On sums and convex combinations of projectors onto convex sets

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Abstract

The projector onto the Minkowski sum of closed convex sets is generally not equal to the sum of individual projectors. In this work, we provide a complete answer to the question of characterizing the instances where such an equality holds. Our results unify and extend the case of linear subspaces and Zarantonello's results for projectors onto cones. A detailed analysis in the case of convex combinations is carried out, and we also establish the partial sum property for projectors onto convex cones.

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1 Introduction

Throughout this paper, we assume that

$$\mathcal{H} \text{ is a real Hilbert space} \tag{1}$$

with inner product $\langle \cdot \mid \cdot \rangle$ and induced norm $\|\cdot\|$. Now assume that[1]

$$(C_i)_{i\in I} \text{ is a finite family of nonempty closed convex subsets of } \mathcal{H} \tag{2}$$

with corresponding projectors

$$(P_{C_i})_{i\in I} \tag{3}$$

and that

$$(\alpha_i)_{i\in I} \text{ are real numbers.} \tag{4}$$

In this paper, we analyze carefully the question: *When is $\sum_{i\in I}\alpha_i P_{C_i}$ a projector?* This allows us to provide a complete answer to the question "*When is the sum of projectors also a projector?*" (In view of Proposition 2.4(iii), an affirmative answer to this question requires the sum $\sum_{i\in I} C_i$ to be closed. This happens, for instance, when each set is bounded.) It is known that, in the case of linear subspaces, $\sum_{i\in I} P_{C_i}$ is a projector onto a closed linear subspace if and only if $(C_i)_{i\in I}$ is pairwise orthogonal; see [15, Theorem 2,

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[1] For basic Convex Analysis, we refer the reader to [5, 24, 26, 27].

p. 46]. This question is also of interest in Quantum Mechanics [18, p. 50]. In 1971, Zarantonello [27] answered this question in the case of convex cones, i.e., if $(C_i)_{i\in I}$ are cones, then $\sum_{i\in I} P_{C_i}$ is a projector if and only if $(P_{C_i})_{i\in I}$ is pairwise orthogonal in the sense that, for every $(i,j) \in I \times I$ with $i \neq j$, we have $(\forall x \in \mathcal{H})\ \langle P_{C_i}x \mid P_{C_j}x\rangle = 0$. However, the question remains open in the general convex case. Therefore, one goal of this paper is to provide necessary and sufficient conditions for $\sum_{i\in I} \alpha_i P_{C_i}$ to be a projector without any further assumption on the sets $(C_i)_{i\in I}$. As a consequence, we answer entirely the question "When is the sum of projectors also a projector?" Our results unify the two aforementioned results and make a connection with the recent work [2] where it was proven that, if the sum of a family of proximity operators is a proximity operator, then every partial sum remains a proximity operator. Interestingly, we shall see that this property is still valid in the class of projectors onto convex cones; in other words, if a finite sum of projectors onto convex cones is a projector, then so are its partial sums. Nevertheless, this result fails outside the world of convex cones. Another goal is to characterize the instances where a convex average of $(P_{C_i})_{i\in I}$ is again a projector. Complementary to a result in 1963 by Moreau [21], which states that a convex average of proximity operators is always a proximity operator, we shall see in Theorem 4.3 that taking convex combinations does not preserve the class of projectors onto convex sets (see Theorem 4.3 for the rigorous statement). Our main results are summarized as follows:

- We provide a new characterization of proximity operators in Theorem 3.1 (for a list of other characterizations, see [12]). In turn, we derive a new characterization of projectors (Theorem 3.2), which is a pillar of this paper and a variant of [27, Theorem 4.1]. Furthermore, we also partially answer an open question by Zarantonello regarding [27, Theorem 4.1].

- Theorem 3.10 characterizes (without any additional assumptions on the underlying sets) when $\sum_{i\in I} \alpha_i P_{C_i}$ is a projector; Theorem 3.12 concerns the sum $\sum_{i\in I} P_{C_i}$.

- By specifying our analysis to the case of convex average in Theorem 4.3, we explicitly determine families of closed convex sets that are preserved under taking convex combinations.

- We present the *partial sum property* (see [2, Theorem 4.2]) for projectors onto convex cones in Theorem 5.7, whose proof is based on Theorem 5.3 and [2, Theorem 4.2]. We also recover [27, Theorems 5.3 and 5.5].

