ON THE FINITE CONVERGENCE OF THE DOUGLAS–RACHFORD ALGORITHM FOR SOLVING (NOT NECESSARILY CONVEX) FEASIBILITY PROBLEMS IN EUCLIDEAN SPACES*

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Abstract. Solving feasibility problems is a central task in mathematics and the applied sciences. One particularly successful method is the Douglas–Rachford algorithm. In this paper, we provide many new conditions sufficient for *finite* convergence. Numerous examples illustrate our results.

Key words. averaged alternating reflections, Douglas–Rachford algorithm, feasibility problem, finite convergence, projector, reflector

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1. Introduction.

The Douglas–Rachford algorithm (DRA) was first introduced in [25] as an operator splitting technique to solve partial differential equations arising in heat conduction. As a result of findings by Lions and Mercier [36] in the monotone operator setting, the method has been extended to find solutions of the sum of two maximally monotone operators. When specialized to normal cone operators, the method is very useful in solving feasibility problems. To fix our setting, we assume throughout that

$$X \text{ is a Euclidean space,} \tag{1}$$

i.e, a finite-dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Given closed subsets A and B of X with nonempty intersection, we consider the fundamental feasibility problem

$$\text{find a point in } A\cap B \tag{2}$$

which frequently arises in science and engineering applications. A common approach for solving (2) is to use projection algorithms that employ projectors onto the underlying sets; see, e.g., [5] [6], [18], [20], [19], [21], [22], [31], and the references therein. Among those algorithms, the Douglas–Rachford algorithm applied to (2) has attracted much attention; see, e.g., [3] and [23] and the references therein for further information.

In the convex case, it is known, see, e.g., Lions and Mercier [36] and Svaiter [39], that the sequence generated by the DRA always converges while the "shadow sequence" converges to a point of the intersection. Even when the convex feasibility problem is inconsistent, i.e., $A\cap B=\varnothing$, it was shown in [7] that the "shadow sequence" is bounded and its cluster points solve a best approximation problem; the entire sequence converges if one of the sets is an affine subspace [8].

Although the Douglas–Rachford algorithm has been applied successfully to various problems involving one or more nonconvex sets, the theoretical justification is far from complete. Recently, in the case of a Euclidean sphere and a line, Borwein and Sims [17] have proved local convergence of the DRA at points of the intersection, while Aragón Artacho and Borwein [1] have given a region of convergence for this model in the plane; moreover, Benoist [15] has even shown that the DRA sequence converges in norm to a point of the intersection except when the starting point belongs to the hyperplane of symmetry. In another direction, [13] proved local convergence for finite unions of convex sets.

On the convergence rate, it has been shown by Hesse, Luke and Neumann [33] that the DRA for two subspaces converges linearly. Furthermore, the rate is then actually the cosine

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of the Friedrichs angle between the subspaces [4]. In the potentially nonconvex case, under transversality assumptions, Hesse and Luke [32] proved local linear convergence of the DRA for a superregular set and an affine subspace, while Phan [38] obtained such a rate for two superregular sets. Specialized to the convex setting, the result in [38] implies linear convergence of the DRA for two convex sets whose the relative interiors have a nonempty intersection; see also [14]. It is worth mentioning that the linear convergence of the DRA may fail even for simple settings in the Euclidean plane, as shown in [10]. Based on Hölder regularity properties, Borwein, Li, and Tam [16] established sublinear convergence for two convex basic semi-algebraic sets. For the linear convergence of the DRA in the framework of optimization problems involving a sum of two functions, we refer the reader to, e.g., Giselsson's [29], [28], Li and Pong [34], Liang, Faili, Peyré, and Luke [35], Patrinos, Stella, and Bemporad's [37], and the references therein.

Davis and Yin [24] observed that the DRA may converge arbitrarily slowly in infinite dimensions; however, in finite dimensions, it often works extremely well. Very recently, the globally finite convergence of the DRA has been shown in [9] for an affine subspace and a locally polyhedral set, or for a hyperplane and an epigraph, and then by Aragón Artacho, Borwein, and Tam [2] for a finite set and a halfspace.

The goal of this paper is to provide various finite-convergence results. The sufficient conditions we present are new and complementary to existing conditions.

After presenting useful results on projectors and the DRA (Section 2) and on locally identical sets (Section 3), we specifically derive results related to the following five scenarios:

- **R1** A is a halfspace and B is an epigraph of a convex function; A is either a hyperplane or a halfspace, and B is a halfspace (see Section 4).
- **R2** A and B are supersets or modifications of other sets where the DRA is better understood (see Section 5).
- **R3** A and B are subsets of other sets where the DRA is better understood (see Section 6).
- **R4** B is a finite, hence nonconvex, set (see Section 7).
- **R5** A is an affine subspace and B is a polyhedron in the absence of Slater's condition (see Section 8).

The paper concludes with a list of open problem in Section 9.

Before we start our analysis, let us note that our notation and terminology is standard and follows, e.g., [6]. The nonnegative integers are $\mathbb{N}$, and the real numbers are $\mathbb{R}$, while $\mathbb{R}_+ := \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$, $\mathbb{R}_{++} := \{\alpha \in \mathbb{R} \mid \alpha > 0\}$, and $\mathbb{R}_- := \{\alpha \in \mathbb{R} \mid \alpha \leq 0\}$. Let C be a subset of X. Then the closure of C is $\overline{C}$, the interior of C is $\operatorname{int} C$, the boundary of C is $\operatorname{bdry} C$, and the smallest affine and linear subspaces containing C are, respectively, $\operatorname{aff} C$ and $\operatorname{span} C$. The relative interior of C, $\operatorname{ri} C$, is the interior of C relative to $\operatorname{aff} C$. The smallest convex cone containing C is $\operatorname{cone} C$, the orthogonal complement of C is $C^\perp := \{y \in X \mid (\forall x \in C)\ \langle x, y\rangle = 0\}$, and the dual cone of C is $C^\oplus := \{y \in X \mid (\forall x \in C)\ \langle x, y\rangle \geq 0\}$. The normal cone operator of C is denoted by N_C, i.e., $N_C(x) = \{y \in X \mid (\forall c \in C)\ \langle y, c - x\rangle \leq 0\}$ if $x \in C$, and $N_C(x) = \varnothing$ otherwise. If $x \in X$ and $\rho \in \mathbb{R}_{++}$, then $\operatorname{ball}(x; \rho) := \{y \in X \mid \|x - y\| \leq \rho\}$ is the closed ball centered at x with radius ρ.

2. Auxiliary results.

For the reader's convenience we recall in this section preliminary concepts and auxiliary results which are mostly well known and which will be useful later.

Let A be a nonempty closed subset of X. The *distance function* of A is

$$d_A\colon X \to \mathbb{R}\colon x \mapsto \min_{a \in A} \|x - a\|. \tag{3}$$

The *projector* onto A is the mapping

$$P_A\colon X \rightrightarrows A\colon x \mapsto \operatorname*{argmin}_{a \in A} \|x - a\| = \{a \in A \mid \|x - a\| = d_A(x)\}, \tag{4}$$

and the *reflector* across A is defined by

$$R_A := 2P_A - \operatorname{Id}, \tag{5}$$

where Id is the identity operator. Note that closedness of the set A is necessary and sufficient for A to be proximinal, i.e., $(\forall x \in X)\ P_A x \neq \varnothing$ (see, e.g., [6, Corollary 3.13]). In the following, we shall write $P_A x = a$ if $P_A x = \{a\}$ is a singleton.

FACT 1 (Projection onto a convex set). *Let A be a nonempty closed convex subset of X, and let x and p be in X. Then the following hold:*

(i) *P_A is single-valued and*

$$p = P_A x \quad \Leftrightarrow \quad [p \in A \text{ and } (\forall y \in A)\ \langle x-p, y-p\rangle \leq 0] \quad \Leftrightarrow \quad x - p \in N_A(p). \tag{6}$$

(ii) *P_A is* firmly nonexpansive*, i.e.,*

$$(\forall x \in X)(\forall y \in X) \quad \|P_A x - P_A y\|^2 + \|(\mathrm{Id} - P_A)x - (\mathrm{Id} - P_A)y\|^2 \leq \|x - y\|^2. \tag{7}$$

(iii) *R_A is* nonexpansive*, i.e.,*

$$(\forall x \in X)(\forall y \in X) \quad \|R_A x - R_A y\| \leq \|x - y\|. \tag{8}$$

In particular, P_A and R_A are continuous on X.

Proof. (i): [6, Theorem 3.14 and Proposition 6.46]. (ii): [6, Proposition 4.8]. (iii): [6, Corollary 4.10]. ☐

LEMMA 2. *Let A and B be closed subsets of X such that $A \subseteq B$, and let $x \in X$. Then the following hold:*

(i) *$A \cap P_B x \subseteq P_A x$.*
(ii) *$(\forall p \in A)$ $P_B^{-1} p \subseteq P_A^{-1} p$.*
(iii) *If $P_B x = p \in A$, then $P_A x = P_B x$.*
(iv) *If B is convex and $P_B x \in A$, then $P_A x = P_B x$.*

Proof. (i): The conclusion is obvious if $A \cap P_B x = \varnothing$. Assume $A \cap P_B x \neq \varnothing$, and let $p \in A \cap P_B x$. Then $\|x - p\| \leq \|x - y\|$ for all $y \in B$, and so for all $y \in A$ since $A \subseteq B$. This combined with $p \in A$ gives $p \in \operatorname{argmin}_{y\in A} \|x - y\| = P_A x$.

(ii): Let $p \in A$. For all $x \in P_B^{-1} p$, we have $p \in P_B x$, and by (i), $p \in A \cap P_B x \subseteq P_A x$, which implies $x \in P_A^{-1} p$.

(iii): Assume that $P_B x = p \in A$. Using (i), we have $p \in P_A x$, and so

$$P_A x = \{y \in A \mid \|x-y\| = \|x-p\|\} \subseteq \{y \in B \mid \|x-y\| = \|x-p\|\} = P_B x = \{p\}. \tag{9}$$

It follows that $P_A x = P_B x = \{p\}$.

(iv): By Fact 1(i), if B is convex, then $P_B x$ is a singleton, and if additionally $P_B x \in A$, then by (iii), $P_A x = P_B x$. ☐

Example 3 (Projection onto an affine subspace). Let Y be a real Hilbert space, let L be a linear operator from X to Y, let $v \in \operatorname{ran} L$, and set $A = \{x \in X \mid Lx = v\}$. Then

$$(\forall x \in X) \quad P_A x = x - L^\dagger(Lx - v), \tag{10}$$

where $L^\dagger$ denotes the Moore–Penrose inverse of L.

Proof. This follows from [11, Lemma 4.1], see also [6, Example 28.14]. ☐

Example 4 (Projection onto a hyperplane or a halfspace). Let $u \in X \smallsetminus \{0\}$, and let $\eta \in \mathbb{R}$. Then the following hold:

(i) If $A = \{x \in X \mid \langle x, u\rangle = \eta\}$, then

$$(\forall x \in X) \quad P_A x = x - \frac{\langle x, u\rangle - \eta}{\|u\|^2} u. \tag{11}$$

(ii) If $A = \{x \in X \mid \langle x, u\rangle \leq \eta\}$, then

$$(\forall x \in X) \quad P_A x = \begin{cases} x & \text{if } \langle x, u\rangle \leq \eta, \\ x - \frac{\langle x,u\rangle - \eta}{\|u\|^2} u & \text{if } \langle x, u\rangle > \eta. \end{cases} \tag{12}$$

Proof. (i): [6, Example 28.15]. (ii): [6, Example 28.16]. ☐

Example 5 (Projection onto a ball). Let $B = \operatorname{ball}(u;\rho)$ with $u \in X$ and $\rho \in \mathbb{R}_{++}$. Then

$$(\forall x \in X) \quad P_B x = u + \frac{\rho}{\max\{\|x-u\|, \rho\}}(x-u). \tag{13}$$

Proof. Let $x \in X$. We have to prove $P_B x = x$ if $\|x-u\| \leq \rho$, and $P_B x = b := u + \frac{\rho}{\|x-u\|}(x-u)$ otherwise. Indeed, if $\|x-u\| \leq \rho$, then $x \in B$, and thus $P_B x = x$. Assume that $\|x-u\| > \rho$. On the one hand, for all $y \in B$, by using $\|y-u\| \leq \rho$ and the triangle inequality,

$$\|x-b\| = \|x-u\| - \rho \leq \|x-u\| - \|y-u\| \leq \|x-y\|. \tag{14}$$

On the other hand, $\|b-u\| = \rho$, and so $b \in \operatorname{ball}(u;\rho)$, then by combining with the convexity of B and the above inequality, $P_B x = b$, which completes the formula. ☐

Example 6 (Projection onto an epigraph). Let $f\colon X \to \mathbb{R}$ be convex and continuous, set $B = \operatorname{epi} f := \{(x,\rho) \in X \times \mathbb{R} \mid f(x) \leq \rho\}$, and let $(x,\rho) \in (X \times \mathbb{R}) \smallsetminus B$. Then there exists $p \in X$ such that $P_B(x,\rho) = (p, f(p))$,

$$x \in p + \big(f(p) - \rho\big)\partial f(p) \text{ and } \rho < f(p) \leq f(x) \tag{15}$$

and

$$(\forall y \in X) \quad \langle y-p, x-p\rangle \leq \big(f(y) - f(p)\big)\big(f(p) - \rho\big). \tag{16}$$

Proof. See [9, Lemma 5.1]. ☐

In order to solve the feasibility problem (2), where A and B are closed subsets of X with nonempty intersection, we employ the *Douglas–Rachford algorithm* (also called *averaged alternating reflections*) that generates a sequence $(x_n)_{n\in\mathbb{N}}$ by

$$(\forall n \in \mathbb{N}) \quad x_{n+1} \in T_{A,B} x_n, \quad \text{where } x_0 \in X, \tag{17}$$

and where

$$T_{A,B} := \tfrac{1}{2}(\operatorname{Id} + R_B R_A) \tag{18}$$

is the *Douglas–Rachford operator* associated with the ordered pair (A,B). The sequence $(x_n)_{n\in\mathbb{N}}$ in (17) is called a *DRA sequence with respect to* (A,B), with starting point x_0. By Fact 1(i), when A and B are convex, then P_A, P_B and hence $T_{A,B}$ are single-valued. Notice that

$$(\forall x \in X) \quad T_{A,B} x = \tfrac{1}{2}(\operatorname{Id} + R_B R_A)x = \{x - a + P_B(2a - x) \mid a \in P_A x\}, \tag{19}$$

and if P_A is single-valued then

$$T_{A,B} = \tfrac{1}{2}(\operatorname{Id} + R_B R_A) = \operatorname{Id} - P_A + P_B R_A. \tag{20}$$

In the sequel we adopt the convention that in the case where $P_A x$ is not a singleton, $(P_A x, P_B R_A x) = \{(a, P_B(2a-x)) \mid a \in P_A x\}$.

