INTRIGUING MAXIMALLY MONOTONE OPERATORS DERIVED FROM NONSUNNY NONEXPANSIVE RETRACTIONS

HEINZ H. BAUSCHKE[1,*], LEVI MILLER, WALAA M. MOURSI[2,3]

[1]*Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada*
[2]*Electrical Engineering, Stanford University, Stanford, CA 94305, USA*
[3]*Mansoura University, Faculty of Science, Mathematics Department, Mansoura 35516, Egypt*

Abstract. Monotone operator theory and fixed point theory for nonexpansive mappings are central areas in modern nonlinear analysis and optimization. Although these areas are fairly well developed, almost all examples published are based on subdifferential operators, linear relations, or combinations thereof.

In this paper, we construct an intriguing maximally monotone operator induced by a certain nonexpansive retraction. We analyze this operator, which does not appear to be assembled from subdifferential operators or linear relations, in some detail. Particular emphasis is placed on duality and strong monotonicity.

Keywords. Firmly nonexpansive mapping; Maximally monotone operator; Nonexpansive mapping; Paramonotone operator; Resolvent; Resolvent average; Retraction; Strongly monotone operator.

2010 Mathematics Subject Classification. 47H05, 47H09, 90C25.

1. Introduction

Suppose that

$$X \text{ is a real Hilbert space,} \tag{1.1}$$

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Throughout, we assume that $X \neq \{0\}$ and that

$$e \in X \text{ and } \|e\| \leq 1. \tag{1.2}$$

Now define

$$R_e \colon X \to X \colon x \mapsto \|x\| \cdot e. \tag{1.3}$$

*Corresponding author.

E-mail addresses: heinz.bauschke@ubc.ca (H.H. Bauschke), levi@levimiller.ca (L. Miller), wmoursi@stanford.edu (W.M. Moursi).

Received May 23, 2018; Accepted June 5, 2018.

The mapping R_e is nonexpansive and thus induces an associated firmly nonexpansive mapping T_e as well as a maximally monotone operator A_e.

The goal of this paper is to present fundamental properties of R_e, T_e, and A_e. These operators are neither subdifferential operators nor linear relations; consequently, they provide a new testing ground for properties in monotone operator theory and fixed point theory.

The paper is organized as follows. In Section 2, we focus on R_e, the induced firmly nonexpansive mapping T_e and the maximally monotone operator A_e. Duality and stronger notions of monotonicity are considered in Section 3. We conclude the paper in Section 4 with a discussion on the resolvent iteration and the resolvent average.

The notation employed is standard and follows, e.g., [2]. Finally, we assume the reader is familiar with basic monotone operator theory and fixed point theory, as can be found in e.g., [2], [4], [5], [6], [7], [11], [12], [13], [14], [15], [16], or [17].

2. R_e, T_e, AND A_e

We start by collecting some properties of R_e. Item (i) of the following result states that $\mathbb{R}_+ \cdot e$ is a nonexpansive retract of X, via R_e. (See [9] for more on nonexpansive retracts.)

Proposition 2.1.

(i) *If $\|e\| = 1$, then R_e is a nonexpansive retraction of $\mathbb{R}_+ \cdot e$ and R_e is not a Banach contraction.*
(ii) *If $\|e\| < 1$, then R_e is a Banach contraction with optimal Lipschitz constant $\|e\|$ and* $\operatorname{Fix} R_e = \{0\}$.
(iii) *If $0 < \|e\| \leq 1$ and $f \in \{e\}^\perp$ with $\|f\| = 1$, then R_e is not sunny.*
(iv) *R_e is nonexpansive.*

Proof. We have

$$(\forall x \in X)(\forall y \in X) \quad \|R_e x - R_e y\| = \big\|\|x\|e - \|y\|e\big\| = \big|\|x\| - \|y\|\big| \cdot \|e\| \leq \|e\| \cdot \|x - y\|, \tag{2.1}$$

which shows that R_e is Lipschitz continuous with constant $\|e\|$; moreover,

$$(\forall x \in \mathbb{R}_+ \cdot e)(\forall y \in \mathbb{R}_+ \cdot e) \quad \|R_e x - R_e y\| = \|e\| \cdot \|x - y\|. \tag{2.2}$$

