The Douglas–Rachford algorithm for a hyperplane and a doubleton

Heinz H. Bauschke[*], Minh N. Dao[†] and Scott B. Lindstrom[‡]

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Abstract

The Douglas–Rachford algorithm is a popular algorithm for solving both convex and nonconvex feasibility problems. While its behaviour is settled in the convex inconsistent case, the general nonconvex inconsistent case is far from being fully understood. In this paper, we focus on the most simple nonconvex inconsistent case: when one set is a hyperplane and the other a doubleton (i.e., a two-point set). We present a characterization of cycling in this case which — somewhat surprisingly — depends on whether the ratio of the distance of the points to the hyperplane is rational or not. Furthermore, we provide closed-form expressions as well as several concrete examples which illustrate the dynamical richness of this algorithm.

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1 Introduction

The Douglas–Rachford (DR) algorithm is a popular algorithm for finding minimizers of the sum of two functions, defined on a real Hilbert space and possibly nonsmooth. Its convergence properties are fairly well understood in the case when the function are convex; see [6], [8], [4], [9], [10], and [13]. When specialized to indicator functions, the DR algorithm aims to solve a feasibility problem.

The *goal* of this paper is to analyze an instructive — and perhaps the most simple — nonconvex setting: when one set is a hyperplane and the other is a doubleton (i.e., it consists of just two distinct points). Our analysis reveals interesting dynamic behaviour whose *periodicity* depends on whether or not a certain ratio distances is rational (Theorem 4.2). We also provide *explicit closed-form expressions* for the iterates in various circumstances (Theorem 5.2). Our work can be regarded as complementary to the recently rapidly growing body of works on the DR algorithm in nonconvex settings including [1], [7], [11], and [16].

[*]Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: heinz.bauschke@ubc.ca.
[†]CARMA, University of Newcastle, Callaghan, NSW 2308, Australia. E-mail: daonminh@gmail.com.
[‡]CARMA, University of Newcastle, Callaghan, NSW 2308, Australia. E-mail: scott.lindstrom@uon.edu.au.

The remainder of the paper is organized as follows. In Section 2, we recall the necessary background material to start our analysis. The case when one set contains not just 2 but finitely many points is considered in Section 3. Section 4 provides a characterization of cycling occurs while Section 5 presents closed-form expressions and various examples. We conclude the paper with Section 6.

2 The set up

Throughout we assume that

$$X \text{ is a finite-dimensional real Hilbert space} \tag{1}$$

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and

$$A \text{ and } B \text{ are nonempty closed subsets of } X. \tag{2}$$

To solve the feasibility problem

$$\text{find a point in } A \cap B, \tag{3}$$

we employ the *Douglas–Rachford algorithm* (also called *averaged alternating reflections*) that uses the *Douglas–Rachford operator*, associated with the ordered pair (A, B),

$$T := \tfrac{1}{2}(\mathrm{Id} + R_B R_A) \tag{4}$$

to generate a *DR sequence* $(x_n)_{n \in \mathbb{N}}$ with starting point $x_0 \in X$ by

$$(\forall n \in \mathbb{N}) \quad x_{n+1} \in T x_n, \tag{5}$$

where Id is the identity operator, P_A and P_B are the projectors, and $R_A := 2P_A - \mathrm{Id}$ and $R_B := 2P_B - \mathrm{Id}$ are the reflectors with respect to A and B, respectively. Here the projection $P_A x$ of a point $x \in X$ is the nearest point of x in the set A, i.e.,

$$P_A x := \operatorname*{argmin}_{a \in A} \|x - a\| = \{a \in A \mid \|x - a\| = d_A(x)\}, \tag{6}$$

where $d_A(x) := \min_{a \in A} \|x - a\|$ is the distance from x to the set A. Note from [3, Corollary 3.15] that closedness of the set A is necessary and sufficient for A being proximinal, i.e., $(\forall x \in X)$ $P_A x \neq \varnothing$. According to [3, Theorem 3.16], if A and B are convex, then P_A, P_B and hence T are single-valued. We also note that

$$(\forall x \in X) \quad Tx = \tfrac{1}{2}(\mathrm{Id} + R_B R_A)x = \{x - a + P_B(2a - x) \mid a \in P_A x\}, \tag{7}$$

and if P_A is single-valued then

$$T = \tfrac{1}{2}(\mathrm{Id} + R_B R_A) = \mathrm{Id} - P_A + P_B R_A. \tag{8}$$

For further information on the DR algorithm in the classical case (when A and B are both convex), see [6], [8], [4], [9], [10], and [13]. Results complementary to the rapidly growing body of work on the DR algorithm in nonconvex settings can be found in [1], [7], [11], [16], and the references therein.

The notation and terminology used is standard and follows, e.g., [3]. The nonnegative integers are $\mathbb{N}$, the positive integers are $\mathbb{N}^*$, and the real numbers are $\mathbb{R}$, while $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{++} := \{x \in \mathbb{R} \mid x > 0\}$. We are now ready to start deriving the results we announced in Section 1.

3 Hyperplane and finitely many points

We focus on the case when B is a finite set, and start with the following observation.

