$

5 Compatibility with Zarantonello's partial order

Zarantonello introduced in [23, 24] a partial ordering of the set of projectors onto nonempty closed convex cones contained in X via

$$\mathrm{P}_C \preceq_Z \mathrm{P}_D \quad :\Leftrightarrow \quad \mathrm{P}_C \mathrm{P}_D = \mathrm{P}_C\,. \tag{21}$$

He established various nice properties which we collect in the following result.

Fact 5.1. **(Zarantonello)** *Let C and D be nonempty closed convex cones in X. Then the following hold:*

(i) $\mathrm{P}_C \preceq_Z \mathrm{P}_D \Leftrightarrow \mathrm{P}_D - \mathrm{P}_C$ *is a projector, in which case*[4] $\mathrm{P}_D - \mathrm{P}_C = \mathrm{P}_{D \cap C^\ominus}$.
(ii) $\mathrm{P}_C \preceq_Z \mathrm{P}_D \Leftrightarrow [\mathrm{P}_C \mathrm{P}_D = \mathrm{P}_D \mathrm{P}_C$ *and* $(\forall x \in X)\ \langle x, \mathrm{P}_C x\rangle \leq \langle x, \mathrm{P}_D x\rangle]$.
(iii) $\mathrm{P}_C \preceq_Z \mathrm{P}_D \Rightarrow C \subseteq D$.
(iv) $\mathrm{P}_C \preceq_Z \mathrm{P}_D \Rightarrow \mathrm{P}_C, \mathrm{P}_D, \mathrm{P}_{C^\ominus}, \mathrm{P}_{D^\ominus}$ *pairwise commute with their products being the projectors onto the intersection of their ranges* ($\mathrm{P}_C \mathrm{P}_D = \mathrm{P}_C \mathrm{P}_D = \mathrm{P}_{C \cap D}$, *etc.*).
(v) *Suppose that C and D are subspaces. Then* $\mathrm{P}_C \preceq_Z \mathrm{P}_D \Leftrightarrow C \subseteq D$.

Proof. (i)–(iv): See [23, Lemma 5.12] and [24, page 347]. (v): If $C \subseteq D$, then $D = C \oplus (D \cap C^\perp)$, which implies that $\mathrm{P}_D - \mathrm{P}_C = \mathrm{P}_{D \cap C^\perp}$ is a projector and $\mathrm{P}_C \preceq_Z \mathrm{P}_D$ by (i). The other implication is (iii). ■

Remark 5.2. *Generalizing* (21) *and hoping that Fact 5.1(i) holds by just replacing projectors by proximal mappings will not work: indeed,* $\mathrm{P}_f - \mathrm{P}_f = 0 = \mathrm{P}_{\{0\}}$ *is a proximal map and a projector yet* $\mathrm{P}_f \mathrm{P}_f \neq \mathrm{P}_f$.

Next, let us show that Zarantonello's order is compatible with the order from Definition 3.1:

[4] Here $C^\ominus := \{x \in X \mid \sup \langle C, x\rangle = 0\}$ is the *polar cone* of C.

Lemma 5.3. (compatibility with Zarantonello's order) *Let C and D be nonempty closed convex cones in X. Then $\mathrm{P}_C \preceq_Z \mathrm{P}_D \Leftrightarrow \mathrm{P}_C \preceq \mathrm{P}_D$.*

Proof. "$\Rightarrow$": Assume that $\mathrm{P}_C \preceq_Z \mathrm{P}_D$. By Fact 5.1(i), $\mathrm{P}_D - \mathrm{P}_C$ is a projector, hence a proximal mapping and thus firmly nonexpansive. Therefore, $\mathrm{P}_C \preceq \mathrm{P}_D$. "$\Leftarrow$": Assume that $\mathrm{P}_C \preceq \mathrm{P}_D$, i.e., $\mathrm{P}_D - \mathrm{P}_C$ is firmly nonexpansive. By Theorem 3.5, $\mathrm{P}_D - \mathrm{P}_C$ is a proximal mapping. Hence, for

$$f := \tfrac{1}{2}d_C^2 - \tfrac{1}{2}d_D^2, \tag{22}$$

there exists $h \in \Gamma_0(X)$ such that

$$\nabla f = (\mathrm{Id} - \mathrm{P}_C) - (\mathrm{Id} - \mathrm{P}_D) = \mathrm{P}_D - \mathrm{P}_C = \mathrm{P}_h\,. \tag{23}$$