It is clear that $\operatorname{Fix} R_e \subseteq \operatorname{ran} R_e = \mathbb{R}_+ e$. If $x \in \mathbb{R}_+ \cdot e$, say $x = \rho e$, where $\rho \in \mathbb{R}_+$, then $R_e x = \|x\|e = \|\rho e\|e = \rho\|e\|e = \|e\|x$. Hence

$$(\forall x \in X) \quad x = R_e x \Leftrightarrow [x = 0 \text{ or } \|e\| = 1]. \tag{2.3}$$

(i): In view of (2.1) and the assumption that $\|e\| = 1$, it is clear that R_e is nonexpansive. From (2.3) we deduce that $\operatorname{Fix} R_e = \mathbb{R}_+ \cdot e$; thus, R_e is a nonexpansive retract of $\mathbb{R}_+ \cdot e$.

(ii): Combine (2.1) with (2.3).

(iii): Set $x = \|e\|^{-1}e + \sqrt{3}f$. Then $\|x\|^2 = 1+3 = 4$ and so $R_e x = \|x\|e = 2e$. Now consider $y = (x+R_e x)/2 = 2^{-1}(2+\|e\|^{-1})e + 2^{-1}\sqrt{3}f \in [x, R_e x]$. Then

$$\|y\|^2 = \big(2^{-1}(2+\|e\|^{-1})\big)^2\|e\|^2 + \big(2^{-1}\sqrt{3}\big)^2 = 1+\|e\|+\|e\|^2 \le 3 < 4 = \|x\|^2. \tag{2.4}$$

We deduce that $R_e x = \|x\|e \neq \|y\|e = R_e y = \|y\|e$ and thus R_e is not sunny.

(iv): Combine (i) and (ii). ■

Proposition 2.1(iv) shows that

$$T_e \colon X \to X \colon x \mapsto \tfrac{1}{2}x + \tfrac{1}{2}R_e x = \tfrac{1}{2}x + \tfrac{1}{2}\|x\|e \tag{2.5}$$

is firmly nonexpansive and hence the resolvent $J_{A_e} = (\mathrm{Id} + A_e)^{-1}$ of the maximally monotone operator

$$A_e = T_e^{-1} - \mathrm{Id}\,. \tag{2.6}$$

Our next task is to provide an explicit formula for A_e. It turns out that A_e behaves quite differently, depending on whether $\|e\| = 1$ or $\|e\| < 1$.

Theorem 2.2 (A_e **for** $\|e\| = 1$)**.** *Suppose that $\|e\| = 1$. Then the maximally monotone operator A_e is given by*

$$(\forall x \in X) \quad A_e x = \begin{cases} x - \frac{\|x\|^2}{\langle e,x\rangle}\cdot e, & \text{if } \langle e,x\rangle > 0;\\ \mathbb{R}_- \cdot e, & \text{if } x = 0;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{2.7}$$

Proof. Denote the right-hand side of (2.7) by B, i.e., set

$$(\forall x \in X) \quad Bx = \begin{cases} x - \frac{\|x\|^2}{\langle e,x\rangle}\cdot e, & \text{if } \langle e,x\rangle > 0;\\ \mathbb{R}_- \cdot e, & \text{if } x = 0;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{2.8}$$

Our job is to show that $B = A_e$, and for that it suffices to show that $\operatorname{gra} B = \operatorname{gra} A_e = \big\{(T_e x, x - T_e x) \;\big|\; x \in X\big\}$ by the Minty parametrization [10]. For convenience, we also write T instead of T_e.

First, let $x \in X$. We need to consider two cases.