The set of fixed points of $T_{A,B}$ is defined by $\operatorname{Fix} T_{A,B} := \{x \in X \mid x \in T_{A,B} x\}$. It follows from $T_{A,B} x = x - P_A x + P_B R_A x$ that

$$x \in \operatorname{Fix} T_{A,B} \quad \Leftrightarrow \quad P_A x \cap P_B R_A x \neq \varnothing, \tag{21}$$

and that modified for clarity

$$\left.\begin{array}{c} x \in \operatorname{Fix} T_{A,B} \\ P_A x \text{ is a singleton} \end{array}\right\} \quad \Rightarrow \quad P_A x \in A \cap B. \tag{22}$$

For the convex case, the basic convergence result of the DRA sequence $(x_n)_{n\in\mathbb{N}}$ and the "shadow sequence" $(P_A x_n)_{n\in\mathbb{N}}$ is as follows.

FACT 7 (Convergence of DRA in the convex consistent case). *Let A and B be closed convex subsets of X with $A \cap B \neq \varnothing$, and let $(x_n)_{n\in\mathbb{N}}$ be a DRA sequence with respect to (A, B). Then the following hold:*

(i) $x_n \to x \in \operatorname{Fix} T_{A,B} = (A \cap B) + N_{A-B}(0)$ *and* $P_A x_n \to P_A x \in A \cap B$.

(ii) *If* $0 \in \operatorname{int}(A - B)$, *then* $x_n \to x \in A \cap B$; *the convergence is finite provided that* $x \in A \cap \operatorname{int} B$.

Proof. (i): This follows from [36, Theorem 1] and [39, Theorem 1]; see also [7, Corollary 3.9 and Theorem 3.13]. (ii): Clear from [9, Lemma 3.2]. ∎

3. Locally identical sets.

DEFINITION 8. *Let A and B be subsets of X such that $A\cap B \neq \varnothing$. Then A and B are called* locally identical *around $c \in A\cap B$ if there exists $\varepsilon \in \mathbb{R}_{++}$ such that $A \cap \operatorname{ball}(c; \varepsilon) = B \cap \operatorname{ball}(c; \varepsilon)$. We say that A and B are locally identical around a set $C \subseteq A \cap B$ if they are locally identical around every point in C. When A and B are locally identical around a point c (respectively, a set C), we also say that (A, B) is locally identical around c (respectively, C).*

LEMMA 9. *Let A and B be subsets of X such that $A \cap B \neq \varnothing$. Then the following hold:*

(i) *A and B are locally identical around* $\operatorname{int}(A \cap B)$.

(ii) *If A and B are locally identical around $c \in A\cap B$, then A, B and $A\cap B$ are also locally identical around c.*

(iii) *If $A \subseteq B$, and c is a point in A such that $d_{B\smallsetminus A}(c) > 0$, then A and B are locally identical around c.*

(iv) *If A is closed convex, and C is a closed subset of A such that A and C are locally identical around C, then $A = C$.*

(v) *If A and B are closed convex and locally identical around $A \cap B$, then $A = B$.*

Proof. (i): Let $c \in \operatorname{int}(A \cap B)$. Then there exists $\varepsilon \in \mathbb{R}_{++}$ such that $\operatorname{ball}(c; \varepsilon) \subseteq A \cap B$, which implies $A\cap\operatorname{ball}(c; \varepsilon) = \operatorname{ball}(c; \varepsilon) = B\cap\operatorname{ball}(c; \varepsilon)$, so A and B are locally identical around c.

(ii): Note that if $A \cap \operatorname{ball}(c; \varepsilon) = B \cap \operatorname{ball}(c; \varepsilon)$ then $A \cap \operatorname{ball}(c; \varepsilon) = B \cap \operatorname{ball}(c; \varepsilon) = (A \cap B) \cap \operatorname{ball}(c; \varepsilon)$.

(iii): Since $d_{B\smallsetminus A}(c) > 0$, there exists $\varepsilon \in \mathbb{R}_{++}$ such that $(B \smallsetminus A) \cap \operatorname{ball}(c; \varepsilon) = \varnothing$. Combining with $A \subseteq B$, we get $A \cap \operatorname{ball}(c; \varepsilon) = (A \cap \operatorname{ball}(c; \varepsilon)) \cup ((B \smallsetminus A) \cap \operatorname{ball}(c; \varepsilon)) = B \cap \operatorname{ball}(c; \varepsilon)$.

(iv): Let $c \in C$. It suffices to show that

$$(\forall \varepsilon \in \mathbb{R}_{++}) \quad A \cap \operatorname{ball}(c; \varepsilon) = C \cap \operatorname{ball}(c; \varepsilon)\,. \tag{23}$$

Suppose to the contrary that (23) does not hold. Since A and C are locally identical around C which includes c,

$$0 < \bar{\varepsilon} := \sup\{\varepsilon \in \mathbb{R}_{++} \mid A \cap \operatorname{ball}(c; \varepsilon) = C \cap \operatorname{ball}(c; \varepsilon)\} < +\infty. \tag{24}$$

Then $(\forall \varepsilon \in\,]\bar{\varepsilon}, +\infty[)$ $A \cap \operatorname{ball}(c; \varepsilon) \supsetneqq C \cap \operatorname{ball}(c; \varepsilon)$. Now let $\varepsilon_n \downarrow \bar{\varepsilon}$ and

$$(\forall n \in \mathbb{N}) \quad a_n \in A \cap \operatorname{ball}(c; \varepsilon_n) \smallsetminus C. \tag{25}$$

By the boundedness of $(a_n)_{n\in\mathbb{N}}$ and the closedness of A, we assume without loss of generality that $a_n \to a \in A$. It follows from $\|a_n - c\| \leq \varepsilon_n$ that $\varepsilon := \|a - c\| \leq \bar{\varepsilon}$. By the convexity of A, $(\forall \lambda \in]0, 1[)$ $a_\lambda = \lambda a + (1 - \lambda)c \in A$, and $\|a_\lambda - c\| = \lambda\|a - c\| = \lambda\varepsilon < \bar{\varepsilon}$, which yields $a_\lambda \in A \cap \operatorname{ball}(c; \lambda\varepsilon) = C \cap \operatorname{ball}(c; \lambda\varepsilon)$, using the definition of $\bar{\varepsilon}$. From $a_\lambda \in C$ and the closedness of C, letting $\lambda \to 1^-$, we obtain $a \in C$, thus A and C are locally identical around a, i.e., $A \cap \operatorname{ball}(a; \rho) = C \cap \operatorname{ball}(a; \rho)$ for some $\rho \in \mathbb{R}_{++}$. Since $a_n \to a$, we find $n_0 \in \mathbb{N}$ satisfying $a_{n_0} \in \operatorname{ball}(a; \rho)$. Then $a_{n_0} \in A \cap \operatorname{ball}(a; \rho) = C \cap \operatorname{ball}(a; \rho) \subseteq C$, which contradicts the fact that $(\forall n \in \mathbb{N})$ $a_n \notin C$. Therefore, (23) holds.

Now pick an arbitrary $a \in A$, and let $\varepsilon > \|a - c\|$. By combining with (23), $a \in A \cap \operatorname{ball}(c; \varepsilon) = C \cap \operatorname{ball}(c; \varepsilon)$, and so $a \in C$. It follows that $A \subseteq C \subseteq A$, which gives $A = C$.

(v): Set $C := A\cap B$. Then C is closed, $C \subseteq A$, $C \subseteq B$, and by (ii), A, B and C are locally identical around C. Now apply (iv). ∎

The following example illustrates that the assumption on convexity of A in Lemma 9(iv) is important.

Example 10. Suppose that $X = \mathbb{R}$, that $A = \{0, 1\}$ and that $C = \{0\}$. Then A and C are closed and locally identical around C, and $C \subseteq A$, but $C \neq A$. This does not contradict Lemma 9(iv) because A is not convex.

LEMMA 11. *Let A and B be closed subsets of X, and assume that A and B are locally identical around some $c \in A \cap B$, say there exists $\varepsilon \in \mathbb{R}_{++}$ such that $A \cap \operatorname{ball}(c; \varepsilon) = B \cap \operatorname{ball}(c; \varepsilon)$. Let*

$$p \in A \cap \operatorname{int}(\operatorname{ball}(c; \varepsilon)) = B \cap \operatorname{int}(\operatorname{ball}(c; \varepsilon)). \tag{26}$$

Then the following hold:

(i) *If $A \subseteq B$, then $(\forall x \in X)$ $P_B x \cap \operatorname{ball}(c; \varepsilon) \subseteq P_A x$.*

(ii) *If A and B are convex, then $(\forall x \in X)$ $p = P_A x \Leftrightarrow p = P_B x$. Equivalently, if A and B are convex then $P_A^{-1} p = P_B^{-1} p$.*

(iii) *If $A \subseteq B$ and B is convex, then $(\forall x \in X)$*
   (a) *$P_B x \in \operatorname{ball}(c; \varepsilon) \Rightarrow P_A x = P_B x$;*
   (b) *$p \in P_A x \quad \Leftrightarrow \quad p = P_B x$;*
   (c) *$p \in P_A x \Rightarrow P_A x = P_B x = p$.*

Proof. (i): Observe that $P_B x \cap \operatorname{ball}(c; \varepsilon) = P_B x \cap (B \cap \operatorname{ball}(c; \varepsilon)) = P_B x \cap (A \cap \operatorname{ball}(c; \varepsilon)) \subseteq A \cap P_B x$. The conclusion follows Lemma 2(i).

To prove (ii) and (iii), note that since $p \in \operatorname{int}(\operatorname{ball}(c; \varepsilon))$, there exists $\rho \in \mathbb{R}_{++}$ such that $\operatorname{ball}(p; \rho) \subseteq \operatorname{ball}(c; \varepsilon)$, which yields

$$A \cap \operatorname{ball}(p; \rho) = B \cap \operatorname{ball}(p; \rho). \tag{27}$$

(ii): By [9, Lemma 2.12], it follows from $p \in A \cap B$ and (27) that $N_A(p) = N_B(p)$. Now using (6), $(\forall x \in X)$ $p = P_A x \Leftrightarrow x - p \in N_A(p) = N_B(p) \Leftrightarrow p = P_B x$. Hence, $P_A^{-1} p = P_B^{-1} p$.

(iii)(a): Let $x \in X$. Assume that $P_B x \in \operatorname{ball}(c; \varepsilon)$. Then $P_B x \in B \cap \operatorname{ball}(c; \varepsilon) = A \cap \operatorname{ball}(c; \varepsilon) \subseteq A$. By Lemma 2(iv), $P_A x = P_B x$.

(iii)(b): Using (27) and applying (ii) for two convex sets $A \cap \operatorname{ball}(p; \rho)$ and B, we obtain $P^{-1}_{A \cap \operatorname{ball}(p;\rho)} p = P_B^{-1} p$. Next applying Lemma 2(ii) for $A \cap \operatorname{ball}(p; \rho) \subseteq A$ and $A \subseteq B$, we have $P_A^{-1} p \subseteq P^{-1}_{A \cap \operatorname{ball}(p;\rho)} p = P_B^{-1} p \subseteq P_A^{-1} p$, and so $P_A^{-1} p = P_B^{-1} p$.

(iii)(c): Now assume $p \in P_A x$. Then (iii)(b) gives $P_B x = p \in A$, and Lemma 2(iv) gives $P_A x = P_B x = p$. ∎

4. Cases involving halfspaces. In this section, we assume that

$$f \colon X \to \mathbb{R} \text{ is convex and continuous,} \tag{28}$$

and that

$$\operatorname{epi} f := \{(x, \rho) \in X \times \mathbb{R} \mid f(x) \le \rho\}. \tag{29}$$

In the space $X \times \mathbb{R}$, we set

$$H := X \times \{0\} \quad \text{and} \quad B := \operatorname{epi} f. \tag{30}$$

Then the projection onto H is given by

$$(\forall (x, \rho) \in X \times \mathbb{R}) \quad P_H(x, \rho) = (x, 0), \tag{31}$$

the projection onto B is described as in Example 6, and the effect of performing each step of the DRA applied to H and B is characterized in the following result.

FACT 12 (One DRA step). *Let $z = (x, \rho) \in X \times \mathbb{R}$, and set $z_+ := (x_+, \rho_+) = T_{H,B}(x, \rho)$. Then the following hold:*

(i) *If* $\rho \le -f(x)$*, then* $z_+ = (x,0) \in H$*. Otherwise, there exists* $x_+^* \in \partial f(x_+)$ *such that*

$$x_+ = x - \rho_+ x_+^*,\ f(x_+) \le f(x),\ \text{and}\ \rho_+ = \rho + f(x_+) > 0; \tag{32}$$

in which either ($\rho \ge 0$ *and* $z_+ \in B$) *or* ($\rho < 0$ *and* $T_{H,B} z_+ \in B$)*.*

(ii) $\operatorname{ran} T_{H,B} \subseteq X \times \mathbb{R}_+$*, or equivalently,* $(\forall z \in X \times \mathbb{R})\ z_+ \in X \times \mathbb{R}_+$*.*

Proof. (i): [9, Corollary 5.3(i)&(ii)]. (ii): Clear from (i). ∎

We have the following result on convergence of the DRA in the case of a hyperplane and an epigraph.