Since P_h is monotone, the function f is convex and so $f \in \Gamma_0(X)$. Now $\nabla f = \mathrm{P}_h = (\mathrm{Id} + \partial h)^{-1} \Rightarrow \partial f^* = (\nabla f)^{-1} = \mathrm{Id} + \partial h = \partial(\mathrm{q} + h)$. Thus, after integrating and noting that (23) is invariant under adding constants to h, we may and do assume that

$$h = f^* - \mathrm{q}\,. \tag{24}$$

Furthermore, combining [4, Example 13.3(ii) and Example 13.24(iii)] yields

$$\big(\tfrac{1}{2}d_C^2\big)^* = \mathrm{q} + \iota_{C^\ominus}. \tag{25}$$

Let us now compute f^* at $u \in X$. By [4, Proposition 14.19],

$$\begin{aligned} f^*(u) &= \sup_{v \in \mathrm{dom}((1/2)d_D^2)^*} \Big(\big(\tfrac{1}{2}d_C^2\big)^*(u+v) - \big(\tfrac{1}{2}d_D^2\big)^*(v) \Big) && \text{(26a)} \\ &= \sup_{v \in D^\ominus} \big(\mathrm{q}(u+v) + \iota_{C^\ominus}(u+v) - \mathrm{q}(v) \big) && \text{(26b)} \\ &= \mathrm{q}(u) + \sup_{v \in D^\ominus} \big(\langle u, v\rangle + \iota_{C^\ominus}(u+v) \big). && \text{(26c)} \end{aligned}$$

Two cases are now conceivable.

Case 1: $(\exists\, v \in D^\ominus)\ u + v \notin C^\ominus$.
Then $f^*(u) = +\infty$ and hence $f^*(u) - q(u) = +\infty$.

Case 2: $u + D^\ominus \subseteq C^\ominus$.
Then

$$f^*(u) - q(u) = \sup_{v \in D^\ominus} \langle u, v\rangle = \iota^*_{D^\ominus}(u) = \iota_{D^{\ominus\ominus}}(u) = \iota_D(u) \in \{0, +\infty\}. \tag{27}$$

Altogether, $h = f^* - q$ takes only values in $\{0, +\infty\}$, i.e., h is an *indicator function* Thus, P_h must be a projector[5]. Therefore, by Fact 5.1(i), $\mathrm{P}_C \preceq_Z \mathrm{P}_D$. ■

[5]We may obtain additional information as follows. Suppose first, as in *Case 2*, that $u + D^\ominus \subseteq C^\ominus$. This case must occur since h is proper. (In passing, note that this precisely states that u is in the so-called *star-difference* $C^\ominus \overset{*}{-} D^\ominus$; see [13].) Since $0 \in D^\ominus$, it is clear that $u \in C^\ominus$. On the other hand, since we are working with *cones*, we have $(\forall \varepsilon > 0)$ $\varepsilon u + D^\ominus = \varepsilon(u + D^\ominus) \subseteq \varepsilon C^\ominus = C^\ominus$. Letting $\varepsilon \to 0^+$, we deduce that $D^\ominus \subseteq C^\ominus$ and thus $C \subseteq D$ as is also guaranteed by Fact 5.1(i). If conversely $u \in C^\ominus$, then $u + D^\ominus \subseteq u + C^\ominus \subseteq C^\ominus$. Altogether, we have shown that u is as in *Case 2* if and only if $u \in C^\ominus$. Therefore, $h = \iota_{D \cap C^\ominus}$, which is consistent with Fact 5.1(i).

6 Compatibility with the Loewner order via resolvents

In this section, we assume that

$$X = \mathbb{S}^n := \left\{ A \in \mathbb{R}^{n\times n} \;\middle|\; A = A^* \right\} \tag{28}$$

is the finite-dimensional Hilbert space of all real symmetric matrices of size $n \times n$ with the inner product $\langle A, B\rangle$ being the trace of AB. We shall focus on the closed convex cone of *positive semidefinite matrices*:

$$\mathbb{S}^n_+ := \left\{ A \in \mathbb{S}^n \;\middle|\; (\forall x \in \mathbb{R}^n)\;\; \langle x, Ax\rangle \geq 0 \right\} = \left\{ A \in \mathbb{S}^n \;\middle|\; A \text{ is monotone} \right\}. \tag{29}$$