Lemma 3.1. *Suppose that A is convex, that B is finite, and that $A \cap B = \varnothing$. Then the DR sequence $(x_n)_{n\in\mathbb{N}}$ is not convergent.*

Proof. Since A is convex, P_A is single-valued and continuous on X. By (8), $T = \frac{1}{2}(\mathrm{Id} + R_B R_A) = \mathrm{Id} - P_A + P_B R_A$, and hence

$$(\forall n \in \mathbb{N}) \quad b_n := x_{n+1} - x_n + P_A x_n \in P_B R_A x_n \subseteq B. \tag{9}$$

Suppose that $x_n \to x \in X$. Then $b_n \to P_A x$. But $(b_n)_{n\in\mathbb{N}}$ lies in B and B is finite, there exists $n_0 \in \mathbb{N}$ such that $(\forall n \geq n_0)$ $b_n = b \in B$. We obtain $P_A x = b \in A \cap B$, which contradicts the assumption that $A \cap B = \varnothing$. ■

From here onwards, we assume that A is a hyperplane and B is a finite subset of X containing m pairwise distinct vectors; more specifically,

$$A = \{u\}^\perp \quad \text{with} \quad u \in X, \|u\| = 1 \tag{10a}$$

and

$$B = \{b_1, \ldots, b_m\} \subseteq X \quad \text{with} \quad \langle b_1, u\rangle \leq \cdots \leq \langle b_m, u\rangle. \tag{10b}$$

Fact 3.2. *Let $x \in X$. Then the following hold:*

(i) $P_A x = x - \langle x, u\rangle u$.
(ii) $R_A x = x - 2\langle x, u\rangle u$.
(iii) $d_A(x) = |\langle x, u\rangle|$.

Proof. This follows from [5, Example 2.4(i)] with noting that $R_A x = 2P_A x - x$ and that $d_A(x) = \|x - P_A x\|$. ■

Let $(x_n)_{n\in\mathbb{N}}$ be a DR sequence with respect to (A, B) with starting point $x_0 \in X$. Since P_A is single-valued, we derive from (8) that

$$(\forall n \in \mathbb{N}^*) \quad x_n - x_{n-1} + P_A x_{n-1} \in T x_{n-1} - x_{n-1} + P_A x_{n-1} = P_B R_A x_{n-1} \subseteq B. \tag{11}$$

Let us set

$$(\forall n \in \mathbb{N}^*) \quad b_{k(n)} := x_n - x_{n-1} + P_A x_{n-1} \in P_B R_A x_{n-1} \subseteq B \quad \text{with} \quad k(n) \in \{1, \ldots, m\}. \tag{12}$$

The following lemma shows that the subsequence $(x_n)_{n\in\mathbb{N}^*}$ lies in the union the lines through the points in B with a common direction vector u.

Lemma 3.3. *For every $n \in \mathbb{N}^*$,*

$$x_n = \langle x_{n-1}, u\rangle u + b_{k(n)} \quad \text{and} \quad \langle x_n, u\rangle = \langle x_{n-1}, u\rangle + \langle b_{k(n)}, u\rangle, \tag{13}$$

where $k(n) \in \{1, \ldots, m\}$. Consequently, the subsequence $(x_n)_{n\in\mathbb{N}^*}$ lies in the union of finitely many (affine) lines:

$$B + \mathbb{R}u = \bigcup_{b\in B} (b + \mathbb{R}u) = \{b + \lambda u \mid b \in B, \lambda \in \mathbb{R}\}. \tag{14}$$

Proof. By combining (12) with Fact 3.2(i),

$$(\forall n \in \mathbb{N}^*) \quad x_n = x_{n-1} - P_A x_{n-1} + b_{k(n)} = \langle x_{n-1}, u\rangle u + b_{k(n)}. \tag{15}$$

Taking the inner product with u yields

$$(\forall n \in \mathbb{N}^*) \quad \langle x_n, u\rangle = \langle \langle x_{n-1}, u\rangle u + b_{k(n)}, u\rangle = \langle x_{n-1}, u\rangle + \langle b_{k(n)}, u\rangle, \tag{16}$$

which completes the proof. ■

Proposition 3.4. *Exactly one of the following holds.*

(i) *B is contained in one of the two closed halfspaces induced by A. Either* (a) *the sequence $(x_n)_{n\in\mathbb{N}}$ converges finitely to a point $x \in \operatorname{Fix} T$ and $P_A x \in A \cap B$, or* (b) *$A \cap B = \varnothing$ and $\|x_n\| \to +\infty$ in which case $(P_A x_n)_{n\in\mathbb{N}}$ converges finitely to a best approximation solution $a \in A$ relative to A and B in the sense that $d_B(a) = \min d_B(A)$.*

(ii) *B is not contained in one of the two closed halfspaces induced by A. Then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. If additionally $A \cap B = \varnothing$, then $(x_n)_{n\in\mathbb{N}}$ is not convergent and*

$$(\forall n \in \mathbb{N}) \quad \|x_n - x_{n+1}\| \geq \min d_A(B) > 0. \tag{17}$$

Proof. (i): This follows from [5, Theorem 7.5].

(ii): Since B is not a subset of one in two closed halfspaces induced by A, it follows from (10b) that

$$\langle b_1, u\rangle < 0 < \langle b_m, u\rangle. \tag{18}$$

Combining Fact 3.2(ii) with Lemma 3.3 yields

$$\begin{aligned}(\forall n \in \mathbb{N}^*) \quad R_A x_n &= x_n - 2\langle x_n, u\rangle u && \text{(19a)}\\ &= \big(\langle x_{n-1}, u\rangle u + b_{k(n)}\big) - \big(\langle x_{n-1}, u\rangle + \langle b_{k(n)}, u\rangle\big)u - \langle x_n, u\rangle u && \text{(19b)}\\ &= -(\langle x_n, u\rangle + \langle b_{k(n)}, u\rangle)u + b_{k(n)}. && \text{(19c)}\end{aligned}$$