FACT 13 (Finite convergence of DRA in the (hyperplane,epigraph) case). *Suppose that*

$$A = H \quad \text{and} \quad B = \operatorname{epi} f \ \text{with}\ \inf_X f < 0. \tag{33}$$

Given a starting point $z_0 = (x_0, \rho_0) \in X \times \mathbb{R}$*, generate the DRA sequence* $(z_n)_{n\in\mathbb{N}}$ *by*

$$(\forall n \in \mathbb{N}) \quad z_{n+1} = (x_{n+1}, \rho_{n+1}) = T_{A,B} z_n. \tag{34}$$

Then $(z_n)_{n\in\mathbb{N}}$ *converges* finitely *to a point in* $A \cap B$*.*

Proof. See [9, Theorem 5.4]. ∎

In view of Fact 13, it is natural to ask about the convergence of the DRA when A is a halfspace instead of a hyperplane.

THEOREM 14 (Finite convergence of DRA in the (halfspace,epigraph) case). *Suppose that either*

(i) $A = H_+ := X \times \mathbb{R}_+$ *and* $B = \operatorname{epi} f$*, or*

(ii) $A = H_- := X \times \mathbb{R}_-$ *and* $B = \operatorname{epi} f$ *with* $\inf_X f < 0$*.*

Then the DRA sequence (34) *converges* finitely *to a point in* $A \cap B$*.*

Proof. (i): Let $z = (x,\rho) \in X \times \mathbb{R}$. If $z \in H_-$, then $P_A z = P_H z$, and so $z_+ := T_{A,B} z = T_{H,B} z \in H_+$ due to Fact 12(ii). If $z \in H_+ \cap B = A \cap B$, we are done. If $z \in H_+ \smallsetminus B$, then $P_A z = z$, $R_A z = z$, and by Example 6, $P_B R_A z = P_B z = (x_+, f(x_+))$ with $f(x_+) > \rho \ge 0$, which implies $z_+ = z - P_A z + P_B R_A z = (x_+, f(x_+)) \in H_+ \cap B = A \cap B$. We deduce that the DRA sequence (34) converges in at most two steps.

(ii): If $z_0 \in H_- = A$, then $P_A z_0 = z_0$, $R_A z_0 = z_0$, and $z_1 = P_B R_A z_0 = (x_1, f(x_1)) \in B$, which gives $z_1 \in A \cap B$ if $f(x_1) \le 0$, and $z_1 \in H_+$ otherwise. It is thus sufficient to consider the case $z_0 \in H_+$. Then $P_A z_0 = P_H z_0$, and so $z_1 = T_{A,B} z_0 = T_{H,B} z_0 \in H_+$ due to Fact 12(ii). This implies that

$$(\forall n \in \mathbb{N}) \quad z_n \in H_+ \quad \text{and} \quad z_{n+1} = T_{H,B} z_n. \tag{35}$$

Now apply Fact 13. ∎

The following example whose special cases can be found in [10] illustrates that the Slater's condition $\inf_X f < 0$ in Fact 13 and Theorem 14(ii) is important.

Example 15. Suppose that either $A = H$ or $A = H_- := X \times \mathbb{R}_-$, that $B = \operatorname{epi} f$ with $\inf_X f \ge 0$, and that f is differentiable at its minimizers (if they exist). Let $z_0 = (x_0, \rho_0) \in B$, where x_0 is not a minimizer of f, and generate the DRA sequence $(z_n)_{n\in\mathbb{N}}$ as in (34). Then $(P_A z_n)_{n\in\mathbb{N}}$ and thus also $(z_n)_{n\in\mathbb{N}}$ do not converge finitely.

Proof. Firstly, we claim that if $z = (x,\rho) \in B$, where x is not a minimizer of f, then $z_+ := T_{A,B} z = T_{H,B} z = (x_+, \rho_+) \in B$ and x_+ is not a minimizer of f. Indeed, by assumption, $\rho > 0$, so $P_A z = P_H z$, and then $z_+ = T_{A,B} z = T_{H,B} z$. By using Fact 12(i), $z_+ \in B$ and

$$x_+ = x - \rho_+ x_+^* \quad \text{with} \quad x_+^* \in \partial f(x_+), \quad \text{and} \quad \rho_+ = \rho + f(x_+) > 0. \tag{36}$$

If x_+ is a minimizer of f, then $x_+^* = \nabla f(x_+) = 0$, and by (36), $x = x_+$ is a minimizer, which is absurd. Hence, the claim holds. As a result,

$$(\forall n \in \mathbb{N}) \quad x_n \text{ is not a minimizer of } f. \tag{37}$$

Now assume that $(P_A z_n)_{n\in\mathbb{N}} = (x_n, 0)_{n\in\mathbb{N}}$ converges finitely. Then there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$. Using again (36), we get $x^*_{n+1} = 0 \in \partial f(x_{n+1})$, which contradicts (37). ☐

THEOREM 16 (Finite convergence of DRA in (hyperplane or halfspace,halfspace) case). *Suppose that A is either a hyperplane or a halfspace, that B is a halfspace of X, and that $A\cap B \neq \varnothing$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges* finitely *to a point x, where $x \in A\cap B$ or $(\forall n \in \mathbb{N})$ $x_n = x \in B$ with $P_A x \in A \cap B$.*

Proof. If $\dim X = 0$, i.e., $X = \{0\}$, then the result is trivial, so we will work in the space $X\times\mathbb{R}$ with $\dim X \geq 0$, and denote by $(z_n)_{n\in\mathbb{N}}$ the DRA sequence. After rotating the sets if necessary, we can and do assume that $A = X\times\mathbb{R}_-$, and $B = \{(x,\rho) \in X \times \mathbb{R} \mid \langle (x,\rho),(u,\nu)\rangle \leq \eta\}$, with $(u,\nu) \in X\times\mathbb{R}\smallsetminus\{(0,0)\}$ and $\eta \in \mathbb{R}$. Noting that $\langle (x,\rho),(u,\nu)\rangle = \langle x,u\rangle+\rho\nu$, we distinguish the following three cases.

Case 1: $\nu < 0$. Then

$$B = \{(x,\rho) \in X \times \mathbb{R} \mid \frac{\eta - \langle x,u\rangle}{\nu} \leq \rho\} \tag{38}$$

is the epigraph of the linear function

$$f\colon X \to \mathbb{R}\colon x \mapsto \frac{\eta - \langle x,u\rangle}{\nu}. \tag{39}$$

If $\inf_X f < 0$, we are done due to Theorem 14(ii). Assume that $\inf_X f \geq 0$. Then $u = 0 \in X$ since $u \in X \smallsetminus \{0\}$ implies $\inf_X f \leq \inf_{\lambda\in\mathbb{R}_-} f(\lambda u) = \inf_{\lambda\in\mathbb{R}_-} \frac{\eta-\lambda\|u\|^2}{\nu} = -\infty$. Now in turn, $(\forall x \in X)$ $f(x) = \frac{\eta}{\nu}$, and so $\frac{\eta}{\nu} = \inf_X f \geq 0$, which gives $\eta \leq 0$. By the assumption that $A\cap B \neq \varnothing$, we must have $\eta = 0$, and then $B = X\times\mathbb{R}_+$. Let $z = (x,\rho) \in X\times\mathbb{R}$. If $z \in B$, then $R_A z = (x,-\rho)$, and $R_B R_A z = (x,\rho) = z$, which gives $T_{A,B} z = z$, i.e., $z \in \operatorname{Fix} T_{A,B}$, in which case $P_A z = (x,0) \in A\cap B$. If $z \notin B$, then $z \in A$ and $R_A z = z$, $R_B R_A z = R_B z = (x,-\rho)$, so

$$T_{A,B} z = \tfrac{1}{2}(z + R_B R_A z) = (x,0) \in A \cap B. \tag{40}$$

Case 2: $\nu > 0$. Then

$$B = \{(x,\rho) \in X \times \mathbb{R} \mid \frac{\eta - \langle x,u\rangle}{\nu} \geq \rho\}. \tag{41}$$

After reflecting the sets across the hyperplane $X \times \{0\}$, we have $A = X \times \mathbb{R}_+$, and B is the epigraph of a linear function. Now apply Theorem 14(i).

Case 3: $\nu = 0$. Then $u \in X \smallsetminus \{0\}$ and

$$B = \{(x,\rho) \in X \times \mathbb{R} \mid \langle x,u\rangle \leq \eta\} = \{x \in X \mid \langle x,u\rangle \leq \eta\} \times \mathbb{R}. \tag{42}$$

Let $z = (x,\rho) \in X \times \mathbb{R}$. If $z \in A\cap B$, we are done. If $z \in A \smallsetminus B$, then $\rho \in \mathbb{R}_-$, $R_A z = P_A z = z \notin B$, and by Example 4(ii),

$$P_B R_A z = P_B z = \left(x - \frac{\langle x,u\rangle - \eta}{\|u\|^2}u, \rho\right) \in B, \tag{43}$$

which is also in $A = X \times \mathbb{R}_-$ and which yields

$$T_{A,B} z = z - P_A z + P_B R_A z = P_B R_A z \in A \cap B. \tag{44}$$

Now assume that $z \notin A$. We have $P_A z = (x,0)$ and $R_A z = (x,-\rho)$. If $(x,-\rho) \in B$, then $R_B R_A z = (x,-\rho)$, and $T_{A,B} z = \frac{1}{2}(z + R_B R_A z) = (x,0) \in A\cap B$. Finally, if $(x,-\rho) \notin B$, then again by Example 4(ii),

$$P_B R_A z = P_B(x,-\rho) = (x,-\rho) - \frac{\langle x,u\rangle - \eta}{\|u\|^2}(u,0), \tag{45}$$

and thus,

$$T_{A,B} z = z - P_A z + P_B R_A z = \left(x - \frac{\langle x,u\rangle - \eta}{\|u\|^2}u, 0\right) \in A. \tag{46}$$

Moreover, $\left\langle x - \frac{\langle x,u\rangle-\eta}{\|u\|^2}u, u\right\rangle = \eta$, so $T_{A,B} z \in B$, and we get $T_{A,B} z \in A\cap B$.

The proof for the (hyperplane,halfspace) case is similar and uses Fact 13. ☐

5. Expanding and modifying sets.

LEMMA 17 (Expanding sets). *Let A and B be closed (not necessarily convex) subsets of X such that $A \cap B \neq \varnothing$, and let x_0 be in X. Suppose that the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B), with starting point x_0, converges to $x \in \operatorname{Fix} T_{A,B}$. Suppose further that there exist two closed convex sets A' and B' in X such that $A \subseteq A'$, $B \subseteq B'$, and that both (A, A') and (B, B') are locally identical around some $c \in P_A x$. Then $P_A x = P_{A'} x$, $x \in \operatorname{Fix} T_{A',B'}$ and*

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A',B'} x_n, \tag{47}$$

i.e., $(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \quad T^n_{A,B} x_{n_0} = T^n_{A',B'} x_{n_0}$.

Proof. By assumption and Lemma 11(iii)(c), $P_A x = P_{A'} x = c$, and so $R_A x = R_{A'} x$. Since $x \in \operatorname{Fix} T_{A,B}$, it follows from (21) that $c \in P_B R_A x = P_B R_{A'} x$. Using again Lemma 11(iii)(c), $P_B R_{A'} x = P_{B'} R_{A'} x = c$. We get $P_{A'} x = P_{B'} R_{A'} x = c$, and again by (21), $x \in \operatorname{Fix} T_{A',B'}$. Now by the definition of A' and B', there exists $\varepsilon \in \mathbb{R}_{++}$ such that

$$A \cap \operatorname{ball}(c; \varepsilon) = A' \cap \operatorname{ball}(c; \varepsilon) \quad \text{and} \quad B \cap \operatorname{ball}(c; \varepsilon) = B' \cap \operatorname{ball}(c; \varepsilon). \tag{48}$$

There exists $n_0 \in \mathbb{N}$ such that

$$(\forall n \geq n_0) \quad \|x_n - x\| < \varepsilon. \tag{49}$$

Let $n \geq n_0$. Since $P_{A'}$, $P_{B'}$ are (firmly) nonexpansive and $R_{A'}$ is nonexpansive (Fact 1(ii)&(iii)),

$$\|P_{A'} x_n - c\| = \|P_{A'} x_n - P_{A'} x\| \leq \|x_n - x\| < \varepsilon, \tag{50}$$

and also

$$\|P_{B'} R_{A'} x_n - c\| = \|P_{B'} R_{A'} x_n - P_{B'} R_{A'} x\| \leq \|x_n - x\| < \varepsilon. \tag{51}$$

Thus, $P_{A'} x_n \in \operatorname{ball}(c; \varepsilon)$ and $P_{B'} R_{A'} x_n \in \operatorname{ball}(c; \varepsilon)$. By Lemma 11(iii)(a), $P_A x_n = P_{A'} x_n$ and $P_B R_{A'} x_n = P_{B'} R_{A'} x_n$, which implies $R_A x_n = R_{A'} x_n$ and $P_B R_A x_n = P_{B'} R_{A'} x_n$. We deduce that $x_{n+1} = T_{A,B} x_n = T_{A',B'} x_n$. ∎

If the assumption that $A \subseteq A'$ and $B \subseteq B'$ in Lemma 17 is replaced by the assumption on convexity of A and B, then (47) still holds, as shown in the following lemma. We shall now look at situations where (A', B') are modifications of (A, B) that preserve local structure.

LEMMA 18. *Let A and B be closed convex subsets of X such that $A \cap B \neq \varnothing$, and let $(x_n)_{n\in\mathbb{N}}$ be the DRA sequence with respect to (A, B), with starting point $x_0 \in X$. Suppose that there exist two closed convex sets A' and B' in X such that both (A, A') and (B, B') are locally identical around $P_A x \in A \cap B$, where $x \in \operatorname{Fix} T_{A,B}$ is the limit of $(x_n)_{n\in\mathbb{N}}$. Then*

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A',B'} x_n, \tag{52}$$

i.e., $(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \quad T^n_{A,B} x_{n_0} = T^n_{A',B'} x_{n_0}$.