Case 1: $\langle e,x\rangle > 0$.
Set $u = x - \|x\|^2\langle e,x\rangle^{-1} e \in Bx$ and $y = x+u = 2x - \|x\|^2\langle e,x\rangle^{-1}e$. Then

$$\|u\|^2 = \|x\|^2 + \|x\|^4\langle e,x\rangle^{-2}\|e\|^2 - 2\|x\|^2\langle e,x\rangle^{-1}\langle x,e\rangle = \|x\|^4\langle e,x\rangle^{-2} - \|x\|^2 \tag{2.9}$$

and

$$\langle x,u\rangle = \left\langle x, x-\|x\|^2\langle e,x\rangle^{-1}e\right\rangle = \langle x,x\rangle - \|x\|^2\langle e,x\rangle^{-1}\langle x,e\rangle = 0. \tag{2.10}$$

Hence $\|y\|^2 = \|x\|^2+\|u\|^2+2\langle x,u\rangle = \|x\|^2+(\|x\|^4\langle e,x\rangle^{-2}-\|x\|^2)+2\cdot 0 = \|x\|^4\langle e,x\rangle^{-2}$ and thus $\|y\| = \|x\|^2\langle e,x\rangle^{-1}$. Hence

$$\begin{aligned} Ty &= \tfrac12 y+\tfrac12\|y\|e = \tfrac12(x+u)+\tfrac12\|x\|^2\langle e,x\rangle^{-1}e && \text{(2.11a)}\\ &= \tfrac12\big(x+(x-\|x\|^2\langle e,x\rangle^{-1}e)+\|x\|^2\langle e,x\rangle^{-1}e\big) && \text{(2.11b)}\\ &= x && \text{(2.11c)}\end{aligned}$$

and

$$y-Ty = (x+u)-x = u. \tag{2.12}$$

We have shown that $(x,u) = (Ty, y-Ty)\in \operatorname{gra} A_e$ as required.

Case 2: $x=0$.
Here we set $u=\eta e$, where $\eta\leq 0$, and again $y=x+u=\eta e$. Then $\|y\|=\|u\|=\|\eta e\|=|\eta|\cdot\|e\|=|\eta|=-\eta$. Hence $Ty=\frac12 y+\frac12\|y\|e=\frac12\eta e+\frac12(-\eta)e=0=x$ and $y-Ty=\eta e-x=u-0=u$. Again, we have shown that $(x,u)=(Ty,y-Ty)\in\operatorname{gra}A_e$, as claimed.

Combining *Case 1* and *Case 2*, we obtain the conclusion

$$\operatorname{gra} B\subseteq \operatorname{gra} A_e. \tag{2.13}$$

Conversely, let $y\in X$. Set $x=Ty=\frac12 y+\frac12\|y\|e$ and note that $y-Ty=\frac12 y-\frac12\|y\|e$. Furthermore, $2\langle e,x\rangle = \langle y,e\rangle+\|y\|\langle e,e\rangle = \langle y,e\rangle+\|y\|\geq -\|y\|\cdot\|e\|+\|y\|=0$ with equality if and only if $y\in\mathbb{R}_-\cdot e$, i.e., $x=0$, by Cauchy–Schwarz. Moreover, $\|x\|^2=\|\frac12 y+\frac12\|y\|e\|^2=\frac14\|y\|^2+\frac14\|y\|^2+\frac12\|y\|\langle y,e\rangle=\frac12(\|y\|^2+\|y\|\langle y,e\rangle)$ and $\langle e,x\rangle=\langle e,\frac12 y+\frac12\|y\|e\rangle=\frac12\langle e,y\rangle+\frac12\|y\|$.

We now consider two conceivable alternatives.

Case 1: $\langle e,x\rangle>0$.
Then

$$\begin{aligned} Bx &= x-\frac{\|x\|^2}{\langle e,x\rangle}\cdot e = \tfrac12 y+\tfrac12\|y\|e-\frac{\frac12(\|y\|^2+\|y\|\langle y,e\rangle)}{\frac12\langle e,y\rangle+\frac12\|y\|}\cdot e && \text{(2.14a)}\\ &= \tfrac12 y-\tfrac12\|y\|e && \text{(2.14b)}\\ &= y-Ty. && \text{(2.14c)}\end{aligned}$$

Therefore, $(Ty,y-Ty)=(x,Bx)\in\operatorname{gra}B$.