For any $n \in \mathbb{N}^*$ and any distint indices $i, j \in \{1, \ldots, m\}$, we have the following equivalences:

$$\begin{aligned} &\|b_i - R_A x_n\| \leq \|b_j - R_A x_n\| && \text{(20a)}\\ \Leftrightarrow\ &\|(\langle x_n, u\rangle + \langle b_{k(n)}, u\rangle)u + (b_i - b_{k(n)})\|^2 \leq \|(\langle x_n, u\rangle + \langle b_{k(n)}, u\rangle)u + (b_j - b_{k(n)})\|^2 && \text{(20b)}\\ \Leftrightarrow\ &\|b_i - b_{k(n)}\|^2 - \|b_j - b_{k(n)}\|^2 \leq 2(\langle x_n, u\rangle + \langle b_{k(n)}, u\rangle)\langle b_j - b_i, u\rangle && \text{(20c)}\\ \Leftrightarrow\ &\begin{cases} \langle x_n, u\rangle \geq \beta_{i,j,n} & \text{if } \langle b_i, u\rangle < \langle b_j, u\rangle,\\ \|b_i - b_{k(n)}\| \leq \|b_j - b_{k(n)}\| & \text{if } \langle b_i, u\rangle = \langle b_j, u\rangle,\\ \langle x_n, u\rangle \leq \beta_{i,j,n} & \text{if } \langle b_i, u\rangle > \langle b_j, u\rangle, \end{cases} && \text{(20d)}\end{aligned}$$

where

$$\beta_{i,j,n} := \frac{\|b_i - b_{k(n)}\|^2 - \|b_j - b_{k(n)}\|^2}{2\langle b_j - b_i, u\rangle} - \langle b_{k(n)}, u\rangle. \tag{21}$$

We shall now show that $(\langle x_n, u\rangle)_{n\in\mathbb{N}}$ is bounded above. Setting

$$r := \max\{k \in \{1,\ldots,m\} \mid \langle b_k, u\rangle = \langle b_1, u\rangle\}, \tag{22}$$

we see that $r < m$ due to (18) and that, by (10b),

$$\langle b_1, u\rangle = \cdots = \langle b_r, u\rangle < \langle b_{r+1}, u\rangle \leq \cdots \leq \langle b_m, u\rangle. \tag{23}$$

Now let $n \in \mathbb{N}^*$ and set

$$I(n) := \{i \in \{1,\ldots,r\} \mid (\forall j \in \{1,\ldots,r\}) \quad \|b_i - b_{k(n)}\| \leq \|b_j - b_{k(n)}\|\}. \tag{24}$$

Then $I(n) = \{k(n)\}$ whenever $k(n) \in \{1,\ldots,r\}$ and, by (20),

$$(\forall i \in I(n))(\forall j \in \{1,\ldots,r\}) \quad \|b_i - R_A x_n\| \leq \|b_j - R_A x_n\|. \tag{25}$$

Define

$$\beta_n := \max\{\beta_{i,j,n} \mid i \in I(n), j \in \{r+1,\ldots,m\}\}. \tag{26}$$

If $\langle x_n, u\rangle > \beta_n$, then (10b) and (20) yield

$$(\forall i \in I(n))(\forall k \in \{r+1,\ldots,m\}) \quad \|b_i - R_A x_n\| < \|b_k - R_A x_n\|, \tag{27}$$

which together with (25) implies that $k(n+1) \in I(n) \subseteq \{1,\ldots,r\}$ and, by (16), (18) and (23),

$$\langle x_{n+1}, u\rangle = \langle x_n, u\rangle + \delta \quad \text{with} \quad \delta := \langle b_{k(n+1)}, u\rangle = \langle b_1, u\rangle < 0. \tag{28}$$

Noting that (28) holds whenever $\langle x_n, u\rangle > \beta_n$ and that the sequence $(\beta_n)_{n\in\mathbb{N}}$ is bounded since the set $\{\beta_{i,j,n} \mid i \in I(n), j \in \{r+1,\ldots,m\}, n \in \mathbb{N}^*\}$ is finite, we deduce that $(\langle x_n, u\rangle)_{n\in\mathbb{N}}$ is bounded above. By a similar argument, $(\langle x_n, u\rangle)_{n\in\mathbb{N}}$ is also bounded below. Combining with (15), we get boundedness of $(x_n)_{n\in\mathbb{N}}$.

Finally, if $A \cap B = \varnothing$, then, by Lemma 3.1, $(x_n)_{n\in\mathbb{N}}$ is not convergent and, by the Cauchy–Schwarz inequality, Lemma 3.3, and Fact 3.2(iii),

$$(\forall n \in \mathbb{N}) \quad \|x_{n+1} - x_n\| \geq |\langle x_{n+1} - x_n, u\rangle| = |\langle b_{k(n+1)}, u\rangle| = d_A(b_{k(n+1)}) \geq \min d_A(B) > 0. \tag{29}$$

The proof is complete. ■

4 Hyperplane and doubleton: characterization of cycling

From now on, we assume that B is a doubleton where the two points do not belong to the same closed halfspace induced by A; more precisely,

$$B = \{b_1, b_2\} \subseteq X \quad \text{with} \quad \langle b_1, u\rangle < 0 < \langle b_2, u\rangle. \tag{30}$$

Set

$$\beta_1 := \langle b_1, u\rangle < 0, \quad \beta_2 := \langle b_2, u\rangle > 0, \quad \text{and} \quad \beta := \frac{\|b_1 - b_2\|^2}{2(\beta_1 - \beta_2)} = -\frac{\|b_1 - b_2\|^2}{2\langle b_2 - b_1, u\rangle} < 0. \tag{31}$$

Proposition 4.1. *The following holds for the DR sequence* $(x_n)_{n\in\mathbb{N}}$.