Proof. Recall from Fact 7(i) that $x_n \to x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A \cap B$. Setting $c := P_A x = P_B R_A x$, from the assumption on A' and B', there is $\varepsilon \in \mathbb{R}_{++}$ such that

$$A \cap \operatorname{ball}(c; \varepsilon) = A' \cap \operatorname{ball}(c; \varepsilon) \quad \text{and} \quad B \cap \operatorname{ball}(c; \varepsilon) = B' \cap \operatorname{ball}(c; \varepsilon). \tag{53}$$

Furthermore, there exists $n_0 \in \mathbb{N}$ such that

$$(\forall n \geq n_0) \quad \|x_n - x\| < \varepsilon. \tag{54}$$

Let $n \geq n_0$. According to Fact 1(ii)&(iii), P_A, P_B are (firmly) nonexpansive and R_A is nonexpansive, so

$$\|P_A x_n - c\| = \|P_A x_n - P_A x\| \leq \|x_n - x\| < \varepsilon, \tag{55}$$

and also

$$\|P_BR_Ax_n - c\| = \|P_BR_Ax_n - P_BR_Ax\| \leq \|x_n - x\| < \varepsilon. \tag{56}$$

Therefore, $P_Ax_n \in A \cap \operatorname{int}\operatorname{ball}(c;\varepsilon)$ and $P_BR_Ax_n \in B \cap \operatorname{int}\operatorname{ball}(c;\varepsilon)$. Using Lemma 11(ii), $P_Ax_n = P_{A'}x_n$ and $P_BR_Ax_n = P_{B'}R_Ax_n$. Hence $R_Ax_n = R_{A'}x_n$ and $P_BR_Ax_n = P_{B'}R_{A'}x_n$. We obtain that $x_{n+1} = T_{A,B}x_n = T_{A',B'}x_n$. ∎

THEOREM 19 (Modifying sets). *Let A and B be closed convex subsets of X such that $A \cap B \neq \varnothing$. Suppose that there exist two closed convex sets A' and B' in X such that both (A, A') and (B, B') are locally identical around $A \cap B$. Then for any DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) in (17),*

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A',B'}x_n, \tag{57}$$

and this is still true when exchanging the roles of $T_{A,B}$ and $T_{A',B'}$ in (17) and (57).

Proof. By Fact 7(i), $x_n \to x \in \operatorname{Fix} T_{A,B}$ with $P_Ax \in A \cap B$. Now apply Lemma 18.

Let us exchange the roles of $T_{A,B}$ and $T_{A',B'}$ in (17) and (57), i.e., $(\forall n \in \mathbb{N})$ $x_{n+1} = T_{A',B'}x_n$, and we shall prove that

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A,B}x_n. \tag{58}$$

By the assumption on A' and B', we have $A \cap B \subseteq A'$, $A \cap B \subseteq B'$, and for all $c \in A \cap B$, there exists $\varepsilon \in \mathbb{R}_{++}$ such that

$$A \cap \operatorname{ball}(c;\varepsilon) = A' \cap \operatorname{ball}(c;\varepsilon) \quad \text{and} \quad B \cap \operatorname{ball}(c;\varepsilon) = B' \cap \operatorname{ball}(c;\varepsilon). \tag{59}$$

Then

$$(A \cap B) \cap \operatorname{ball}(c;\varepsilon) = (A' \cap B') \cap \operatorname{ball}(c;\varepsilon). \tag{60}$$

Therefore, $A\cap B$ and $A'\cap B'$ are locally identical around $A\cap B$. Noting that $A\cap B$ and $A'\cap B'$ are closed convex, and $A \cap B \subseteq A' \cap B'$, Lemma 9(iv) gives $A \cap B = A' \cap B'$. Next again by Fact 7(i), $x_n \to x \in \operatorname{Fix} T_{A',B'}$ with $P_{A'}x \in A' \cap B' = A \cap B$. By assumption, both (A, A') and (B, B') are locally identical around $P_{A'}x$, and hence the proof is completed by applying Lemma 18. ∎

In the following, we say that the DRA applied to (A, B) *converges finitely globally* if the sequence $(T^n_{A,B}x)_{n\in\mathbb{N}}$ converges finitely for all $x \in X$.

THEOREM 20. *Let A and B be nonempty closed convex subsets of X. Then the DRA applied to (A, B) converges* finitely globally *provided one of the following holds:*

(i) *$A \cap B \neq \varnothing$ and $A \cap \operatorname{bdry} B = \varnothing$; equivalently, $A \subseteq \operatorname{int} B$.*

(ii) *$A \cap \operatorname{bdry} B \neq \varnothing$ and there exist two closed convex sets A' and B' in X such that both (A, A') and (B, B') are locally identical around $A \cap \operatorname{bdry} B$, and that the DRA applied to (A', B') converges finitely globally when $A' \cap B' \neq \varnothing$.*

(iii) *$A \cap \operatorname{int} B \neq \varnothing$, $A \cap \operatorname{bdry} B \neq \varnothing$ and there exist two closed convex sets A' and B' in X such that both (A, A') and (B, B') are locally identical around $A \cap \operatorname{bdry} B$, and that the DRA applied to (A', B') converges finitely globally when $A' \cap \operatorname{int} B' \neq \varnothing$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a DRA sequence with respect to (A, B).

(i): It follows from $A \cap B \neq \varnothing$, $A \cap \operatorname{bdry} B = \varnothing$ and the closedness of B that $A \cap \operatorname{int} B = A \cap B \neq \varnothing$, and so $0 \in \operatorname{int}(A - B)$. By Fact 7(ii), $x_n \to x \in A \cap B = A \cap \operatorname{int} B$ finitely.

Now if $A \subseteq \operatorname{int} B$, then $A \cap B = A \neq \varnothing$, and $A \cap \operatorname{bdry} B \subseteq \operatorname{int} B \cap \operatorname{bdry} B = \varnothing$, which implies $A \cap \operatorname{bdry} B = \varnothing$. Conversely, assume that $A \cap B \neq \varnothing$ and $A \cap \operatorname{bdry} B = \varnothing$. Let $a \in A$. Then $a \notin \operatorname{bdry} B$. We have to show $a \in \operatorname{int} B$. Suppose to the contrary that $a \notin \operatorname{int} B$. Pick $b \in A \cap B$. By convexity, $[a, b] := \{\lambda a + (1 - \lambda)b \mid 0 \leq \lambda \leq 1\} \subseteq A$, and so $[a, b] \cap \operatorname{bdry} B = \varnothing$, which is impossible since $a \notin B$ and $b \in B$. Hence, $a \in \operatorname{int} B$ for all $a \in A$. This means $A \subseteq \operatorname{int} B$.

(ii): By assumption, $A \cap \operatorname{bdry} B \subseteq A' \cap B'$, and so the DRA applied to (A', B') converges finitely globally. If $A \cap \operatorname{int} B = \varnothing$, then both (A, A') and (B, B') are locally identical around $A \cap \operatorname{bdry} B = A \cap B$ (using the closedness of B), and using Theorem 19,

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A',B'} x_n, \tag{61}$$

which implies the finite convergence of $(x_n)_{n\in\mathbb{N}}$ due to the finite convergence of the DRA applied to (A', B').

Next assume that $A \cap \operatorname{int} B \neq \varnothing$. Then $\operatorname{int}(A - B) \neq \varnothing$. By Fact 7(ii), $x_n \to x \in A \cap B$, and this convergence is finite when $x \in A \cap \operatorname{int} B$. It thus suffices to consider the case when $x \in A \cap \operatorname{bdry} B$. Then (A, A') and (B, B') are locally identical around $x = P_A x$, and by Lemma 18, (61) holds. Using again the finite convergence of the DRA applied to (A', B'), we are done.

(iii): First, we show that $A' \cap \operatorname{int} B' \neq \varnothing$. Let $c \in A \cap \operatorname{bdry} B$. By the assumption on A' and B', there is $\varepsilon \in \mathbb{R}_{++}$ such that

$$A \cap \operatorname{ball}(c; \varepsilon) = A' \cap \operatorname{ball}(c; \varepsilon) \quad \text{and} \quad B \cap \operatorname{ball}(c; \varepsilon) = B' \cap \operatorname{ball}(c; \varepsilon). \tag{62}$$

Now let $d \in A \cap \operatorname{int} B$. Then $c \neq d$, and by the convexity of A and B, [6, Proposition 3.35] implies $]c, d] := \{\lambda c + (1-\lambda)d \mid 0 \leq \lambda < 1\} \subseteq A \cap \operatorname{int} B$. Therefore, $]c, d] \cap \operatorname{int} \operatorname{ball}(c; \varepsilon) \subseteq A \cap \operatorname{ball}(c; \varepsilon) = A' \cap \operatorname{ball}(c; \varepsilon) \subseteq A'$ and $]c, d] \cap \operatorname{int} \operatorname{ball}(c; \varepsilon) \subseteq \operatorname{int}(B \cap \operatorname{ball}(c; \varepsilon)) = \operatorname{int}(B' \cap \operatorname{ball}(c; \varepsilon)) \subseteq \operatorname{int} B'$. We deduce that $A' \cap \operatorname{int} B' \neq \varnothing$. By assumption, the DRA applied to (A', B') converges finitely globally. Now argue as the case where $A \cap \operatorname{int} B \neq \varnothing$ in the proof of part (ii). ∎

Corollary 21. *Let A and B be closed convex subsets of X such that $A \cap B \neq \varnothing$. Suppose that both $(A, \operatorname{aff} A)$ and $(B, \operatorname{aff} B)$ are locally identical around $A \cap \operatorname{bdry} B$ when $A \cap \operatorname{bdry} B \neq \varnothing$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges* linearly *with rate $c_F(\operatorname{aff} A - \operatorname{aff} A, \operatorname{aff} B - \operatorname{aff} B)$ to a point $x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A \cap B$, where $c_F(U, V)$ is the* cosine of the Friedrichs angle *between two subspaces U and V defined by*

$$c_F(U, V) := \sup\{|\langle u, v\rangle| \mid u \in U \cap (U \cap V)^\perp, v \in V \cap (U \cap V)^\perp, \|u\| \leq 1, \|v\| \leq 1\}. \tag{63}$$

Proof. If $A \cap \operatorname{bdry} B = \varnothing$, then by Theorem 20(i), we are done. Now assume that $A \cap \operatorname{bdry} B \neq \varnothing$. By assumption and Theorem 19,

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{\operatorname{aff} A, \operatorname{aff} B} x_n. \tag{64}$$

Since we work with a finite-dimensional space, [4, Corollary 4.5] completes the proof. ∎

Example 22. Suppose that $X = \mathbb{R}^3$, that $A = [(2, 1, 2), (-2, 1, -2)]$, and that $B = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid |\alpha| \leq 2, |\beta| \leq 2, \gamma = 1\}$. Then $A \cap B = \{(1, 1, 1)\} \in \operatorname{ri} A \cap \operatorname{ri} B$. By [38, Theorem 4.14], every DRA sequence with respect to (A, B) converges linearly. Furthermore, $\operatorname{aff} A = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha - \gamma = 0, \beta = 1\}$, $\operatorname{aff} B = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \gamma = 1\}$, $\operatorname{aff} A - \operatorname{aff} A = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha - \gamma = 0, \beta = 0\}$, $\operatorname{aff} B - \operatorname{aff} B = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \gamma = 0\}$, and both $(A, \operatorname{aff} A)$ and $(B, \operatorname{aff} B)$ are locally identical around $A \cap \operatorname{bdry} B = A \cap B$. By applying Corollary 21, the linearly rate is $c_F(\operatorname{aff} A - \operatorname{aff} A, \operatorname{aff} B - \operatorname{aff} B) = 1/\sqrt{2}$.

Proposition 23 (Finite convergence of the DRA in the (hyperplane or halfspace,ball) case). *Let A be either a hyperplane or a halfspace, and B be a closed ball of X such that $A \cap \operatorname{int} B \neq \varnothing$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges in* finitely many steps *to a point in $A \cap B$.*

Proof. If $\dim X = 0$, i.e., $X = \{0\}$, then the result is trivial, so we will work in the space $X \times \mathbb{R}$ with $\dim X \geq 0$, and denote by $(z_n)_{n\in\mathbb{N}}$ the DRA sequence. We just prove the the result for the case when A is a hyperplane because the case when A is a halfspace is similar. Without loss of generality, we assume that $A = X \times \{0\}$ and that $B = \operatorname{ball}((0, \theta); 1)$ is the closed ball of radius 1 and center $(0, \theta) \in X \times \mathbb{R}$ with $0 \leq \theta < 1$. Nothing that

$$B = \{(x, \rho) \in X \times \mathbb{R} \mid \theta - \sqrt{1 - \|x\|^2} \leq \rho \leq \theta + \sqrt{1 - \|x\|^2}\}, \tag{65}$$

we write $B = B_- \cup B_+$, where

$$B_- = \{(x,\rho) \in X \times \mathbb{R} \mid \theta - \sqrt{1-\|x\|^2} \leq \rho \leq \theta\}, \tag{66a}$$

$$B_+ = \{(x,\rho) \in X \times \mathbb{R} \mid \theta \leq \rho \leq \theta + \sqrt{1-\|x\|^2}\}. \tag{66b}$$

We distinguish two cases.

Case 1: $\theta = 0$. Then the two halves $B_- \subseteq X \times \mathbb{R}_-$ and $B_+ \subseteq X \times \mathbb{R}_+$ of the ball B are symmetric with respect to the hyperplane A. By symmetry, we can and do assume that $z_0 = (x_0, \rho_0) \in X \times \mathbb{R}_+$. Now for any $z = (x,\rho) \in X \times \mathbb{R}_+$, we have $P_A z = (x, 0)$, $R_A z = (x, -\rho)$, and by Example 5,

$$P_B R_A z = \delta(x, -\rho) \quad \text{with} \quad \delta := \frac{1}{\max\{\sqrt{\|x\|^2 + \rho^2}, 1\}} \leq 1, \tag{67}$$

which gives

$$T_{A,B} z = z - P_A z + P_B R_A z = (x,\rho) - (x,0) + \delta(x,-\rho) = (\delta x, (1-\delta)\rho) \in X \times \mathbb{R}_+. \tag{68}$$

Hence, $(\forall n \in \mathbb{N})$ $z_n \in X \times \mathbb{R}_+$. From $B_- = \{(x,\rho) \in X \times \mathbb{R} \mid -\sqrt{1-\|x\|^2} \leq \rho \leq 0\}$, we have

$$B_- \subseteq B' := \operatorname{epi} f := \{(x,\rho) \in X \times \mathbb{R} \mid f(x) \leq \rho\}, \tag{69}$$

where $f\colon X \to \mathbb{R}\colon x \mapsto -\sqrt{1-\|x\|^2}$. Since $R_A z_n \in X \times \mathbb{R}_-$, $P_B R_A z_n = P_{B_-} R_A z_n = P_{B'} R_A z_n$, and so

$$(\forall n \in \mathbb{N}) \quad z_{n+1} := T_{A,B} z_n = T_{A,B'} z_n. \tag{70}$$

According to Fact 13, $(z_n)_{n\in\mathbb{N}}$ converges finitely to a point in $A \cap B' = A \cap B$.