Case 2: $x=0$.
Then $y=\eta e$, where $\eta\in\mathbb{R}_-$. Thus $y-Ty=\frac12 y-\frac12\|y\|e=\frac12\eta e-\frac12|\eta|e=\eta e\in B0=Bx$. Therefore, $(y,Ty)\in\{0\}\times B0\subseteq\operatorname{gra}B$.

Combining *Case 1* with *Case 2*, we deduce that

$$\operatorname{gra} A_e \subseteq \operatorname{gra} B. \tag{2.15}$$

Finally, (2.13) and (2.15) yield the result. ■

Theorem 2.3 (A_e **for** $\|e\| < 1$)**.** *Suppose that* $\|e\| < 1$. *Then the maximally monotone operator* A_e *is given by*

$$(\forall x \in X) \quad A_e x = x + \frac{2\langle e,x\rangle - 2\sqrt{(1-\|e\|^2)\|x\|^2 + \langle e,x\rangle^2}}{1-\|e\|^2} \cdot e. \tag{2.16}$$

Proof. Denote the right-hand side of (2.16) by B, i.e., set

$$(\forall x \in X) \quad Bx = x + \frac{2\langle e,x\rangle - 2\sqrt{(1-\|e\|^2)\|x\|^2 + \langle e,x\rangle^2}}{1-\|e\|^2} \cdot e. \tag{2.17}$$

We also write

$$Bx = x + \rho(x) \cdot e \tag{2.18}$$

and observe that $\rho(x)$ is the *nonpositive* root of the quadratic equation

$$(1-\|e\|^2)\rho^2 - 4\langle e,x\rangle \rho - 4\|x\|^2 = 0. \tag{2.19}$$

Once again, our job is to show that $B = A_e$, and for that it suffices to show that $\operatorname{gra} B = \operatorname{gra} A_e = \big\{(T_e x, x - T_e x) \;\big|\; x \in X\big\}$ by the Minty parametrization [10]. For convenience, we also abbreviate $T = T_e$ and $\rho = \rho(x)$.

First, let $x \in X$ and set $y = x + Bx = 2x + \rho e$. Then $\|y\|^2 = \|2x+\rho e\|^2 = 4\|x\|^2 + 4\rho\langle x,e\rangle + \rho^2\|e\|^2 = 4\|x\|^2 + 4\rho\langle x,e\rangle + \rho^2(\|e\|^2-1) + \rho^2 = 0 + \rho^2 = \rho^2$ by (2.19); thus, $\|y\| = |\rho| = -\rho$. Hence

$$Ty = \tfrac{1}{2}y + \tfrac{1}{2}\|y\|e = \tfrac{1}{2}(2x+\rho e) + \tfrac{1}{2}(-\rho)e = x \tag{2.20}$$

and so

$$y - Ty = (2x+\rho e) - x = x + \rho e = Bx. \tag{2.21}$$

It follows that $(x, Bx) = (Ty, y - Ty) \in \operatorname{gra} A_e$ and thus

$$\operatorname{gra} B \subseteq \operatorname{gra} A_e. \tag{2.22}$$

Conversely, let $y \in X$, set $x = Ty = \frac{1}{2}y + \frac{1}{2}\|y\|e$ and note that $y - Ty = \frac{1}{2}y - \frac{1}{2}\|y\|e$. We have $\|x\|^2 = \|\frac{1}{2}y + \frac{1}{2}\|y\|e\|^2 = \frac{1}{4}\|y\|^2 + \frac{1}{4}\|y\|^2\|e\|^2 + \frac{1}{2}\|y\|\langle y,e\rangle$ and $\langle e,x\rangle = \langle e, \frac{1}{2}y + \frac{1}{2}\|y\|e\rangle = \frac{1}{2}\langle e,y\rangle + \frac{1}{2}\|y\|\|e\|^2$. Hence

$$\begin{aligned}
&(1-\|e\|^2)(-\|y\|)^2 - 4\langle e,x\rangle(-\|y\|) - 4\|x\|^2 && (2.23a)\\
&= (1-\|e\|^2)\|y\|^2 + 2\|y\|\langle e, y + \|y\|e\rangle - \big(\|y\|^2 + \|y\|^2\|e\|^2 + 2\|y\|\langle y,e\rangle\big) && (2.23b)\\
&= 0. && (2.23c)
\end{aligned}$$