(i) $(x_n)_{n\in\mathbb{N}}$ *is bounded but not convergent with*

$$(\forall n \in \mathbb{N}) \quad \|x_n - x_{n+1}\| \geq \min\{d_A(b_1), d_A(b_2)\} > 0. \tag{32}$$

(ii) *For every* $n \in \mathbb{N}^*$,

$$x_n = \langle x_{n-1}, u\rangle u + b_{k(n)} \quad \text{and} \quad \langle x_n, u\rangle = \langle x_{n-1}, u\rangle + \langle b_{k(n)}, u\rangle, \tag{33}$$

where $k(n) \in \{1, 2\}$ *and where*

$$k(n) = 1 \;\&\; \langle x_n, u\rangle > \beta - \langle b_1, u\rangle \implies k(n+1) = 1, \tag{34a}$$

$$k(n) = 1 \;\&\; \langle x_n, u\rangle < \beta - \langle b_1, u\rangle \implies k(n+1) = 2, \tag{34b}$$

$$k(n) = 2 \;\&\; \langle x_n, u\rangle > -\beta - \langle b_2, u\rangle \implies k(n+1) = 1, \tag{34c}$$

$$k(n) = 2 \;\&\; \langle x_n, u\rangle < -\beta - \langle b_2, u\rangle \implies k(n+1) = 2. \tag{34d}$$

(iii) *There exist increasing (a.k.a. "nondecreasing") sequences* $(l_{1,n})_{n\in\mathbb{N}}$ *and* $(l_{2,n})_{n\in\mathbb{N}}$ *in* $\mathbb{N}$ *such that*

$$(\forall n \in \mathbb{N}) \quad \langle x_n, u\rangle = \langle x_0, u\rangle + l_{1,n}\langle b_1, u\rangle + l_{2,n}\langle b_2, u\rangle \quad \text{and} \quad l_{1,n} + l_{2,n} = n. \tag{35}$$

Moreover,

$$\frac{l_{1,n}}{n} \to \frac{\langle b_2, u\rangle}{\langle b_2 - b_1, u\rangle} \in \,]0,1[\quad \text{and} \quad \frac{l_{2,n}}{n} \to \frac{\langle b_1, u\rangle}{\langle b_1 - b_2, u\rangle} \in \,]0,1[\quad \text{as } n \to +\infty. \tag{36}$$

Proof. (i): By assumption, $b_1, b_2 \notin A$, and hence $A \cap B = \varnothing$. The conclusion follows from Proposition 3.4(ii).

(ii): We get (33) from Lemma 3.3. The equivalences (20) in the proof of Proposition 3.4(ii) state

$$\|b_1 - R_A x_n\| \leq \|b_2 - R_A x_n\| \Leftrightarrow \langle x_n, u\rangle \geq \frac{\|b_1 - b_{k(n)}\|^2 - \|b_2 - b_{k(n)}\|^2}{2\langle b_2 - b_1, u\rangle} - \langle b_{k(n)}, u\rangle, \tag{37}$$

which implies (34).

(iii): Using (33), we find increasing sequences $(l_{1,n})_{n\in\mathbb{N}}$ and $(l_{2,n})_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$(\forall n \in \mathbb{N}) \quad \langle x_n, u\rangle = \langle x_0, u\rangle + l_{1,n}\langle b_1, u\rangle + l_{2,n}\langle b_2, u\rangle \tag{38}$$

and that

$$(\forall n \in \mathbb{N}) \quad l_{1,n} + l_{2,n} = n. \tag{39}$$

Combining with (i), we obtain that

$$l_{1,n}\langle b_1, u\rangle + (n - l_{1,n})\langle b_2, u\rangle = l_{1,n}\langle b_1, u\rangle + l_{2,n}\langle b_2, u\rangle = \langle x_n, u\rangle - \langle x_0, u\rangle \tag{40}$$

is bounded. It follows that

$$\frac{l_{1,n}}{n}\langle b_1 - b_2, u\rangle + \langle b_2, u\rangle \to 0 \quad \text{as } n \to +\infty, \tag{41}$$

which yields

$$\frac{l_{1,n}}{n} \to \frac{\langle b_2, u\rangle}{\langle b_2 - b_1, u\rangle} \in \,]0,1[\quad \text{and} \quad \frac{l_{2,n}}{n} = 1 - \frac{l_{1,n}}{n} \to \frac{-\langle b_1, u\rangle}{\langle b_2 - b_1, u\rangle} \in \,]0,1[\tag{42}$$

as $n \to +\infty$. ∎

Theorem 4.2 (cycling and rationality). *The DR sequence $(x_n)_{n\in\mathbb{N}}$ cycles after a certain number of steps regardless of the starting point if and only if $d_A(b_1)/d_A(b_2) \in \mathbb{Q}$.*

Proof. First, by Fact 3.2(iii), $d_A = |\langle \cdot, u\rangle|$, which yields

$$d_A(b_1) = -\langle b_1, u\rangle \quad \text{and} \quad d_A(b_2) = \langle b_2, u\rangle. \tag{43}$$

We also note from Proposition 4.1(i)–(ii) that

$$(|\langle x_n, u\rangle|)_{n\in\mathbb{N}} \text{ is bounded,} \tag{44}$$

that

$$(\forall n \in \mathbb{N}^*) \quad x_n = \langle x_{n-1}, u\rangle u + b_{k(n)}, \tag{45}$$

and that

$$(\forall n \in \mathbb{N}^*) \quad \langle x_n, u\rangle = \langle x_{n-1}, u\rangle + \langle b_{k(n)}, u\rangle, \tag{46}$$

where $k(n) \in \{1, 2\}$.