Case 2: $0 < \theta < 1$. Let $B' := \operatorname{epi} f$, where $f\colon X \to \mathbb{R}\colon x \mapsto \theta - \sqrt{1-\|x\|^2}$. Then $B \subseteq B'$, $A \cap B = A \cap B' = \{(x,0) \in X \times \mathbb{R} \mid \theta - \sqrt{1-\|x\|^2} \leq 0\} \subseteq A = X \times \{0\}$, and $B' \smallsetminus B = \{(x,\rho) \in X \times \mathbb{R} \mid \theta + \sqrt{1-\|x\|^2} < \rho\} \subseteq X \times \mathbb{R}_{++}$, which implies $(\forall c \in A \cap B)$ $d_{B' \smallsetminus B}(c) > 0$. Following Lemma 9(iii), B and B' are locally identical around $A \cap B$. By using Theorem 19,

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad z_{n+1} = T_{A,B'} z_n, \tag{71}$$

and again by Fact 13, we are done. ☐

Remark 24. It follows from Example 15 that the conclusion of Proposition 23 no longer holds without Slater's condition $A \cap \operatorname{int} B \neq \varnothing$.

Proposition 25. *Let $A = \bigcap_{i\in I} A_i$ and $B = \bigcap_{j\in J} B_j$ be finite intersections of closed convex sets in X such that $A \cap B \neq \varnothing$. Suppose that $(\forall x \in \operatorname{Fix} T_{A,B})(\exists i \in I)(\exists j \in J)$ both (A, A_i) and (B, B_j) are locally identical around $P_A x$. Then the following holds for any DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A,B):*

$$(\exists i \in I)(\exists j \in J)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A_i,B_j} x_n. \tag{72}$$

Proof. Since A and B are closed convex, Fact 7(i) gives $x_n \to x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A \cap B$. By assumption, $(\exists i \in I)(\exists j \in J)$ both (A, A_i) and (B, B_j) are locally identical around $P_A x$. Noting that $A \subseteq A_i$, $B \subseteq B_j$, the conclusion follows from Lemma 17. ☐

Corollary 26. *Let $A = \bigcap_{i\in I} A_i$ and $B = \bigcap_{j\in J} B_j$ be finite intersections of closed convex sets in X such that $0 \in \operatorname{int}(A - B)$. Suppose that $(\forall x \in A \cap B)(\exists i \in I)(\exists j \in J)$ both (A, A_i) and (B, B_j) are locally identical around x. Then (72) holds for any DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A,B).*

Proof. Since $0 \in \operatorname{int}(A - B)$, Fact 7(ii) implies $x_n \to x \in A \cap B$. Then $P_A x = x$, and Proposition 25 completes the proof. ☐

Corollary 27. *Let A be a hyperplane or a halfspace, and $B = \bigcap_{j\in J} B_j$ be a finite intersection of closed balls in X. Suppose that $A \cap \operatorname{int} B \neq \varnothing$, and for all $x \in A \cap \operatorname{bdry} B$, there exists a unique $j \in J$ such that $x \in \operatorname{bdry} B_j$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges* finitely *to a point in $A \cap B$.*

Proof. From $A \cap \operatorname{int} B \neq \varnothing$, we immediately have $0 \in \operatorname{int}(A - B)$. Let $x \in A \cap B$. If $x \in \operatorname{int} B$, then $(\forall j \in J)$ $x \in \operatorname{int} B_j$, and so B and B_j are locally identical around x, following Lemma 9(i). If $x \in \operatorname{bdry} B$, then by assumption, there exists a unique $j \in J$ such that $x \in \operatorname{bdry} B_j$, which implies that B and B_j are locally identical around x. Now using Corollary 26,

$$(\exists j \in J)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A,B_j} x_n. \tag{73}$$

Since $B \subseteq B_j$, we also have $A \cap \operatorname{int} B_j \neq \varnothing$, and so $(x_n)_{n\in\mathbb{N}}$ converges finitely due to Proposition 23. ◻

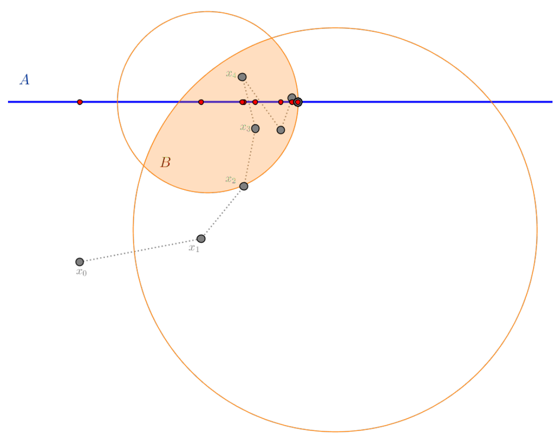

Fig. 1. *A* `GeoGebra` *snapshot that illustrates Corollary 27.*

Corollary 28. *Let A be a closed convex set, and B be a closed ball in $\mathbb{R}^2$ such that $A \cap \operatorname{int} B \neq \varnothing$. Suppose that A is locally identical with some polyhedral set around $A \cap \operatorname{bdry} B$, and that no vertex of A lies in $\operatorname{bdry} B$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges* finitely *to a point in $A \cap B$.*

Proof. By Theorem 20(i)&(iii), it is sufficient to consider the case where A is a polyhedral set in $\mathbb{R}^2$ satisfying $A \cap \operatorname{int} B \neq \varnothing$. Then $0 \in \operatorname{int}(A - B)$, and using Fact 7(ii), $x_n \to x \in A \cap B$, and this convergence is finite if $x \in A \cap \operatorname{int} B$. It thus suffices to consider the case where $x \in A \cap \operatorname{bdry} B$. We can write $A = \bigcap_{i=1}^{m} A_i$, where each A_i is a halfplane in $\mathbb{R}^2$. Since all vertices of A are not in $\operatorname{bdry} B$, we deduce that x is not a vertex of A. Hence, A and A_i are locally identical around x for some i. Now using Lemma 17,

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A_i,B} x_n. \tag{74}$$

Moreover, $A_i \cap \operatorname{int} B \neq \varnothing$, and by Proposition 23, $(x_n)_{n\in\mathbb{N}}$ converges finitely. ◻

6. Shrinking sets. In this section we focus on cases where we use information of the DRA for (A, B) to understand the DRA for (A', B') where $A' \subseteq A$ and $B' \subseteq B$.

Lemma 29 (Shrinking sets). *Let A be a closed convex subset and B be a closed (not necessarily convex) subset of X such that $A \cap B \neq \varnothing$, and let x_0 be in X. Suppose that the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B), with starting point x_0, converges to $x \in X$. Suppose further that there exist two closed sets A' and B' in X such that $A' \subseteq A$, $B' \subseteq B$, and that both (A', A) and (B', B) are locally identical around $c := P_A x \in A' \cap B'$. Then*

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} \in T_{A',B'} x_n. \tag{75}$$

Proof. By assumption, there exists $\varepsilon \in \mathbb{R}_{++}$ such that

$$A \cap \operatorname{ball}(c;\varepsilon) = A' \cap \operatorname{ball}(c;\varepsilon) \quad \text{and} \quad B \cap \operatorname{ball}(c;\varepsilon) = B' \cap \operatorname{ball}(c;\varepsilon). \tag{76}$$

Then, there is $n_0 \in \mathbb{N}$ such that

$$(\forall n \geq n_0) \quad \|x_n - x\| < \varepsilon/3. \tag{77}$$

Let $n \geq n_0$. Since P_A is (firmly) nonexpansive (Fact 1(ii)),

$$\|P_A x_n - c\| = \|P_A x_n - P_A x\| \leq \|x_n - x\| < \varepsilon/3, \tag{78}$$

which implies $P_A x_n \in \operatorname{ball}(c;\varepsilon)$. Using the convexity of A and applying Lemma 11(iii)(a) for $A' \subseteq A$, we have $P_{A'} x_n = P_A x_n$, and also $R_{A'} x_n = R_A x_n$. Noting that $x_{n+1} - x_n + P_A x_n \in P_B R_A x_n$ and

$$\|x_{n+1} - x_n + P_A x_n - c\| \leq \|x_{n+1} - x\| + \|x_n - x\| + \|P_A x_n - c\| < \varepsilon, \tag{79}$$

we get $x_{n+1} - x_n + P_A x_n \in P_B R_A x_n \cap \operatorname{ball}(c;\varepsilon)$, and then applying Lemma 11(i) for $B' \subseteq B$ yields $x_{n+1} - x_n + P_A x_n \in P_{B'} R_A x_n = P_{B'} R_{A'} x_n$. Hence, $x_{n+1} \in x_n - P_{A'} x_n + P_{B'} R_{A'} x_n = T_{A',B'} x_n$. ∎

Remark 30. If A' and B' in Lemma 29 are convex, then $T_{A',B'}$ is single-valued, and we have the conclusion that

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A',B'} x_n, \tag{80}$$

i.e., $(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}) \quad T^n_{A,B} x_{n_0} = T^n_{A',B'} x_{n_0}$.

COROLLARY 31. *Let A be a closed convex subset and $B = \bigcup_{j \in J} B_j$ be a finite union of disjoint closed convex sets in X such that $A \cap B \neq \varnothing$, and let x_0 be in X. Suppose that the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B), with starting point x_0, is bounded and* asymptotically regular, *i.e., $x_n - x_{n+1} \to 0$. Then $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in \operatorname{Fix} T_{A,B}$, and there exists $j \in J$ such that*

$$P_A x \in A \cap B_j \quad \textit{and} \quad (\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A,B_j} x_n. \tag{81}$$

Proof. According to [13, Theorem 2], $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in \operatorname{Fix} T_{A,B}$. Since A is convex, $P_A x$ is a singleton, and by (22), $P_A x \in A \cap B$. Then there exists $j \in J$ such that $P_A x \in A \cap B_j$. By assumption, there exists $\varepsilon \in \mathbb{R}_{++}$ such that $(\forall k \in J \smallsetminus \{j\})$ $B_k \cap \operatorname{ball}(P_A x;\varepsilon) = \varnothing$. This implies $B \cap \operatorname{ball}(P_A x;\varepsilon) = B_j \cap \operatorname{ball}(P_A x;\varepsilon)$, so B and B_j are locally identical around $P_A x$. Now apply Lemma 29. ∎

COROLLARY 32. *Let A be a hyperplane or a halfspace, and $B = \bigcup_{j \in J} B_j$ be a finite union of disjoint closed balls in X such that $A \cap B \neq \varnothing$, and $A \cap \operatorname{int} B_j \neq \varnothing$ whenever $A \cap B_j \neq \varnothing$. Let x_0 be in X. Suppose that the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B), with starting point x_0, is bounded and* asymptotically regular, *i.e., $x_n - x_{n+1} \to 0$. Then $(x_n)_{n\in\mathbb{N}}$ converges* finitely *to a point $x \in A \cap B$.*

Proof. Using Corollary 31, $x_n \to x \in \operatorname{Fix} T_{A,B}$, and there is $j \in J$ such that

$$P_A x \in A \cap B_j \quad \text{and} \quad (\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_{n+1} = T_{A,B_j} x_n. \tag{82}$$

Then $A \cap B_j \neq \varnothing$, and by assumption, $A \cap \operatorname{int} B_j \neq \varnothing$. Now by Proposition 23, the convergence of $(x_n)_{n\in\mathbb{N}}$ to x is finite, and $x \in A \cap B_j \subseteq A \cap B$. ∎

7. When one set is finite. If the B_j in Corollary 31 are singletons and A is either an affine subspace or a halfspace, then it is possible to obtain stronger conclusions.

THEOREM 33. *Let A be an affine subspace or a halfspace, and B be a finite subset of X such that $A \cap B \neq \varnothing$, and let x_0 be in X. Suppose that the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B), with starting point x_0, is asymptotically regular, i.e., $x_n - x_{n+1} \to 0$. Then $(x_n)_{n\in\mathbb{N}}$ converges* finitely *to a point $x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A \cap B$.*

Proof. Observe that P_A is single-valued as A is convex. According to (22), it suffices to show that $x_n \to x \in \operatorname{Fix} T_{A,B}$ finitely. Set

$$(\forall n \in \mathbb{N}) \quad b_n := x_{n+1} - x_n + P_A x_n \in P_B R_A x_n \subseteq B. \tag{83}$$

Let us first consider the case when A is an affine subspace. Then we can represent $A = \{x \in X \mid Lx = v\}$, where L is a linear operator from X to a real Hilbert space Y, and $v \in \operatorname{ran} L$. Denoting by $L^\dagger$ the Moore–Penrose inverse of L, Example 3 gives

$$(\forall n \in \mathbb{N}) \quad P_A x_n = x_n - L^\dagger(Lx_n - v), \tag{84}$$

and so

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n - P_A x_n + b_n = L^\dagger(Lx_n - v) + b_n. \tag{85}$$

Since $L^\dagger L L^\dagger = L^\dagger$ (see [30, Chapter II, Section 2]), we get

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad L^\dagger(Lx_{n+1} - v) &= L^\dagger L(L^\dagger(Lx_n - v) + b_n) - L^\dagger v && \text{(86a)}\\ &= L^\dagger(Lx_n - v) + L^\dagger(Lb_n - v), && \text{(86b)}\end{aligned}$$

and then (84) gives

$$(\forall n \in \mathbb{N}) \quad P_A x_{n+1} = x_{n+1} - L^\dagger(Lx_{n+1} - v) = -L^\dagger(Lb_n - v) + b_n. \tag{87}$$

Now in turn,

$$(\forall n \in \mathbb{N}) \quad x_{n+2} = x_{n+1} - P_A x_{n+1} + b_{n+1} = x_{n+1} + L^\dagger(Lb_n - v) - b_n + b_{n+1}. \tag{88}$$

Using the asymptotic regularity of $(x_n)_{n\in\mathbb{N}}$, (88) and (86) yield

$$\begin{aligned} L^\dagger(Lb_n - v) &= L^\dagger L(x_{n+1} - x_n) \to 0, && \text{(89a)}\\ b_{n+1} - b_n &= x_{n+2} - x_{n+1} - L^\dagger(Lb_n - v) \to 0. && \text{(89b)}\end{aligned}$$

Since $(b_n)_{n\in\mathbb{N}}$ lies in B and B is finite, there exists $n_0 \in \mathbb{N}$ such that $(\forall n \geq n_0)$ $b_{n+1} = b_n = b \in B$. Then by (89a), $L^\dagger(Lb - v) = 0$, which together with (88) gives

$$(\forall n \geq n_0) \quad x_{n+2} = x_{n+1} + L^\dagger(Lb - v) = x_{n+1}, \tag{90}$$

and $(x_n)_{n\in\mathbb{N}}$ thus converges finitely.