In view of (2.19) and the fact that $-\|y\| \leq 0$, it follows that $-\|y\| = \rho(x)$. Hence

$$Bx = x + \rho(x)e = \left(\tfrac{1}{2}y + \tfrac{1}{2}\|y\|e\right) - \|y\|e = \tfrac{1}{2}y - \tfrac{1}{2}\|y\|e = y - Ty. \tag{2.24}$$

Hence $(Ty, y-Ty) = (x, Bx) \in \operatorname{gra} B$ and thus

$$\operatorname{gra} A_e \subseteq \operatorname{gra} B. \tag{2.25}$$

The conclusion now follows by combining (2.22) with (2.25). ■

3. Duality and stronger notions of monotonicity

If $A\colon X \rightrightarrows X$ is maximally monotone, then its *dual* operator is A^{-1}, and the corresponding dual objects of the resolvent and reflected resolvent are $\operatorname{Id} - J_A$ and $-R_A$, respectively [3]. We now identify the dual objects, which have pleasant explicit formulae, as well as some other interesting properties.

Lemma 3.1. *Recall that A_e, T_e, and R_e are defined in (2.6), (2.5), and (1.3), respectively. Then the following hold:*

(i) **(duality)** $-R_e = R_{-e}$, $\operatorname{Id} - T_e = T_{-e}$, *and* $(A_e)^{-1} = A_{-e}$.
(ii) **(cone)** $(\forall x \in X)(\forall \lambda \in \mathbb{R}_{++})\ A_e(\lambda x) = \lambda A_e x$. *Consequently,* $\operatorname{gra} A$ *is a (nonconvex) cone.*
(iii) *If* $\|e\| = 1$, *then* $(\forall (x,u) \in \operatorname{gra} A_e)\ \langle x, u\rangle = 0$.
(iv) *If* $\|e\| = 1$, *then* $\operatorname{zer} A_e = (A_e)^{-1}(0) = \mathbb{R}_+ \cdot e$.

Proof. (i): It is clear that $R_{(A_e)^{-1}} = -R_{A_e} = -R_e = -\|\cdot\|e = R_{-e}$. Hence, $(A_e)^{-1} = A_{-e}$. Consequently, by the inverse resolvent identity and (2.5), we have $\operatorname{Id} - T_e = \operatorname{Id} - J_{A_e} = J_{(A_e)^{-1}} = J_{A_{-e}} = T_{-e}$. (ii): This can be directly verified using (2.7) and (2.16). (iii): Let $x \in \operatorname{dom} A$. If $x = 0$ then $(\forall u \in Ax = \mathbb{R}_- \cdot e)$ we have $\langle x, u\rangle = 0$. Now suppose that $\langle e, x\rangle > 0$. Then Ax is a singleton and $\langle x, Ax\rangle = \left\langle x, x - \frac{\|x\|^2}{\langle e,x\rangle}\cdot e\right\rangle = \|x\|^2 - \|x\|^2 = 0$. (iv): Indeed, $\operatorname{zer} A_e = A_e^{-1}(0) = A_{-e}(0) = \mathbb{R}_- \cdot (-e) = \mathbb{R}_+ \cdot e$. ■

Strong monotonicity, paramonotonicity, cocoercivity, and 3^* monotonicity are perhaps the most important properties a maximally monotone operators can have. The following two results provide a complete characterization of these properties. Once again, the norm $\|e\|$ plays a crucial role.

Proposition 3.2. *Suppose that $\|e\| = 1$ and that X is not one-dimensional. Then the following hold:*

(i) *A_e is* not *paramonotone. Consequently, A_e is neither strictly nor strongly monotone.*
(ii) *A_e is* not 3^* *monotone.*

Proof. (i): Let $\alpha \in \left]1, +\infty\right[$, let $x \in \operatorname{dom} A \smallsetminus \mathbb{R}_+ \cdot e$ and set $y = \alpha x$. Note that $\langle e, x\rangle > 0$, that $Ay = A(\alpha x) = \alpha Ax \neq 0$ by Lemma 3.1(ii)&(iv), and that Ax (and consequently Ay) is a singleton. Therefore, by Lemma 3.1(iii) $\langle x-y, Ax-Ay\rangle = \langle (1-\alpha)x, (1-\alpha)Ax\rangle = (1-\alpha)^2\langle x, Ax\rangle = 0$. However, $(x, Ay) \notin \operatorname{gra} A$, hence A is not paramonotone.