"$\Leftarrow$": Assume that $d_A(b_1)/d_A(b_2) \in \mathbb{Q}$. Then there exist $q_1, q_2 \in \mathbb{N}^*$ such that $q_1 d_A(b_1) = q_2 d_A(b_2)$, or equivalently (using (43)),

$$q_1\langle b_1, u\rangle + q_2\langle b_2, u\rangle = 0. \tag{47}$$

It follows from Proposition 4.1(iii) that

$$(\forall n \in \mathbb{N}) \quad \langle x_n, u\rangle = \langle x_0, u\rangle + l_{1,n}\langle b_1, u\rangle + l_{2,n}\langle b_2, u\rangle \tag{48}$$

with $(l_{1,n}, l_{2,n}) \in \mathbb{N}^2$. By (47), whenever $l_{1,n} \geq q_1$ and $l_{2,n} \geq q_2$, we have

$$\langle x_n, u\rangle = \langle x_0, u\rangle + (l_{1,n} - q_1)\langle b_1, u\rangle + (l_{2,n} - q_2)\langle b_2, u\rangle. \tag{49}$$

We can thus restrict to considering the sequences $l'_{1,n}, l'_{2,n}$ satisfying (48) and also the additional stipulation that $l'_{1,n} < q_1$ or $l'_{2,n} < q_2$. Then $l'_{1,n}\langle b_1, u\rangle$ or $l'_{2,n}\langle b_2, u\rangle$ is bounded. This together with (44) and (48) implies that both $l'_{1,n}\langle b_1, u\rangle$ and $l'_{2,n}\langle b_2, u\rangle$ are bounded, and so are $l'_{1,n}$ and $l'_{2,n}$. Hence, there exist $L_1, L_2 \in \mathbb{N}$ such that

$$(\forall n \in \mathbb{N}) \quad 0 \leq l'_{1,n} \leq L_1 \quad \text{and} \quad 0 \leq l'_{2,n} \leq L_2. \tag{50}$$

By combining with (45) and (48), $(\forall n \in \mathbb{N}^*)$ $x_n \in S$, where

$$S := \{\langle x_0, u\rangle u + l'_1\langle b_1, u\rangle u + l'_2\langle b_2, u\rangle u + b_k \mid l'_1 = 0, \dots, L_1,\ l'_2 = 0, \dots, L_2,\ k = 1, 2\}, \tag{51}$$

Since S is a finite set, there exist $n_0 \in \mathbb{N}$ and $m \in \mathbb{N}^*$ such that $x_{n_0} = x_{n_0+m}$. It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ cycles between m points $x_{n_0}, \dots, x_{n_0+m-1}$ from n_0 onwards.

"$\Rightarrow$": Assume that $(x_n)_{n\in\mathbb{N}}$ cycles between m points from $n_0 \in \mathbb{N}$ onwards, i.e., $(\forall n \geq n_0)$ $x_{n+m} = x_n$. By (46),

$$\langle x_{n_0}, u\rangle + \sum_{n=n_0}^{n_0+m-1} \langle b_{k(n)}, u\rangle = \langle x_{n_0}, u\rangle. \tag{52}$$

There thus exist $q_1, q_2 \in \mathbb{N}$ such that $q_1 + q_2 = m > 0$ and $q_1\langle b_1, u\rangle + q_2\langle b_2, u\rangle = 0$. Combining with (43) implies that $q_1, q_2 \neq 0$ and that $d_A(b_1)/d_A(b_2) = q_2/q_1 \in \mathbb{Q}$. ■

5 Hyperplane and doubleton: closed-form expressions

In this final section, we refine the previously considered case with the aim of obtaining *closed-form* expressions for the terms of the DR sequence $(x_n)_{n\in\mathbb{N}}$.

Recall from Proposition 4.1(ii) that

$$(\forall n \in \mathbb{N}^*) \quad x_n = \langle x_{n-1}, u\rangle u + b_{k(n)} \quad \text{and} \quad \langle x_n, u\rangle = \langle x_{n-1}, u\rangle + \langle b_{k(n)}, u\rangle, \tag{53}$$

where $k(n) \in \{1, 2\}$ and where

$$k(n) = 1 \;\&\; \langle x_n, u\rangle > \beta - \beta_1 \implies k(n+1) = 1, \tag{54a}$$

$$k(n) = 1 \;\&\; \langle x_n, u\rangle < \beta - \beta_1 \implies k(n+1) = 2, \tag{54b}$$

$$k(n) = 2 \;\&\; \langle x_n, u\rangle > -\beta - \beta_2 \implies k(n+1) = 1. \tag{54c}$$

We note here that if $k(n) = 1$ and $\langle x_n, u\rangle = \beta - \beta_1$, then both 1 and 2 are acceptable values for $k(n+1)$; for the sake of simplicity, we choose $k(n+1) = 2$ in this case. Define

$$S_1 := \left\{ x_n \;\middle|\; n \in \mathbb{N}^*,\ k(n) = 1,\ \langle x_n, u\rangle \in \left]\beta, \beta + \beta_2\right] \right\}, \tag{55a}$$

$$S_2 := \left\{ x_n \;\middle|\; n \in \mathbb{N}^*,\ k(n) = 2,\ \langle x_n, u\rangle \in \left]\beta + \beta_2, \beta - \beta_1 + \beta_2\right] \right\}. \tag{55b}$$

Proposition 5.1. *Let $n \in \mathbb{N}^*$. Then the following hold:*

(i) *If $k(n) = 1$ and $\langle x_n, u\rangle \in \left]\beta - \beta_1, \beta + \beta_2\right]$, then*

$$k(n+1) = 1 \quad \textit{and} \quad \langle x_{n+1}, u\rangle = \langle x_n, u\rangle + \beta_1 \in \left]\beta, \beta + \beta_2\right]. \tag{56}$$