Now consider the case when A is a halfspace. Without loss of generality, we assume that $A = \{x \in X \mid \langle x, u\rangle \leq 0\}$, where $u \in X$ and $\|u\| = 1$. Using Example 4(ii), we have

$$(\forall n \in \mathbb{N}) \quad P_A x_n = \begin{cases} x_n & \text{if } x_n \in A,\\ x_n - \langle x_n, u\rangle\, u & \text{if } x_n \notin A,\end{cases} \tag{91}$$

and by (83),

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \begin{cases} b_n & \text{if } x_n \in A,\\ \langle x_n, u\rangle\, u + b_n & \text{if } x_n \notin A.\end{cases} \tag{92}$$

If $(\exists n \in \mathbb{N})$ $x_n \in A$ and $b_n \in A$, then (92) gives $x_{n+1} = b_n \in A \cap B$, and we are done. Assume that $(\forall n \in \mathbb{N})$ $x_n \notin A$ or $b_n \notin A$. By using (92), $(\forall n \in \mathbb{N})$ $x_n \in A \Rightarrow x_{n+1} = b_n \notin A$. Thus, the set $\{n \in \mathbb{N} \mid x_n \notin A\}$ is infinite, and denoted by $(n_k)_{k\in\mathbb{N}}$ the enumeration of that set, we have

$$(\forall k \in \mathbb{N}) \quad x_{n_k} \notin A, \text{ i.e., } \langle x_{n_k}, u\rangle > 0, \quad \text{and} \quad n_{k+1} - n_k \in \{1, 2\}. \tag{93}$$

Then $x_{n_{k+1}} - x_{n_k} = x_{n_k+1} - x_{n_k}$ or $x_{n_{k+1}} - x_{n_k} = (x_{n_k+2} - x_{n_k+1}) + (x_{n_k+1} - x_{n_k})$, and the asymptotic regularity of $(x_n)_{n\in\mathbb{N}}$ implies the one of $(x_{n_k})_{k\in\mathbb{N}}$ and also of $(x_{n_k+1})_{k\in\mathbb{N}}$. Since $x_{n_k} \notin A$, (92) gives

$$x_{n_k+1} = \langle x_{n_k}, u\rangle\, u + b_{n_k}, \tag{94}$$

and so

$$b_{n_{k+1}} - b_{n_k} = (x_{n_{k+1}+1} - x_{n_k+1}) - \big\langle x_{n_{k+1}} - x_{n_k}, u\big\rangle\, u \to 0. \tag{95}$$

But $(b_{n_k})_{k\in\mathbb{N}}$ is in the finite set B, there exists $k_0 \in \mathbb{N}$ such that

$$(\forall k \geq k_0) \quad b_{n_{k+1}} = b_{n_k} =: b \in B. \tag{96}$$

On the other hand, (94) implies

$$(\forall k \in \mathbb{N}) \quad \langle x_{n_k+1}, u\rangle = \langle x_{n_k}, u\rangle + \langle b_{n_k}, u\rangle\,, \tag{97}$$

and then

$$\langle b_{n_k}, u\rangle = \langle x_{n_k+1} - x_{n_k}, u\rangle \to 0, \tag{98}$$

which yields $\langle b, u\rangle = 0$, and thus $b \in A \cap B$. Let $k \geq k_0$. It follows from (96) and (97) that

$$\langle x_{n_k+1}, u\rangle = \langle x_{n_k}, u\rangle + \langle b, u\rangle = \langle x_{n_k}, u\rangle\,. \tag{99}$$

Hence $x_{n_k+1} \notin A$ as $x_{n_k} \notin A$. We obtain $n_{k+1} = n_k + 1$, and by combining with (94) and (96),

$$x_{n_k+2} = \langle x_{n_k+1}, u\rangle\, u + b = \langle x_{n_k}, u\rangle + b = x_{n_k+1}, \tag{100}$$

which completes the proof. ◻

The following examples illustrate that without asymptotic regularity a DRA sequence with respect to (A, B) may fail to converge.

Example 34. Suppose that $X = \mathbb{R}^2$, $A = \mathbb{R} \times \{0\}$ and $B = \{(0, -2), (1, 2), (-2, 0)\}$. Then $A \cap B \neq \varnothing$ but the DRA sequence with respect to (A, B) with starting point $x_0 = (0, -1)$ does not converge since it cycles between two points $x_0 = (0, -1)$ and $x_1 = (1, 1)$.

Example 35. Suppose that $X = \mathbb{R}^2$, that $A = \mathbb{R} \times \mathbb{R}_-$ is a halfspace, and that $B = \{(2, 5), (20, -20), (8, 7), (-20, 0)\}$ is a finite set. Then $A \cap B \neq \varnothing$ but when started at $x_0 = (2, 17)$, the DRA cycles between four points $x_0 = (2, 17)$, $x_1 = (20, -3)$, $x_2 = (8, 7)$ and $x_3 = (2, 12)$, as shown in Figure 2 which was created by `GeoGebra` [27].

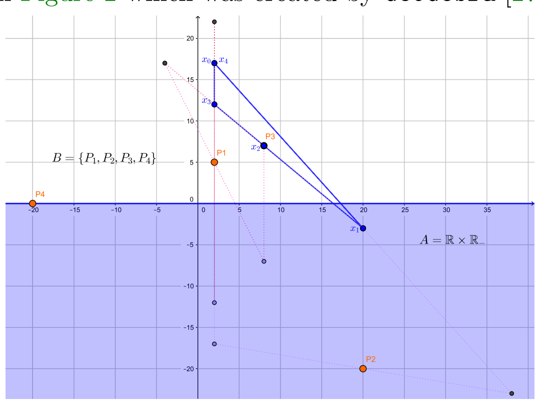

FIG. 2. *A 4-cycle of the DRA for a halfspace and a finite set.*

Remark 36 (Order matters). Notice that if A is a halfspace and B is a finite subset of X such that $A \cap B \neq \varnothing$, then every DRA sequence with respect to (B, A) converges finitely due to [2, Theorem 4.2]. Recall from [12] that if we work with an affine subspace instead of a halfspace, then the quality of convergence of the DRA sequence with respect to (A, B) is the same as the one with respect to (B, A).

THEOREM 37. *Let A be either a hyperplane or a halfspace of X, and B be a finite subset of one in two halfspaces generated by A, and let x_0 be in X. Then either: (i) the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A,B), with starting point x_0, converges* finitely *to a point $x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A\cap B$, or (ii) $A\cap B = \varnothing$ and $\|x_n\| \to +\infty$ in which case $(P_A x_n)_{n\in\mathbb{N}}$ converges* finitely *to a best approximation solution $a \in A$ relative to A and B in the sense that $d_B(a) = \min_{a'\in A} d_B(a')$.*

Proof. Case 1: A is a hyperplane. Without loss of generality, we assume that

$$A = H := \{x \in X \mid \langle x, u\rangle = 0\} \quad \text{with} \quad u \in X, \|u\| = 1, \tag{101a}$$

and that

$$(\forall b \in B) \quad \langle b, u\rangle \geq 0. \tag{101b}$$

By Example 4(i),

$$(\forall x \in X) \quad P_A x = x - \langle x, u\rangle\, u. \tag{102}$$

Therefore,

$$(\forall x \in X) \quad R_A x = 2P_A x - x = x - 2\langle x, u\rangle\, u, \tag{103}$$

and also

$$(\forall x \in X) \quad d_A(x) = \|x - P_A x\| = |\langle x, u\rangle|. \tag{104}$$

Now setting

$$(\forall n \in \mathbb{N}) \quad b_n := x_{n+1} - x_n + P_A x_n \in P_B R_A x_n \subseteq B, \tag{105}$$

we have

$$\begin{aligned}
&(\forall n \in \mathbb{N}) \quad x_{n+1} = T_{A,B} x_n = x_n - P_A x_n + P_B R_A x_n = \langle x_n, u\rangle\, u + b_n, && \text{(106a)}\\
&\langle x_{n+1}, u\rangle = \langle \langle x_n, u\rangle\, u + b_n, u\rangle = \langle x_n, u\rangle + \langle b_n, u\rangle \geq \langle x_n, u\rangle, && \text{(106b)}\\
&P_A x_{n+1} = x_{n+1} - \langle x_{n+1}, u\rangle\, u = b_n - \langle b_n, u\rangle\, u, && \text{(106c)}\\
&R_A x_{n+1} = x_{n+1} - 2\langle x_{n+1}, u\rangle\, u = b_n - (\langle x_n, u\rangle + 2\langle b_n, u\rangle)u, && \text{(106d)}
\end{aligned}$$

and so

$$(\forall n \in \mathbb{N}) \quad x_{n+2} = (\langle x_n, u\rangle + \langle b_n, u\rangle)u + b_{n+1} = x_{n+1} + \langle b_n, u\rangle\, u + b_{n+1} - b_n. \tag{107}$$

It follows that $b_n - R_A x_{n+1} = (\langle x_n, u\rangle + 2\langle b_n, u\rangle)u$, and

$$\begin{aligned}
&\|b_{n+1} - R_A x_{n+1}\|^2 = \|(b_{n+1} - b_n) + (b_n - R_A x_{n+1})\|^2 && \text{(108a)}\\
&= \|b_{n+1} - b_n\|^2 + 2(\langle x_n, u\rangle + 2\langle b_n, u\rangle)\langle b_{n+1} - b_n, u\rangle + \|b_n - R_A x_{n+1}\|^2. && \text{(108b)}
\end{aligned}$$

From $b_{n+1} = P_B R_A x_{n+1}$ and $b_n \in B$, we have $\|b_{n+1} - R_A x_{n+1}\| \leq \|b_n - R_A x_{n+1}\|$, which yields

$$\begin{aligned}
0 \leq \|b_{n+1} - b_n\|^2 &\leq 2(\langle x_n, u\rangle + 2\langle b_n, u\rangle)\langle b_n - b_{n+1}, u\rangle && \text{(109a)}\\
&= 2(\langle x_n, u\rangle + 2\langle b_n, u\rangle)(\langle b_n, u\rangle - \langle b_{n+1}, u\rangle). && \text{(109b)}
\end{aligned}$$

Case 1.1: $(\forall n \in \mathbb{N})\ \langle x_n, u\rangle \leq 0$. By combining with (106b), the sequence $(\langle x_n, u\rangle)_{n\in\mathbb{N}}$ converges, and so

$$\langle b_n, u\rangle = \langle x_{n+1}, u\rangle - \langle x_n, u\rangle \to 0. \tag{110}$$

But $(b_n)_{n\in\mathbb{N}}$ lies in the finite set B; hence, there exists $n_0 \in \mathbb{N}$ such that $(\forall n \geq n_0)\ \langle b_n, u\rangle = 0$, equivalently, $b_n \in A$. Then (109) implies $(\forall n \geq n_0)\ b_{n+1} = b_n$, and by (107), $x_{n+2} = x_{n+1} \in \operatorname{Fix} T_{A,B}$.

Case 1.2: $(\exists n_0 \in \mathbb{N})\ \langle x_{n_0}, u\rangle > 0$. Then (106b) and (101b) give

$$(\forall n \geq n_0) \quad \langle x_n, u\rangle + 2\langle b_n, u\rangle > 0. \tag{111}$$

Combining with (109), this implies

$$(\forall n \in \mathbb{N}) \quad 0 \leq \langle b_{n+1}, u\rangle \leq \langle b_n, u\rangle, \tag{112}$$

and the sequence $(\langle b_n, u\rangle)_{n\in\mathbb{N}} \subseteq B$ thus converges. Since again B is finite, there exists $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that $(\forall n \geq n_1)\ \langle b_{n+1}, u\rangle = \langle b_n, u\rangle$, which yields $b_{n+1} = b_n =: b \in B$ due to (109). By combining with (106c),

$$(\forall n \geq n_1) \quad P_A x_{n+1} = b - \langle b, u\rangle u \quad \text{and} \quad \|P_A x_{n+1} - b\| = |\langle b, u\rangle| = \langle b, u\rangle, \tag{113}$$

so $(P_A x_n)_{n\in\mathbb{N}}$ converges finitely. Furthermore, if $\langle b, u\rangle = 0$, i.e., $b \in A$, then $b \in A \cap B$, in which case $A \cap B \neq \varnothing$ and by (107), $(\forall n \geq n_1)\ x_{n+2} = x_{n+1} \in \operatorname{Fix} T_{A,B}$.

Now assume that $\langle b, u\rangle \neq 0$. Then $\langle b, u\rangle > 0$ due to (101b). It follows from (106b) and (106d) that

$$(\forall n \geq n_1) \quad R_A x_{n+1} = b - (\langle x_{n_1}, u\rangle + (n - n_1 + 2)\langle b, u\rangle)u. \tag{114}$$

Let $n \geq n_1$, and let $b' \in B$. Since $b = b_{n+1} = P_B R_A x_{n+1}$, we have $\|b - R_A x_{n+1}\| \leq \|b' - R_A x_{n+1}\|$, and so

$$\|b - R_A x_{n+1}\|^2 \leq \|b' - b\|^2 + 2\langle b' - b, b - R_A x_{n+1}\rangle + \|b - R_A x_{n+1}\|^2, \tag{115}$$

which implies

$$\|b' - b\|^2 \geq 2\langle b - b', (\langle x_{n_1}, u\rangle + (n - n_1 + 2)\langle b, u\rangle)u\rangle \tag{116a}$$

$$= 2(\langle x_{n_1}, u\rangle + (n - n_1 + 2)\langle b, u\rangle)(\langle b, u\rangle - \langle b', u\rangle). \tag{116b}$$

Noting that $\langle x_{n_1}, u\rangle + (n - n_1 + 2)\langle b, u\rangle \to +\infty$, we deduce $\langle b, u\rangle \leq \langle b', u\rangle$. Hence

$$0 < \langle b, u\rangle = \min_{b'\in B} \langle b', u\rangle = \min_{b'\in B} d_A(b'). \tag{117}$$

This yields $A \cap B = \varnothing$, and by (106b),

$$\|x_n\| \geq \langle x_n, u\rangle = \langle x_{n_1}, u\rangle + (n - n_1)\langle b, u\rangle \to +\infty \quad \text{as} \quad n \to +\infty, \tag{118}$$

while by (113), $(\forall n \geq n_1)\ (P_A x_{n+1}, b)$ is a best approximation pair relative to A and B.