Let A and B be in $\mathbb{S}^n_+$. The classical *Loewner* (or Löwner) *order* [15] states

$$B \preceq_{\mathsf{L}} A \quad :\Leftrightarrow \quad A - B \in \mathbb{S}^n_+, \text{ i.e., } A - B \text{ is monotone.} \tag{30}$$

Passing to resolvents, we have

$$B \preceq_{\mathsf{L}} A \quad \Leftrightarrow \quad \mathrm{Id} + B \preceq_{\mathsf{L}} \mathrm{Id} + A \quad \Leftrightarrow \quad (\mathrm{Id} + A)^{-1} \preceq_{\mathsf{L}} (\mathrm{Id} + B)^{-1} \quad \Leftrightarrow \quad \mathrm{J}_A \preceq_{\mathsf{L}} \mathrm{J}_B\,. \tag{31}$$

The question now arises whether the Loewner order for resolvents is compatible with our order from Definition 3.1. Clearly,

$$\mathrm{J}_A \preceq \mathrm{J}_B \quad \Rightarrow \quad \mathrm{J}_B - \mathrm{J}_A \text{ is monotone} \quad \Leftrightarrow \quad \mathrm{J}_A \preceq_{\mathsf{L}} \mathrm{J}_B\,. \tag{32}$$

Conversely, assume that $\mathrm{J}_A \preceq_{\mathsf{L}} \mathrm{J}_B$, i.e., $\mathrm{J}_B - \mathrm{J}_A$ is monotone. On the other hand, $\mathrm{J}_B - \mathrm{J}_A$ is nonexpansive by Lemma 2.3. Altogether, by Fact 2.7, $\mathrm{J}_B - \mathrm{J}_A$ is firmly nonexpansive, i.e., $\mathrm{J}_A \preceq \mathrm{J}_B$. In summary,

$$B \preceq_{\mathsf{L}} A \quad \Leftrightarrow \quad \mathrm{J}_A \preceq_{\mathsf{L}} \mathrm{J}_B \quad \Leftrightarrow \quad \mathrm{J}_A \preceq \mathrm{J}_B, \tag{33}$$

which shows that *the Loewner order and our order are compatible*. (In passing, we note that the comments in this section have extensions to self-adjoint operators on Hilbert space.) Finally, we recall that the Loewner order is *not* a lattice order: indeed, the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{34}$$

do not have a supremum (see, e.g., [6, Exercise 1.2.4]).

7 A connection to Moreau's order

In his seminal work [17], Moreau introduced an order of $\Gamma_0(X)$ via

$$g \preceq_{\mathsf{M}} f \quad :\Leftrightarrow \quad (\exists\, h \in \Gamma_0(X))\;\; f = g + h. \tag{35}$$

In fact, using this notation, we can write the equivalence of (iii) and (iv) in Fact 2.4, which was first observed by Moreau [17], more succinctly as $f \preceq_{\mathsf{M}} \mathrm{q} \Leftrightarrow \mathrm{q} \preceq_{\mathsf{M}} f^*$. To make the connection with our order, let us take f and g from $\Gamma_0(X)$. Using Corollary 2.6 and Theorem 3.5, we have the following equivalences:

$$\begin{aligned}
\operatorname{env}(g) \preceq_{\mathsf{M}} \operatorname{env}(f) &\Leftrightarrow \operatorname{env}(f) - \operatorname{env}(g) \text{ is convex} \Leftrightarrow \operatorname{env}(g^*) - \operatorname{env}(f^*) \text{ is convex} && \text{(36a)}\\
&\Leftrightarrow \mathrm{P}_g - \mathrm{P}_f \text{ is monotone} \Leftrightarrow \operatorname{env}(f) - \operatorname{env}(g) \in \mathrm{M}_0(X) && \text{(36b)}\\
&\Leftrightarrow \mathrm{P}_g - \mathrm{P}_f \in \mathcal{P}(X) \Leftrightarrow \mathrm{P}_f \preceq \mathrm{P}_g\,. && \text{(36c)}
\end{aligned}$$

Hence, *our order is compatible with Moreau's order* when restricted to $\mathrm{M}_0(X)$, the set of envelope functions on X.