(ii) *If $k(n) = 1$ and $\langle x_n, u\rangle \in \left]\beta, \beta - \beta_1\right]$, then*

$$k(n+1) = 2 \quad \textit{and} \quad \langle x_{n+1}, u\rangle = \langle x_n, u\rangle + \beta_2 \in \left]\beta + \beta_2, \beta - \beta_1 + \beta_2\right]. \tag{57}$$

(iii) *If $k(n) = 2$, $\langle x_n, u\rangle \in \left]\beta + \beta_2, \beta - \beta_1 + \beta_2\right]$ and $\beta + \beta_2 \geq 0$, then*

$$k(n+1) = 1 \quad \textit{and} \quad \langle x_{n+1}, u\rangle = \langle x_n, u\rangle + \beta_1 \in \left]\beta + \beta_1 + \beta_2, \beta + \beta_2\right] \subseteq \left]\beta, \beta + \beta_2\right] \tag{58}$$

Consequently,

$$\left(\beta + \beta_2 \geq 0 \textit{ and } x_n \in S_1 \cup S_2\right) \implies x_{n+1} \in S_1 \cup S_2. \tag{59}$$

Proof. Notice from (53) that

$$(\forall n \in \mathbb{N}) \quad \langle x_{n+1}, u\rangle = \langle x_n, u\rangle + \langle b_{k(n+1)}, u\rangle. \tag{60}$$

(i): Combine (54a) and (60) while noting that $\beta + \beta_1 + \beta_2 < \beta + \beta_2$ by (31).

(ii): Combine (54b) and (60).

(iii): By (31) and the Cauchy–Schwarz inequality, we obtain

$$0 < \beta_2 - \beta_1 = \langle b_2 - b_1, u\rangle \leq \|b_2 - b_1\|\|u\| = \|b_2 - b_1\| \tag{61}$$

and

$$\beta_2 - \beta_1 \le \frac{\|b_2 - b_1\|^2}{\beta_2 - \beta_1} = -2\beta. \tag{62}$$

Now assume that $\beta + \beta_2 \ge 0$. Then $\beta_1 + \beta_2 \ge (2\beta + \beta_2) + \beta_2 = 2(\beta + \beta_2) \ge 0$, and hence $]\beta + \beta_1 + \beta_2, \beta + \beta_2] \subseteq]\beta, \beta + \beta_2]$. It follows from $\langle x_n, u\rangle > \beta + \beta_2 \ge 0$ that $\langle x_n, u\rangle > -\beta - \beta_2$. Now use (54c) and (60).

Finally, assume that $x_n \in S_1 \cup S_2$. If $x_n \in S_2$, then we have from (iii) that $x_{n+1} \in S_1$. If $x_n \in S_1$ and $\langle x_n, u\rangle \in]\beta, \beta - \beta_1]$, then, by (ii), $x_{n+1} \in S_2$. If $x_n \in S_1$ and $\langle x_n, u\rangle \in]\beta - \beta_1, \beta + \beta_2]$, then $x_{n+1} \in S_1$ due to (i). Altogether, $x_{n+1} \in S_1 \cup S_2$. ■

Theorem 5.2 (closed-form expressions). *Suppose that $\beta + \beta_2 \ge 0$ and that $x_1 \in S_1 \cup S_2$. Then*

$$(\forall n \in \mathbb{N}^*) \quad \langle x_n, u\rangle = \langle x_0, u\rangle + n\beta_1 + \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor (\beta_2 - \beta_1) \tag{63a}$$

$$= \langle x_0, u\rangle - \left\lfloor \frac{-\langle x_0, u\rangle + \beta - \beta_1 - (n-1)\beta_2}{\beta_2 - \beta_1} \right\rfloor \beta_1 + \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor \beta_2 \tag{63b}$$

and

$$(\forall n \in \mathbb{N}^*) \quad x_n = \langle x_{n-1}, u\rangle u + b_{k(n)}, \tag{64}$$

where

$$(\forall n \in \mathbb{N}^*) \quad k(n) = \begin{cases} 1 & \text{if } \langle x_n, u\rangle \le \beta + \beta_2, \\ 2 & \text{if } \langle x_n, u\rangle > \beta + \beta_2 \end{cases} \tag{65a}$$

$$= \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor - \left\lfloor \frac{-\langle x_0, u\rangle + \beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor + 1. \tag{65b}$$

Proof. Note that (64) follows from (33). According to Proposition 4.1(iii),

$$(\forall n \in \mathbb{N}) \quad \langle x_n, u\rangle = \langle x_0, u\rangle + (n - l_n)\beta_1 + l_n\beta_2 \quad \text{with} \quad l_n \in \mathbb{N}. \tag{66}$$

Since $x_1 \in S_1 \cup S_2$, Proposition 5.1 yields

$$(\forall n \in \mathbb{N}^*) \quad x_n \in S_1 \cup S_2. \tag{67}$$

Let $n \in \mathbb{N}^*$. It follows from (31) and (67) that $\langle x_n, u\rangle \in]\beta, \beta - \beta_1 + \beta_2]$, which combined with (66) gives

$$\frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} - 1 < l_n \le \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1}. \tag{68}$$

Therefore,

$$l_n = \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor \tag{69}$$

and

$$n - l_n = -\left\lfloor \frac{-\langle x_0, u\rangle + \beta - \beta_1 - (n-1)\beta_2}{\beta_2 - \beta_1} \right\rfloor, \tag{70}$$

which imply (63).

To get (64) and (65), we distinguish two cases.