Case 2: A is a halfspace. By assumption, we assume without loss of generality that either

$$A = H_+ := \{x \in X \mid \langle x, u\rangle \geq 0\} \quad \text{and} \quad B \subseteq H_+, \tag{119a}$$

or

$$A = H_- := \{x \in X \mid \langle x, u\rangle \leq 0\} \quad \text{and} \quad B \subseteq H_+, \tag{119b}$$

where $u \in X$ and $\|u\| = 1$.

Case 2.1: (119a) holds. If $(\forall n \in \mathbb{N})\ \langle x_n, u\rangle \leq 0$, i.e. $x_n \in H_-$, then $P_A x_n = P_H x_n$, so

$$x_{n+1} = T_{A,B} x_n = T_{H,B} x_n, \tag{120}$$

and according to *Case 1.1*, we must have $H \cap B \neq \varnothing$ and the finite convergence of $(x_n)_{n\in\mathbb{N}}$. If $(\exists n_0 \in \mathbb{N})\ \langle x_{n_0}, u\rangle \geq 0$, i.e. $x_{n_0} \in H_+$, then $R_A x_{n_0} = P_A x_{n_0} = x_{n_0}$, which yields $x_{n_0+1} = x_{n_0} - P_A x_{n_0} + P_B R_A x_{n_0} = P_B x_{n_0} \in B = A \cap B$, and we are done.

Case 2.2: (119b) holds. If $\langle x_0, u\rangle \leq 0$, i.e. $x_0 \in H_-$, then $R_A x_0 = P_A x_0 = x_0$, and thus $x_1 = x_0 - P_A x_0 + P_B R_A x_0 = P_B x_0 \in B \subseteq H_+$. It is therefore sufficient to consider $\langle x_0, u\rangle \geq 0$, i.e. $x_0 \in H_+$. Then $P_A x_0 = P_H x_0$, $x_1 = T_{A,B} x_0 = T_{H,B} x_0$, and by (106b), $\langle x_1, u\rangle \geq \langle x_0, u\rangle \geq 0$. This yields

$$(\forall n \in \mathbb{N}) \quad x_n \in H_+ \quad \text{and} \quad x_{n+1} = T_{H,B} x_n. \tag{121}$$

Now apply Case 1. ☐

Example 38. Suppose that $X = \mathbb{R}^2$, $A = \mathbb{R}\times\{0\}$ and $B = \{(0,1),(1,2)\}$. Then $A\cap B = \varnothing$, and for starting point $x_0 \in\,]1,+\infty[\, \times \{-1\}$, the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A,B) satisfies $(\forall n \in \{2,3,\dots\})$ $x_n = (0,n)$ and $P_A x_n = (0,0)$. See Figure 3 for an illustration, created with `GeoGebra` [27].

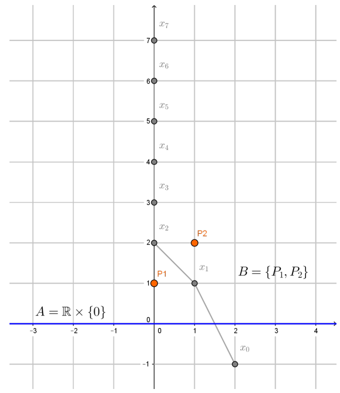

FIG. 3. *An illustration for Example 38 with the starting point* $x_0 = (2,-1)$.

8. When A is an affine subspace and B is a polyhedron. In view of Definition 8, we recall a result on finite convergence of the Douglas–Rachford algorithm under Slater's condition.

FACT 39 (Finite convergence of DRA in the affine-polyhedral case). *Let A be an affine subspace and B be a closed convex subset of X such that* Slater's condition

$$A \cap \operatorname{int} B \neq \varnothing \tag{122}$$

holds. Suppose that B is locally identical with some polyhedral set around $A\cap \operatorname{bdry} B$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A,B) converges finitely *to a point in $A\cap B$.*

Proof. Combine [9, Theorem 3.7 and Definition 2.7] with Definition 8. ◻

A natural question is whether the conclusion of Fact 39 holds when the Slater's condition $A\cap \operatorname{int} B \neq \varnothing$ is replaced by $A\cap B \neq \varnothing$ and $\operatorname{int} B \neq \varnothing$. In the sequel, we shall provide a positive answer in $\mathbb{R}^2$ (Theorem 45) and a negative answer in $\mathbb{R}^3$ (Example 46). For the next little while, we work with

$$X = \mathbb{R}^2 \quad \text{and} \quad A = \mathbb{R}\times\{0\}, \tag{123}$$

and consider the (counter-clockwise) rotator defined by

$$(\forall \theta\in\mathbb{R})\quad \mathcal{R}_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{124}$$

Let $\theta \in [0,\pi]$, and set

$$e_0 := (1,0), \quad e_{\pi/2} := (0,1), \quad e_\theta := (\cos\theta)e_0 + (\sin\theta)e_{\pi/2}. \tag{125}$$

Then $\mathbb{R}_+\times\{0\} = \mathbb{R}_+\cdot e_0$ is the positive x-axis, and $\mathcal{R}_\theta(\mathbb{R}_+\times\{0\}) = \mathbb{R}_+\cdot e_\theta$ is the ray starting at $0\in X$ and making an angle of θ with respect to $\mathbb{R}_+\times\{0\}$ in counter-clockwise direction.

For $x,y\in X$, we write $\angle(x,y) := \theta$ if $y\in\mathbb{R}_+\mathcal{R}_\theta(x)$, and $\angle(x,y) = \theta-\pi$ if $y \in \mathbb{R}_-\mathcal{R}_\theta(x)$.

FACT 40. *Let $\theta\in[0,\pi]$. Then*

$$T_{A,\mathcal{R}_\theta(A)} = \operatorname{Id} - P_A + P_{\mathcal{R}_\theta(A)}R_A = (\cos\theta)\mathcal{R}_\theta. \tag{126}$$

Proof. This follows from [4, Section 5]. ◻

LEMMA 41. *Assume that $\theta\in[0,\pi]$, $B = \mathcal{R}_\theta(\mathbb{R}_+\times\{0\})$, $H = B^\oplus$, and $H' = R_A(H)$. Let $x = (\alpha,\beta)\in X$, and set $x_+ = T_{A,B}x$. Then $x_+ = (0,\beta)$ if $x\notin H'$, and $x_+ = (\cos\theta)\mathcal{R}_\theta(z)$ otherwise. In the latter case, $x_+ = 0$ if $\theta = \pi/2$, and*

$$\angle(x,x_+) = \begin{cases} \theta, & \text{if } \theta < \pi/2; \\ \theta - \pi, & \text{if } \theta > \pi/2. \end{cases} \tag{127}$$

Furthermore,

$$\operatorname{Fix} T_{A,B} = \begin{cases} \mathbb{R}_+ \times \mathbb{R}, & \text{if } \theta = 0; \\ \{0\} \times \mathbb{R}_+, & \text{if } 0 < \theta < \pi; \\ \mathbb{R}_- \times \mathbb{R}, & \text{if } \theta = \pi. \end{cases} \tag{128}$$

Proof. We have $P_A x = (\alpha, 0)$ and $R_A x = (\alpha, -\beta)$. If $x = (\alpha, \beta) \notin H'$, then $R_A x \notin H$, and so $P_B R_A x = (0, 0)$, which yields

$$x_+ = (\mathrm{Id} - P_A + P_B R_A)x = (\alpha, \beta) - (\alpha, 0) + (0, 0) = (0, \beta). \tag{129}$$

Now we consider the case $x \in H'$. Then $R_A x \in H$, so $P_B R_A x = P_{\mathcal{R}_\theta(A)} R_A x$, and by applying Fact 40,

$$x_+ = (\cos\theta)\mathcal{R}_\theta(x). \tag{130}$$

The rest is clear. ☐

Lemma 42. *Let*

$$A = \mathbb{R} \times \{0\} \quad \text{and} \quad B = \mathcal{R}_\theta(\mathbb{R}_+ \times \{0\}), \tag{131}$$

where $\theta \in [0, \pi]$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges to a point $x \in \operatorname{Fix} T_{A,B}$, and the "shadow sequence" $(P_A x_n)_{n\in\mathbb{N}}$ converges to $P_A x \in A \cap B$ in at most N iterations, where

$$N = \begin{cases} \lfloor \frac{\pi}{\theta} \rfloor + 3, & \text{if } \theta \le \pi/2; \\ \lfloor \frac{\pi}{\pi - \theta} \rfloor + 3, & \text{if } \theta > \pi/2. \end{cases} \tag{132}$$

Proof. Set $H = B^\oplus$, and $H' = R_A(H)$. We will study the behavior of the iterations in regions

$$R_1 = \{(\alpha, \beta) \in X \mid (\alpha, \beta) \notin H',\ \beta < 0\}, \tag{133a}$$
$$R_2 = H', \tag{133b}$$
$$R_3 = \{(\alpha, \beta) \in X \mid (\alpha, \beta) \notin H',\ \beta \ge 0\} \tag{133c}$$

as shown in Figure 4.

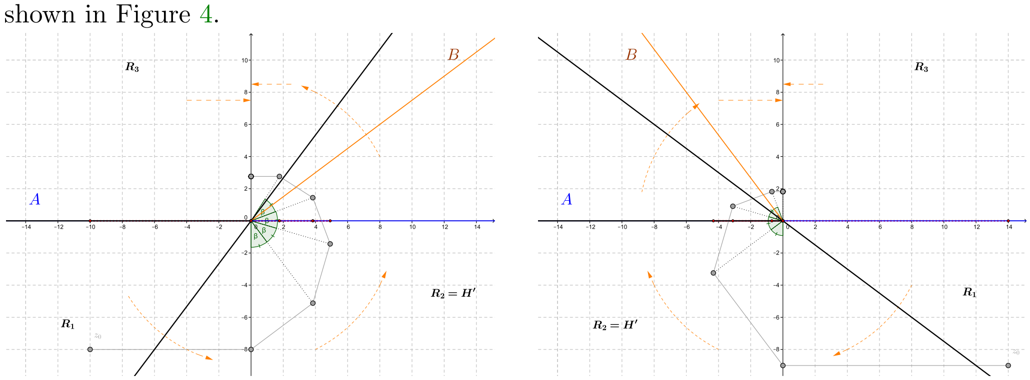

Fig. 4. *The DRA for the case of a line and a ray in the Euclidean plane*

Since $\theta \in [0, \pi]$, we have $0 \times \mathbb{R}_+ \subseteq H$, and so $\{0\} \times \mathbb{R}_- \subseteq H'$. Set $x_0 := (\alpha_0, \beta_0) \in X$. According to Lemma 41, if $x_0 \in R_1$, then $x_1 = (0, \beta_0) \in \{0\} \times \mathbb{R}_- \subseteq H'$; if $x_0 \in R_3$, then $x_1 = (0, \beta_0) \in 0 \times \mathbb{R}_+ \subseteq \operatorname{Fix} T_{A,B}$. So it is sufficient to consider the case $x_0 \in H' = R_2$. If $\theta = \pi/2$, we have immediately $x_1 = 0 \in A \cap B$. Now we assume without loss of generality that $\theta < \pi/2$. Then, (127) yields the implication

$$x_0, \ldots, x_{n-1} \in R_2 \quad \Rightarrow \quad \angle(x_0, x_n) = n\theta. \tag{134}$$

There thus exists $n_0 \in \mathbb{N}$ such that

$$x_0, \dots, x_{n_0-1} \in R_2, \quad \text{and} \quad x_{n_0} \notin R_2, \tag{135}$$

which yields $x_{n_0} \in R_3$. Using again Lemma 41, $x_{n_0+1} = (0, \beta_{n_0}) \in 0 \times \mathbb{R}_+ \subseteq \operatorname{Fix} T$. Noting that

$$\angle(x_0, x_{n_0}) = n_0\theta \leq \pi + \theta, \tag{136}$$

we get $n_0 \leq \lfloor \pi/\theta \rfloor + 1$. Hence, $x_n = x \in \operatorname{Fix} T_{A,B}$ and $P_A x_n = P_A x \in A \cap B$ for all $n \geq \lfloor \pi/\theta \rfloor + 3$ iterations. ∎

LEMMA 43. *Let either $A = \mathbb{R} \times \{0\}$ or $A = \mathbb{R} \times \mathbb{R}_-$, and let B be the convex cone generated by the union of the rays*

$$B_1 = \mathcal{R}_{\theta_1}(\mathbb{R}_+ \times \{0\}) \quad \textit{and} \quad B_2 = \mathcal{R}_{\theta_2}(\mathbb{R}_+ \times \{0\}) \tag{137}$$

with $\theta_1, \theta_2 \in [0, \pi]$. Then the DRA applied to (A, B) converges finitely globally uniformly *in the sense that there exists $N \in \mathbb{N}$ such that $(\forall x \in X)$ the sequence $(T_{A,B}^n x)_{n \in \mathbb{N}}$ converges to a point in $\operatorname{Fix} T_{A,B}$ in at most N iterations.*

Proof. We shall prove this for the case $A = \mathbb{R} \times \{0\}$, the other case being similar. For $i \in \{1, 2\}$, set $H_i = B_i^{\oplus}$, $H_i' = R_A(H_i)$, $B_i' = R_A(B_i)$, and let $B_1'' = \mathcal{R}_{\pi/2}(B_1')$, $B_2'' = \mathcal{R}_{\pi/2}^{-1}(B_2')$. Without loss of generality, we distinguish two cases: $0 \leq \theta_1 < \pi/2 < \theta_2 \leq \pi$ or $0 \leq \theta_1 < \theta_2 \leq \pi/2$.