We conclude this section with a new order on $\Gamma_0(X)$ defined by

$$g \preceq_{\Box} f \quad :\Leftrightarrow \quad (\exists\, h \in \Gamma_0(X))\ \ f = (g \Box h)^{**}. \tag{37}$$

Then the equivalences

$$\begin{aligned}
\operatorname{env}(g) \preceq_{\Box} \operatorname{env}(f) &\Leftrightarrow \big(\operatorname{env}(g)\big)^* \preceq_{\mathsf{M}} \big(\operatorname{env}(f)\big)^* \Leftrightarrow \big(\operatorname{env}(f)\big)^* - \big(\operatorname{env}(g)\big)^* \text{ is convex} && \text{(38a)}\\
&\Leftrightarrow f^* - g^* \text{ is convex} \Leftrightarrow g^* \preceq_{\mathsf{M}} f^* \Leftrightarrow g \preceq_{\Box} f && \text{(38b)}
\end{aligned}$$

reveal that $\preceq_{\Box}$ can be interpreted as the order "dual" to the Moreau order.

8 Ordering monotone operators and convex functions

The classical bijection between the maximally monotone operators on X and the firmly nonexpansive mappings on X, introduced by Minty [16] (see also [10]), is

$$\mathcal{M}(X) \to \mathcal{F}(X) \colon A \mapsto \mathrm{J}_A = (\mathrm{Id} + A)^{-1}. \tag{39}$$

We thus define a new binary relation on $\mathcal{M}(X)$ via

$$B \preceq A \quad :\Leftrightarrow \quad \mathrm{J}_A \preceq \mathrm{J}_B\,. \tag{40}$$

For instance, we have $\mathrm{N}_X = 0 \preceq A \preceq \mathrm{N}_{\{0\}}$ by Lemma 3.2(ii) while Lemma 3.2(iv) gives a statement relating to the resolvent average introduced in [2]. The results in Section 6 show that when $X = \mathbb{R}^n$ and we consider $\mathbb{S}^n_+$, which is a subset of $\mathcal{M}(X)$, then these notions are compatible (see (33)).

Furthermore, we can use this binary relation on $\mathcal{M}(X)$ to define a new binary relation on $\Gamma_0(X)$ via

$$g \preceq f \quad :\Leftrightarrow \quad \partial g \preceq \partial f. \tag{41}$$

With these definitions and using (36), we have the equivalences

$$g \preceq f \;\Leftrightarrow\; \partial g \preceq \partial f \;\Leftrightarrow\; \mathrm{J}_{\partial f} \preceq \mathrm{J}_{\partial g} \;\Leftrightarrow\; \mathrm{P}_f \preceq \mathrm{P}_g \;\Leftrightarrow\; \operatorname{env}(g) \preceq_{\mathsf{M}} \operatorname{env}(f) \tag{42}$$

which show that our binary relation on $\Gamma_0(X)$ plays along nicely with Moreau's order.

Let us present another example. Let A and B be in $\mathbb{S}^n_+$ and define the quadratic forms

$$(\forall x \in \mathbb{R}^n) \quad \mathrm{q}_A(x) := \tfrac{1}{2}\langle x, Ax\rangle \text{ and } \mathrm{q}_B(x) := \tfrac{1}{2}\langle x, Bx\rangle. \tag{43}$$

Then $\nabla \mathrm{q}_A = A$ and $\nabla \mathrm{q}_B = B$. Using (31), we see that

$$\mathrm{q}_B \leq \mathrm{q}_A \text{ (pointwise)} \Leftrightarrow 0 \leq \mathrm{q}_{A-B} \Leftrightarrow A - B \in \mathbb{S}^n_+ \Leftrightarrow B \preceq_{\mathsf{L}} A \Leftrightarrow \mathrm{J}_A \preceq \mathrm{J}_B \tag{44a}$$

$$\Leftrightarrow B \preceq A \Leftrightarrow \nabla \mathrm{q}_B \preceq \nabla \mathrm{q}_A \Leftrightarrow \mathrm{q}_B \preceq \mathrm{q}_A \Leftrightarrow \mathrm{P}_{\mathrm{q}_A} \preceq \mathrm{P}_{\mathrm{q}_B} \tag{44b}$$

$$\Leftrightarrow \operatorname{env}(\mathrm{q}_B) \preceq_{\mathsf{M}} \operatorname{env}(\mathrm{q}_A). \tag{44c}$$

This nicely illustrates the connections between the various orders considered in this paper.