Case 1: $\langle x_n, u\rangle \le \beta + \beta_2$. On the one hand, by (67) we must have $x_n \in S_1$ and $k(n) = 1$. On the other hand, from $\langle x_n, u\rangle \le \beta + \beta_2$ and (63), noting that $\beta_1 < 0$, we obtain that

$$\left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor \le \frac{-\langle x_0, u\rangle + \beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \tag{71a}$$

$$< \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \tag{71b}$$

which yields

$$\left\lfloor \frac{-\langle x_0, u\rangle + \beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor = \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor, \tag{72}$$

hence (64) and (65) hold.

Case 2: $\langle x_n, u\rangle > \beta + \beta_2$. By (67), $x_n \in S_2$ and $k(n) = 2$. Again using (63) and noting that $\beta_1 < 0 < \beta_2$, we derive that

$$\left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor > \frac{-\langle x_0, u\rangle + \beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \tag{73a}$$

$$= \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} + \frac{\beta_1}{\beta_2 - \beta_1} \tag{73b}$$

$$> \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor - 1. \tag{73c}$$

It follows that

$$\left\lfloor \frac{-\langle x_0, u\rangle + \beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor = \left\lfloor \frac{-\langle x_0, u\rangle + \beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor - 1, \tag{74}$$

and we have (64) and (65). The proof is complete. ■

Corollary 5.3. *Suppose that $\beta_1 > \beta \ge -\beta_2$, that $x_0 \in A$, and that $2\langle x_0, b_1 - b_2\rangle > \|b_1\|^2 - \|b_2\|^2$. Then*

$$(\forall n \in \mathbb{N}) \quad \langle x_n, u\rangle = n\beta_1 + \left\lfloor \frac{\beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor (\beta_2 - \beta_1) \tag{75}$$

and

$$(\forall n \in \mathbb{N}^*) \quad x_n = \left((n-1)\beta_1 + \left\lfloor \frac{\beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor (\beta_2 - \beta_1) \right) u + b_{k(n)}, \tag{76}$$

where

$$(\forall n \in \mathbb{N}^*) \quad k(n) = \left\lfloor \frac{\beta - (n+1)\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor - \left\lfloor \frac{\beta - n\beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor + 1. \tag{77}$$

Proof. From $x_0 \in A$, we have that $\langle x_0, u\rangle = 0$ and also $R_A x_0 = P_A x_0 = x_0$. Since $2\langle x_0, b_1 - b_2\rangle > \|b_1\|^2 - \|b_2\|^2$, it holds that $\|b_1 - x_0\|^2 < \|b_2 - x_0\|^2$, which yields $P_B R_A x_0 = P_B x_0 = b_1$. Therefore, $k(1) = 1$, $x_1 = x_0 - P_A x_0 + P_B R_A x_0 = b_1$, and $\langle x_1, u\rangle = \langle b_1, u\rangle = \beta_1$.

On the other hand, it follows from $\beta_1 > \beta \geq -\beta_2$ and $\beta_1 < 0$ that $\beta + \beta_2 \geq 0$ and that $\beta < \beta_1 < 0 \leq \beta + \beta_2$. We deduce that $\langle x_1, u\rangle = \beta_1 \in \,]\beta, \beta+\beta_2[$, which implies that $x_1 \in S_1$. Using Theorem 5.2, we get (75) for all $n \in \mathbb{N}^*$. When $n = 0$, the right-hand side of (75) becomes

$$\left\lfloor \frac{\beta - \beta_1 + \beta_2}{\beta_2 - \beta_1} \right\rfloor (\beta_2 - \beta_1) = 0 = \langle x_0, u\rangle \tag{78}$$

since $0 < \beta - \beta_1 + \beta_2 < \beta_2 - \beta_1$. Hence, (75) holds for all $n \in \mathbb{N}$, which together with the second part of Theorem 5.2 completes the proof. ■

Example 5.4. Suppose that $X = \mathbb{R}$, that $A = \{0\}$, and that $B = \{b_1, b_2\}$ with $b_1 = -1$ and $b_2 = r$, where $r \in \mathbb{R}$, $r > 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a DR sequence with respect to (A, B) with starting point $x_0 = 0$. Then

$$(\forall n \in \mathbb{N}) \quad x_n = -n + \left\lfloor \frac{n}{r+1} + \frac{1}{2} \right\rfloor (r+1). \tag{79}$$

Proof. Let $u = 1$. Then $A = \{u\}^\perp$ and $(\forall x \in \mathbb{R})$ $\langle x, u\rangle = x$. We have that $\beta_1 = \langle b_1, u\rangle = -1 < 0$, $\beta_2 = \langle b_2, u\rangle = r > 0$, and, since $r > 1$,

$$-1 = \beta_1 > \beta = \frac{|b_1 - b_2|^2}{2(\beta_1 - \beta_2)} = -\frac{(r+1)^2}{2(r+1)} = -\frac{r+1}{2} > -\beta_2 = -r. \tag{80}$$