(ii): Indeed, let $\alpha > 0$, let $f \in \{e\}^{\perp}$ such that $\|f\| = 1$, set $x = 2(f-e)$, set $y = 0$ and set $z = 2\alpha f$. We also write T instead of T_e for convenience. Now

$$\begin{aligned}
\langle Tx - Tz, (\mathrm{Id}-T)y - (\mathrm{Id}-T)z\rangle &= -\langle Tx - Tz, (\mathrm{Id}-T)z\rangle && (3.1a)\\
&= -\langle f - e + \|f-e\|e - \alpha(f+e), \alpha(f-e)\rangle && (3.1b)\\
&= -\langle f - e + \|f-e\|e, \alpha(f-e)\rangle + \alpha^2(\|f\|^2 - \|e\|^2) && (3.1c)\\
&= -\alpha\langle f - e + \|f-e\|e, f-e\rangle && (3.1d)\\
&= -\alpha(\|f-e\|^2 - \|f-e\|) = -\alpha(2-\sqrt{2}). && (3.1e)
\end{aligned}$$

Therefore, we conclude that $\inf_{z\in X}\langle Tx - Tz, (\mathrm{Id}-T)y - (\mathrm{Id}-T)z\rangle = -\infty$, hence A is *not* 3^* monotone by [3, Theorem 2.1(xvii)]. ■

Proposition 3.3. *Suppose that $\|e\| < 1$. Then the following hold:*

(i) *A_e is strongly monotone, with sharp constant $\frac{1-\|e\|}{1+\|e\|}$.*
(ii) *A_e is cocoercive, with sharp constant $\frac{1-\|e\|}{1+\|e\|}$.*
(iii) *A_e is paramonotone.*
(iv) *A_e is 3^* monotone.*
(v) *A_e is a displacement map if and only if $\|e\| \le \frac{1}{3}$.*

Proof. (i): Let $(x,y) \in X\times X$. By the Minty parametrization [10] of $\operatorname{gra} A_e$ $(\exists (u,v) \in X\times X)$ such that $(x, Ax) = (T_e u, u - T_e u) = \frac{1}{2}(u + \|u\|e, u - \|u\|e)$ and $(y, Ay) = (T_e v, v - T_e v) = \frac{1}{2}(v + \|v\|e, v - \|v\|e)$. Now, on the one hand, by the reverse triangle inequality we have

$$\begin{aligned}
4\langle x-y, Ax-Ay\rangle &= \langle (u-v) + (\|u\|e - \|v\|e), (u-v) - (\|u\|e - \|v\|e)\rangle && (3.2a)\\
&= \|u-v\|^2 - (\|u\| - \|v\|)^2\|e\|^2 \ge (1-\|e\|^2)\|u-v\|^2. && (3.2b)
\end{aligned}$$

On the other hand, using the triangle inequality and the reverse triangle inequality we have

$$2\|x-y\| = \|(u-v) + (\|u\| - \|v\|)e\| \le \|u-v\| + \big|\|u\| - \|v\|\big|\|e\| \le (1+\|e\|)\|u-v\|. \quad (3.3)$$

Combining (3.2) and (3.3) yields

$$\langle x-y, Ax-Ay\rangle \ge \tfrac{1-\|e\|^2}{(1+\|e\|)^2}\|x-y\|^2 = \tfrac{1-\|e\|}{1+\|e\|}\|x-y\|^2. \quad (3.4)$$