9 New partial orders

In our final section, we introduce a quotient space construction which remedies the lack of antisymmetry observed in Example 3.4.

We start with a simple but useful result.

Lemma 9.1. *Let $T \in \mathcal{F}(X)$ be such that $-T \in \mathcal{F}(X)$. Then T is a constant mapping.*

Proof. Take x and y from X. Then

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \text{ and } \|(-T)x - (-T)y\|^2 \leq -\langle Tx - Ty, x - y\rangle. \tag{45}$$

Thus, $0 \leq \|Tx - Ty\|^2 \leq \min\{\langle Tx - Ty, x - y\rangle, -\langle Tx - Ty, x - y\rangle\} = -|\langle Tx - Ty, x - y\rangle| \leq 0.$ Therefore, $Tx - Ty = 0$. ∎

We now define a binary relation on $\mathcal{F}(X)$ via

$$T_1 \sim T_2 \;:\Leftrightarrow\; T_2 - T_1 \text{ is a constant mapping.} \tag{46}$$

It is straightforward to check that $(\mathcal{F}(X), \sim)$ is an *equivalence relation*. Denote the corresponding *quotient set* by

$$[\mathcal{F}(X)] := \mathcal{F}(X)/\!\sim. \tag{47}$$

Now define a binary relation on $[\mathcal{F}(X)]$ by

$$[T_1] \preceq [T_2] \quad :\Leftrightarrow \quad T_1 \preceq T_2. \tag{48}$$

Then $([\mathcal{F}(X)], \preceq)$ is *reflexive*; moreover, by Lemma 9.1, $([\mathcal{F}(X)], \preceq)$ is *antisymmetric*. If we restrict to proximal mappings, then $([\mathcal{P}(X)], \preceq)$ is also *transitive* by Theorem 3.6. In summary,

$$\big(\,[\mathcal{P}(X)], \preceq\,\big) \textit{ is a partially ordered set.} \tag{49}$$

Finally, let us investigate (46) from the view point of maximally monotone operators via (39).

Lemma 9.2. *Let A and B be in $\mathcal{M}(X)$ and let $c \in X$. Then the following are equivalent:*

(i) $(\forall x \in X)\ \mathrm{J}_B\, x = c + \mathrm{J}_A\, x$.
(ii) $(\forall x \in X)\ Bx = -c + A(x - c)$.
(iii) $\operatorname{gra} B = (c, -c) + \operatorname{gra} A$.

Proof. "(i)⇒(iii)": By assumption, $(\forall x \in X)\ (\mathrm{J}_B\, x, x - \mathrm{J}_B\, x) = (c, -c) + (\mathrm{J}_A\, x, x - \mathrm{J}_A\, x)$. Using the Minty parametrization (see [4, (23.18)]) of the graph, we see that $\operatorname{gra} B = (c, -c) + \operatorname{gra} A$. The other implications are proved similarly. ■

In view of Lemma 9.2, the equivalence relation (46) in $\mathcal{F}(X)$ gives rise to the following equivalence relation on $\mathcal{M}(X)$:

$$A \sim B \quad :\Leftrightarrow \quad (\exists\, c \in X)(\forall x \in X)\ Bx = -c + A(x - c). \tag{50}$$

In turn, we can "integrate" (50) to obtain the following equivalence relation on $\Gamma_0(X)$:

$$f \sim g \quad :\Leftrightarrow \quad (\exists\, c \in X)(\exists\, \gamma \in \mathbb{R})(\forall x \in X)\ g(x) = f(x - c) - \langle c, x \rangle + \gamma. \tag{51}$$

The last equivalence relation induces a quotient set $[\Gamma_0(X)] := \Gamma_0(X)/\sim$. Interpreting (49) in this setting, we obtain the following result.

Theorem 9.3. *Equip the quotient set $[\Gamma_0(X)]$ with the binary relation*

$$[g] \preceq [f] \ :\Leftrightarrow\ [\mathrm{P}_f] \preceq [\mathrm{P}_g] \ \Leftrightarrow\ \mathrm{P}_f \preceq \mathrm{P}_g\,. \tag{52}$$

Then

$$\big(\, [\Gamma_0(X)], \preceq \,\big) \text{ is a partially ordered set.} \tag{53}$$

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