It is clear that $x_0 = 0 \in A$ and that $2\langle x_0, b_1 - b_2\rangle = 0 > 1 - r^2 = |b_1|^2 - |b_2|^2$. Now applying Corollary 5.3 yields

$$(\forall n \in \mathbb{N}) \quad x_n = \langle x_n, u\rangle = -n + \left\lfloor \frac{-\frac{r+1}{2} + (n+1) + r}{r+1} \right\rfloor (r+1), \tag{81}$$

and the conclusion follows. ■

Example 5.5. Suppose that $X = \mathbb{R}^2$, that $A = \mathbb{R} \times \{0\}$, and that $B = \{b_1, b_2\}$ with $b_1 = (0, -1)$ and $b_2 = (1, r)$, where $r \in \mathbb{R}$, $r \geq \sqrt{2}$. Let $(x_n)_{n\in\mathbb{N}}$ be a DR sequence with respect to (A, B) with starting point $x_0 = (\alpha, 0)$, where $\alpha \in \mathbb{R}$, $\alpha < r^2/2$. Then

$$\begin{aligned}(\forall n \in \mathbb{N}^*) \quad x_n = \Bigg(\left\lfloor \frac{n}{r+1} + \frac{r^2+2r}{2(r+1)} \right\rfloor - \left\lfloor \frac{n-1}{r+1} + \frac{r^2+2r}{2(r+1)} \right\rfloor, \\ -n + \left\lfloor \frac{n}{r+1} + \frac{r^2+2r}{2(r+1)} \right\rfloor (r+1) \Bigg).\end{aligned} \tag{82}$$

Proof. In this case, $A = \{u\}^\perp$ with $u = (0, 1)$, $\beta_1 = \langle b_1, u\rangle = -1 < 0$, $\beta_2 = \langle b_2, u\rangle = r > 0$, and

$$\beta_1 = -1 > \beta = \frac{\|b_1 - b_2\|^2}{2(\beta_1 - \beta_2)} = -\frac{1 + (r+1)^2}{2(r+1)} = -1 - \frac{r^2}{2(r+1)}. \tag{83}$$

On the one hand, $\beta + \beta_2 = \frac{r^2-2}{2(r+1)} \geq 0$. On the other hand, it is straightforward to see that $x_0 \in A$ and that $2\langle x_0, b_1 - b_2\rangle = -2\alpha > -r^2 = \|b_1\|^2 - \|b_2\|^2$. Applying Corollary 5.3, we obtain that

$$(\forall n \in \mathbb{N}^*) \quad \langle x_n, u\rangle = -n + \left\lfloor \frac{-1 - \frac{r^2}{2(r+1)} + (n+1) + r}{r+1} \right\rfloor (r+1) \tag{84a}$$

$$= -n + \left\lfloor \frac{n}{r+1} + \frac{r^2+2r}{2(r+1)} \right\rfloor (r+1). \tag{84b}$$

Now for each $n \in \mathbb{N}^*$, writing $x_n = (\alpha_n, \beta_n) \in \mathbb{R}^2$, we observe that $\beta_n = \langle x_n, u\rangle$ and, by (76), α_n is actually the first coordinate of $b_{k(n)}$, that is,

$$\alpha_n = \begin{cases} 0 & \text{if } k(n) = 1, \\ 1 & \text{if } k(n) = 2, \end{cases} \tag{85}$$

which combined with (77) implies that

$$\alpha_n = k(n) - 1 = \left\lfloor \frac{n}{r+1} + \frac{r^2+2r}{2(r+1)} \right\rfloor - \left\lfloor \frac{n-1}{r+1} + \frac{r^2+2r}{2(r+1)} \right\rfloor . \tag{86}$$

The conclusion follows. ■

Let us specialize Example 5.4 further and also illustrate Theorem 4.2.

Example 5.6 (rational case). Suppose that $X = \mathbb{R}$, that $A = \{0\}$, and that $B = \{-1, 2\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a DR sequence with respect to (A, B) with starting point $x_0 = 0$. Then

$$(\forall n \in \mathbb{N}) \quad x_n = -n + 3\left\lfloor \frac{n}{3} + \frac{1}{2} \right\rfloor \tag{87}$$

and $(x_n)_{n\in\mathbb{N}} = \big(0, -1, 1, 0, -1, 1, 0, -1, 1, \dots\big)$ is periodic. (See also [7, Remark 6] for another cyclic example.)

Proof. Apply Example 5.4 with $b_1 = -1$ and $b_2 = 2$. ■

Example 5.7 (irrational case). Suppose that $X = \mathbb{R}$, that $A = \{0\}$, and that $B = \{-1, \sqrt{2}\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a DR sequence with respect to (A, B) with starting point $x_0 = 0$. Then

$$(\forall n \in \mathbb{N}) \quad x_n = -n + \left\lfloor \frac{n}{\sqrt{2}+1} + \frac{1}{2} \right\rfloor (\sqrt{2}+1) \tag{88}$$

and $(x_n)_{n\in\mathbb{N}} = \big(0, -1, -1+\sqrt{2}, -2+\sqrt{2}, -2+2\sqrt{2}, -3+2\sqrt{2}, -4+2\sqrt{2}, -4+3\sqrt{2}, \dots\big)$ which is not periodic.

Proof. Apply Example 5.4 with $b_1 = -1$ and $b_2 = \sqrt{2}$. ■

Remark 5.8. Some comments on the last examples are in order.

(i) We note that the last examples feature terms resembling (inhomogeneous) Beatty sequences; see [12]. In fact, let us disclose that we started this journey by experimentally investigating Example 5.5 and discovering integer sequences [14] and [15] which eventually led to the more general analysis in this paper.

(ii) Finally, let us contrast the DR algorithm to the method of alternating projections (see, e.g., [2] and [3]) in the setting of Example 5.4: indeed, the sequence $(x_0, P_A x_0, P_B P_A x_0, \dots)$ is simply $(0, 0, -1, 0, -1, 0, \dots)$ regardless of whether or not $r > 1$ is irrational.

6 Conclusion

In this paper, we provided a detailed analysis of the Douglas–Rachford algorithm for the case when one set is a hyperplane and the other a doubleton. We characterized cycling of this method in terms of the ratio of the distances of the points to the hyperplane. Moreover, we presented closed-form expressions of the actual iterates. The results obtained show the surprising complexity of this algorithm when compared to, e.g., the method of alternating projections.

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