To show that the strong monotonicity constant is sharp we set $(x,y) = (e,0)$. Now

$$\begin{aligned}
\langle e-0, Ae-A0\rangle = \langle e, Ae\rangle &= \Big\langle e, \Big(1 + \tfrac{2\langle e,e\rangle - 2\sqrt{(1-\|e\|^2)\|e\|^2 + \langle e,e\rangle^2}}{1-\|e\|^2}\Big)e\Big\rangle && (3.5a)\\
&= \Big(1 + \tfrac{2\|e\|^2 - 2\|e\|}{1-\|e\|^2}\Big)\|e\|^2 = \Big(\tfrac{\|e\|^2 - 2\|e\| + 1}{1-\|e\|^2}\Big)\|e\|^2 && (3.5b)\\
&= \tfrac{1-\|e\|}{1+\|e\|}\|e-0\|^2. && (3.5c)
\end{aligned}$$

(ii): It follows from (i) applied with e replaced by $-e$ that A_{-e} is strongly monotone with sharp constant $\frac{1-\|e\|}{1+\|e\|}$. Combining with (i) and [2, Example 22.7] we conclude that $A_e = (A_{-e})^{-1}$ is $\frac{1-\|e\|}{1+\|e\|}$-cocoercive. (iii): Combine (ii) and [2, Example 22.8]. (iv): Combine (ii) and [2, Example 25.20(i)]. (v): It follows from [1, Proposition 4.13] and (ii) in view of [2, Example 22.7] that A_e is a displacement map if and only if $\frac{1-\|e\|}{1+\|e\|} \geq \frac{1}{2}$; equivalently $\|e\| \leq \frac{1}{3}$. ■

Remark 3.4 (**the one-dimensional case for** $\|e\| = 1$)**.** *Now suppose that X is one-dimensional. Then A_e is a subdifferential operator; in fact, $A_e = N_{\mathbb{R}_+ e}$. Hence A_e is both paramonotone (see, e.g.,* [2, Example 22.4(i)]*) and 3^* monotone (see, e.g.,* [2, Example 25.14]*), but neither strictly nor strongly monotone.*

Remark 3.5 (**cyclic monotonicity**)**.** *If X is one-dimensional or $e = 0$, then A_e is clearly cyclically monotone; otherwise, A_e is not a gradient by the formulae provided in Section 2. Based on some numerical experiments, we conjecture that the degree of cyclic monotonicity decreases from $+\infty$ (when $\|e\| = 0$) to 2 (when $\|e\| = 1$) as $\|e\|$ increases; however, we do not know at which values the transitions occur.*

4. Resolvent iteration and average

It is well known that the sequence $((T_e)^n x_0)_{n\in\mathbb{N}}$ converges weakly to some point in $\operatorname{Fix} T_e = \operatorname{zer} A_e$; however, more can be said in our setting:

Proposition 4.1 (**resolvent iteration**)**.** *Suppose that X is not one-dimensional and let $x_0 \in X$. The following hold true:*

(i) $((T_e)^n x_0)_{n\in\mathbb{N}}$ *converges* strongly *to some point in* $\operatorname{Fix} T_e = \operatorname{zer} A_e$.
(ii) *If $\|e\| < 1$, then $(T_e)^n x_0 \to 0$ linearly, with rate $(1+\|e\|)/2$.*
(iii) *If $\|e\| = 1$, then $(T_e)^n x_0 \to \|x_0\| \operatorname{sinc}(\theta_0) e$, where* sinc *is the (unnormalized) sinc function and $\theta_0 \in [0,\pi]$ satisfies $\cos(\theta_0)\|x_0\| = \langle x_0, e\rangle$.*

Proof. (i): On the one hand, the convergence is known to be weak, see, e.g., [2, Proposition 5.16(iii)]. On the other hand, all iterates lie in $\operatorname{span}\{e, x_0\}$. Altogether, the convergence must be strong.