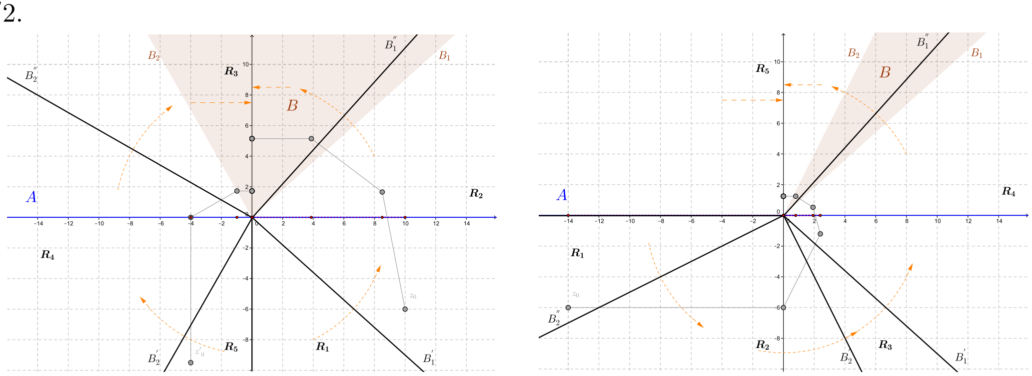

FIG. 5. *The DRA for the case of a line and a cone in the Euclidean plane*

Case 1: $0 \leq \theta_1 < \pi/2 < \theta_2 \leq \pi$. As shown in the left image in Figure 4, we study the behavior of the iterations in regions

$$R_1 = \operatorname{cone}(\{0\} \times \mathbb{R}_- \cup B_1') = R_A(B) \cap (\mathbb{R}_+ \times \mathbb{R}), \tag{138a}$$
$$R_2 = \operatorname{cone}(B_1' \cup B_1'') \subseteq H_1' \smallsetminus R_A(B), \tag{138b}$$
$$R_3 = \operatorname{cone}(B_1'' \cup B_2''), \tag{138c}$$
$$R_4 = \operatorname{cone}(B_2'' \cup B_2') \subseteq H_2' \smallsetminus R_A(B), \tag{138d}$$
$$R_5 = \operatorname{cone}(B_2' \cup \{0\} \times \mathbb{R}_-) = R_A(B) \cap (\mathbb{R}_- \times \mathbb{R}). \tag{138e}$$

Set $x_0 := (\alpha_0, \beta_0)$.

Case 1.1: $x_0 \in R_1 \cup R_5$. Then $P_A x_0 = (\alpha_0, 0)$, and $R_A x_0 = (\alpha_0, -\beta_0) \in B = R_A(R_1 \cup R_5)$, so

$$x_1 = (\operatorname{Id} - P_A + P_B R_A)x_0 = (\alpha_0, \beta_0) - (\alpha_0, 0) + (\alpha_0, -\beta_0) = (\alpha_0, 0) \in R_2 \cup R_4. \tag{139}$$

Case 1.2: $x_0 \in R_2$. Then $x_0 \in H_1' \smallsetminus R_A(B)$, and $R_A x_0 \in H_1 \smallsetminus B$. We also see that $R_A x_0$ belongs to the halfspace with boundary $\operatorname{span} B_1$ and not containing B_2. Thus, $P_B R_A x_0 = P_{B_1} R_A x_0$, and

$$x_1 = T_{A,B} x_0 = T_{A,B_1} x_0. \tag{140}$$

Using Lemma 41, this implies

$$x_0, \dots, x_{n-1} \in R_2 \quad \Rightarrow \quad \angle(x_0, x_n) = n\theta_1. \tag{141}$$

Therefore, as in the proof of Lemma 42, there exists $n_0 \in \mathbb{N}$, $n_0 \le \lfloor \pi/(2\theta_1) \rfloor + 1$ such that $x_{n_0} \in R_3$.

Case 1.3: $x_0 \in R_4$. By an argument similar to the above, we have $x_{n_0} \in R_3$ for some $n_0 \in \mathbb{N}$, $n_0 \le \lfloor \pi/(2\pi - 2\theta_2) \rfloor + 1$.

Case 1.4: $x_0 = (\alpha_0, \beta_0) \in R_3$. Then $\beta_0 \ge 0$ and $R_A x_0 \notin H_1 \cup H_2$ since $R_3 \not\subseteq H_1' \cup H_2'$. Therefore, $P_B R_A x_0 = (0,0)$, and

$$x_1 = (\alpha_0, \beta_0) - (\alpha_0, 0) + (0,0) = (0, \beta_0) \in \{0\} \times \mathbb{R}_+ \subseteq \operatorname{Fix} T_{A,B}. \tag{142}$$

Hence, in all cases, there exists $n_1 \in \mathbb{N}$ such that

$$n_1 \le N := \max\left\{ \left\lfloor \frac{\pi}{2\theta_1} \right\rfloor, \left\lfloor \frac{\pi}{2(\pi - \theta_2)} \right\rfloor \right\} + 3, \tag{143}$$

and $x_{n_1} \in \operatorname{Fix} T_{A,B}$. This shows that $x_n \to x_{n_1} \in \operatorname{Fix} T_{A,B}$ in at most N iterations.

Case 2: $0 \le \theta_1 < \theta_2 \le \pi/2$. Partitioning

$$R_1 = \operatorname{cone}(B_1'' \cup B_2'') \cap (\mathbb{R} \times \mathbb{R}_-), \tag{144a}$$
$$R_2 = \operatorname{cone}(B_2'' \cup B_2') \subseteq H_2' \smallsetminus R_A(B), \tag{144b}$$
$$R_3 = \operatorname{cone}(B_2' \cup B_1') = R_A(B), \tag{144c}$$
$$R_4 = \operatorname{cone}(B_1' \cup B_1'') \subseteq H_1' \smallsetminus R_A(B), \tag{144d}$$
$$R_5 = \operatorname{cone}(B_1'' \cup B_2'') \cap (\mathbb{R} \times \mathbb{R}_+) \tag{144e}$$

(see the right image in Figure 5) and arguing as in the above case, we obtain that $x_n \to x \in \operatorname{Fix} T_{A,B}$ in at most N iterations, where

$$N := \left\lfloor \frac{\pi}{2\theta_1} \right\rfloor + \left\lfloor \frac{\pi}{2\theta_2} \right\rfloor + 5. \tag{145}$$

The proof is complete. ∎

Remark 44. By the same argument, Lemma 43 also remains true when $\theta_1, \theta_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

THEOREM 45. *Let A be either a line or a halfplane, and B be a closed convex set in the Euclidean plane $\mathbb{R}^2$. Suppose that $A \cap B \neq \varnothing$, and that B is locally identical with some polyhedral set around $A \cap \operatorname{bdry} B$. Then every DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges* finitely *to a point $x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A \cap B$.*

Proof. Using Theorem 20, it suffices to prove for the case where B is a polyhedral set in $\mathbb{R}^2$ satisfying $A \cap B \neq \varnothing$. Then $B = \bigcap_{j\in J} B_j$ is a finite intersection of halfplanes B_j. Now by Fact 7(i), $x_n \to x \in \operatorname{Fix} T_{A,B}$ with $P_A x \in A \cap B = A \cap (\bigcap_{j\in J} B_j)$.

Case 1: $P_A x$ is not a vertex of B. Then there exists $j \in J$ such that B and B_j are locally identical around $P_A x$. Applying Lemma 17 for $A' = A$ and $B' = B_j$, we have

$$(\exists n_0 \in \mathbb{N})(\forall n \ge n_0) \quad x_{n+1} = T_{A,B_j} x_n. \tag{146}$$

Since A is either a line or a halfplane, and B_j is a halfplane in $\mathbb{R}^2$, Theorem 16 implies that $x_n \to x$ finitely.

Case 2: $P_A x$ is a vertex of B. Noting that there are exactly two of halfplanes B_j through each vertex of B, it can also represent $B = \bigcap_{j\in J} C_j$, where each C_j is a closed convex cone in $\mathbb{R}^2$. We then find $j \in J$ such that B and C_j are locally identical around $P_A x$. By using again Lemma 17,

$$(\exists n_0 \in \mathbb{N})(\forall n \ge n_0) \quad x_{n+1} = T_{A,C_j} x_n. \tag{147}$$

Here A is either a line or a halfplane through vertex $P_A x$ of the cone C_j. Now apply Lemma 43 and Remark 44. ∎

Example 46. Suppose that $X = \mathbb{R}^3$, that $A = \{x \in X \mid Lx = a\}$, and that $B = \mathbb{R}^3_+$, where

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{148}$$

Then for starting point $x_0 = (1/3, 2/3, 1/3) \in X$, the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (A, B) converges $x_\infty = (1/3, 1, 1/3)$ with $P_A x_\infty = (0, 1, 0) \in A \cap B$, but this convergence is not finite.

Proof. It is easy to see that $A = \{(-\lambda, \lambda + 1, \lambda) \mid \lambda \in \mathbb{R}\}$, and so

$$A \cap B = \{(0, 1, 0)\}. \tag{149}$$

Let $x = (\alpha, \beta, \gamma) \in X$. Noting that the Moore–Penrose inverse of L is given by

$$L^\dagger = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}, \tag{150}$$

we learn from Example 3 that $P_A x = x - L^\dagger(Lx - a)$, and so

$$R_A x = 2P_A x - x = x - 2L^\dagger(Lx - a) = \frac{1}{3}\left(\begin{bmatrix} -1 & -2 & -2 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}\right). \tag{151}$$

By, e.g., [6, Example 6.28], $P_B x = (\max\{\alpha, 0\}, \max\{\beta, 0\}, \max\{\gamma, 0\})$, and thus

$$R_B x = (|\alpha|, |\beta|, |\gamma|). \tag{152}$$

Setting $x_+ := (\alpha_+, \beta_+, \gamma_+) = T_{A,B}x$, we claim that if

$$\frac{2}{3} \le \alpha + \gamma, \tag{153a}$$
$$-\frac{2}{3} \le \alpha - \gamma \le \frac{2}{3}, \tag{153b}$$
$$\frac{2}{3} \le \beta \le \frac{4}{3}, \tag{153c}$$

then $x_+ = \frac{1}{3}(Mx + b)$, where

$$M := \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \tag{154}$$

and (153) also holds for α_+, β_+ and γ_+. Indeed, recall that

$$R_A x = \frac{1}{3}(-\alpha - 2\beta - 2\gamma + 2, -2\alpha - \beta + 2\gamma + 4, -2\alpha + 2\beta - \gamma - 2). \tag{155}$$

It follows from (153) that $\alpha \ge 0$, $\gamma \ge 0$, and

$$-\alpha - 2\beta - 2\gamma + 2 \le -(\alpha + \gamma) - 2\beta + 2 \le -\frac{2}{3} - 2\cdot\frac{2}{3} + 2 = 0, \tag{156a}$$
$$-2\alpha - \beta + 2\gamma + 4 = -2(\alpha - \gamma) - \beta + 4 \ge -2\cdot\frac{2}{3} - \frac{4}{3} + 4 = \frac{4}{3} > 0, \tag{156b}$$
$$-2\alpha + 2\beta - \gamma - 2 \le -(\alpha + \gamma) + 2\beta - 2 \le -\frac{2}{3} + 2\cdot\frac{4}{3} - 2 = 0. \tag{156c}$$

By (152) and a direct computation,

$$x_+ = \frac{1}{2}(x + R_B R_A x) = \frac{1}{3}(Mx + b), \tag{157}$$

which means

$$\alpha_+ = \frac{1}{3}(2\alpha + \beta + \gamma - 1), \tag{158a}$$

$$\beta_+ = \frac{1}{3}(-\alpha + \beta + \gamma + 2), \tag{158b}$$

$$\gamma_+ = \frac{1}{3}(\alpha - \beta + 2\gamma + 1). \tag{158c}$$

Using again (153) we get

$$\alpha_+ + \gamma_+ = \alpha + \gamma \geq \frac{2}{3}, \tag{159a}$$

$$-\frac{2}{3} < -\frac{4}{9} \leq \alpha_+ - \gamma_+ = \frac{1}{3}((\alpha - \gamma) + 2\beta - 2) \leq \frac{4}{9} < \frac{2}{3}, \tag{159b}$$

$$\frac{2}{3} \leq \beta_+ = \frac{1}{3}(-(\alpha - \gamma) + \beta + 2) \leq \frac{4}{3}, \tag{159c}$$

as claimed. Now let $x_0 = (1/3, 2/3, 1/3)$, the above claim implies that

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = T_{A,B}x_n = \frac{1}{3}(Mx_n + b). \tag{160}$$

A direct argument yields

$$(\forall n \in \mathbb{N}) \quad x_{n+3} = \frac{5}{3}x_{n+2} - x_{n+1} + \frac{1}{3}x_n, \tag{161}$$

and then

$$x_n = \left(\frac{1}{3} - \frac{\sqrt{2}}{2}\frac{\sin(n\arctan\sqrt{2})}{3^{\frac{n}{2}+1}}, 1 - \frac{\cos(n\arctan\sqrt{2})}{3^{\frac{n}{2}+1}}, \frac{1}{3} + \frac{\sqrt{2}}{2}\frac{\sin(n\arctan\sqrt{2})}{3^{\frac{n}{2}+1}}\right). \tag{162}$$

Therefore, $x_n \to x_\infty = (1/3, 1, 1/3)$ linearly with rate $1/\sqrt{3}$, but not finitely. ∎

9. Open problems.

We conclude with a list of specific open problems.

P1 Do the conclusions of Fact 13 and Theorem 14 hold when A is any hyperplane or halfspace?

P2 Does Proposition 23 remain true if A is an affine subspace or a polyhedron?

P3 Does Corollary 27 remain true without assumption on the uniqueness?

P4 Do Corollary 28 and Theorem 45 remain true in $\mathbb{R}^n$ with $n > 2$?

P5 Does Fact 39 remain true if we replace "affine subspace" by "halfspace"?

P6 What can be said about convergence of the DRA for two polyhedrons or for two balls?

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