(ii)&(iii): Write $x_n = (T_e)^n x_0$ for every $n \in \mathbb{N}$. The result is clear when $e = 0$ since then $T_e = \frac{1}{2}\operatorname{Id}$. Assume that $e \neq 0$ and that $x_0 \neq 0$, and let us work in "polar coordinates", i.e., pick $f \in \operatorname{span}\{x_0, e\} \cap \{e\}^\perp$ such that $\|f\| = 1$ and write $x_n = \|x_n\|(\cos(\theta_n)\widehat{e} + \sin(\theta_n) f)$, where $\widehat{e} = e/\|e\|$ and $\theta_n \in [0,\pi]$. Then

$$\|x_{n+1}\|^2/\|x_n\|^2 = (1 + 2\cos(\theta_n)\|e\| + \|e\|^2)/4 \tag{4.1}$$

and $\theta_n \to 0^+$. Thus $\|x_{n+1}\|^2/\|x_n\|^2 \to (1+\|e\|^2+2\|e\|)/4 = ((1+\|e\|)/2)^2$ and the result follows when $\|e\| < 1$.

Now assume that $\|e\| = 1$. Then, by (4.1), $\|x_{n+1}\|^2 = \|x_n\|^2(1+\cos(\theta_n))/2 = \|x_n\|^2\cos^2(\theta_n/2)$. Inductively, it follows that

$$x_n = \|x_n\|\big(\cos(\theta_0/2^n)e + \sin(\theta_0/2^n)f\big) \tag{4.2a}$$

$$= \|x_0\|\Big(\prod_{k=1}^{n-1}\cos(\theta_0/2^k)\Big)\big(\cos(\theta_0/2^n)e + \sin(\theta/2^n)f\big) \tag{4.2b}$$

$$\to \|x_0\|\Big(\prod_{k=1}^{\infty}\cos(\theta_0/2^k)\Big)e \tag{4.2c}$$

$$= \|x_0\|\operatorname{sinc}(\theta_0)e, \tag{4.2d}$$

where we use [8, equation (1.3) on page 2] in (4.2d). ■

We conclude this paper with an observation on the resolvent average that follows readily from the definition.

Proposition 4.2 (resolvent average). *Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are in $[0,1]$ such that $\sum_{i=1}^m \lambda_i = 1$ and that $e_1, \ldots, e_m$ are in the unit ball of X. Set $\bar{e} = \sum_{i=1}^m \lambda_i e_i$. Then $\sum_{i=1}^m \lambda_i R_{e_i} = R_{\bar{e}}$ and consequently the resolvent average [1] of $A_{e_1}, \ldots, A_{e_m}$, with parameters $\lambda_1, \ldots, \lambda_m$, is $A_{\bar{e}}$.*

Acknowledgments

HHB was partially supported by the Natural Sciences and Engineering Research Council of Canada. WMM was partially supported by the Natural Sciences and Engineering Research Council of Canada Postdoctoral Fellowship.

References

[1] S. Bartz, H.H. Bauschke, S.M. Moffat, and X. Wang, The resolvent average of monotone operators: dominant and recessive properties, *SIAM Journal on Optimization* 26 (2016), 602–634.
[2] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Second Edition, Springer, 2017.
[3] H.H. Bauschke, S.M. Moffat, and X. Wang, Firmly nonexpansive mappings and maximally monotone operators: correspondence and duality, *Set-Valued and Variational Analysis* 20 (2012), 131–153.
[4] H. Brezis, *Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland/Elsevier, 1973.
[5] R.S. Burachik and A.N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer-Verlag, 2008.
[6] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.
[7] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.
[8] M. Kac, *Statistical Independence in Probability, Analysis and Number Theory*, Wiley, 1959.
[9] E. Kopecká and S. Reich, Nonexpansive retracts in Banach spaces, in *Fixed Point Theory and its Applications*, Banach Center Publications volume 77, 161–174. 2007.
[10] G.J. Minty, Monotone (nonlinear) operators in Hilbert spaces, *Duke Mathematical Journal* 29 (1962), 341–346.
[11] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer-Verlag, corrected 3rd printing, 2009.

[12] S. Simons, *From Hahn-Banach to Monotonicity*, Springer-Verlag, 2008.
[13] S. Simons, *Minimax and Monotonicity*, Springer-Verlag, 1998.
[14] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing, 2002.
[15] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I: Fixed Point Theorems*, Springer-Verlag, 1993.
[16] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/A: Linear Monotone Operators*, Springer-Verlag, 1990.
[17] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators*, Springer-Verlag, 1990.