The paper is organized as follows. In Section 2, we collect miscellaneous results that will be used in the sequel. Our main results are presented in Section 3: Theorem 3.1 provides a characterization of proximity operators, while projectors are dealt with in Theorem 3.2, which is a variant of [27, Theorem 4.1]. This allows us to recover the classical characterization of orthogonal projectors; see, e.g., [25, Theorem 4.29]. In turn, we establish a necessary and sufficient condition for a linear combination of projectors to be a projector in Theorem 3.10 and then particularize to sums of projectors in Theorem 3.12. We then specialize the analysis of Section 3 to convex combinations of projectors in Section 4. In Section 5, we show that, in the case of sums of projectors, Theorem 3.12 covers the result obtained by Zarantonello ([27, Theorem 5.5]) and the case of linear subspaces. Furthermore, we provide Theorem 5.3 and Theorem 5.7 to illustrate the connection between our work and [2, 27]. Finally, we turn to a generalization of the classical result [15, Theorem 2, p. 46] in Section 6. Various examples are given to illustrate the necessity of our assumptions.

The notation used in this paper is standard and mainly follows [5]. We write $A := B$ to indicate that A is defined to be B. We set $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ := [0, +\infty[$, $\mathbb{R}_{++} :=]0, +\infty[$, $\mathbb{R}_- :=]-\infty, 0]$, and $\mathbb{R}_{--} :=]-\infty, 0[$. The closed ball in $\mathcal{H}$ with center $x \in \mathcal{H}$ and radius $\rho \in \mathbb{R}_{++}$ is $\mathrm{B}(x;\rho) := \{y \in \mathcal{H} \mid \|y - x\| \leqslant \rho\}$. It is convenient to set

$$\mathrm{q} := \tfrac{1}{2}\|\cdot\|^2, \tag{5}$$

where $\nabla \mathrm{q} = \mathrm{Id}$ is the *identity operator* on $\mathcal{H}$. Let C be a subset of $\mathcal{H}$. Then we denote by $\overline{C}$ the closure of C (with respect to the norm topology on $\mathcal{H}$), by d_C its *distance function*, by $C^{\ominus}$ its *polar cone*, i.e., $C^{\ominus} := \{u \in \mathcal{H} \mid \sup\langle C \mid u\rangle \leqslant 0\}$, and by $C^{\perp}$ its *orthogonal complement*. Next, the *indicator* and *support functions* of C are

$$\iota_C \colon \mathcal{H} \to [-\infty, +\infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise} \end{cases} \tag{6}$$

and

$$\sigma_C \colon \mathcal{H} \to [-\infty, +\infty] : u \mapsto \sup\langle C \mid u\rangle, \tag{7}$$

respectively. Moreover, if C is convex, closed, and nonempty, then the projector associated with C is denoted by P_C. In turn, we set

$$\texttt{Proj}(\mathcal{H}) := \Big\{ P_C \;\Big|\; \mathcal{H} \supseteq C \text{ is convex, closed, and nonempty} \Big\}. \tag{8}$$

Next, the set of convex, lower semicontinuous, and proper functions from $\mathcal{H}$ to $\left]-\infty, +\infty\right]$ is $\Gamma_0(\mathcal{H})$. The *domain* of a function $f \colon \mathcal{H} \to [-\infty, +\infty]$ is $\operatorname{dom} f := \{x \in \mathcal{H} \mid f(x) < +\infty\}$ with closure $\overline{\operatorname{dom}} f$, its *graph* is denoted by $\operatorname{gra} f$, its *conjugate* is denoted by f^*, and, when f is proper, its *subdifferential* is denoted by ∂f. Furthermore, if $f \in \Gamma_0(\mathcal{H})$, then we denote its *proximity operator* by Prox_f and its *Moreau envelope* by $\operatorname{env} f$, i.e., $\operatorname{env} f := f \,\square\, \mathrm{q} = f \,\boxdot\, \mathrm{q}$, where $\square$ and $\boxdot$ denote the infimal convolution and the exact infimal convolution, respectively. Next, let $T \colon \mathcal{H} \to \mathcal{H}$. The *range* of T is $\operatorname{ran} T$ with closure $\overline{\operatorname{ran}}\, T$. If $T \in \mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, then its *adjoint* is denoted by T^*. Finally, we adopt the convention that *empty sums are zero*.

2 Auxiliary results

In this section, we provide various results that will be useful in the sequel. Let us start with a simple identity in $\mathcal{H}$.

Lemma 2.1 *Let $x \in \mathcal{H}$, let $(x_i)_{i \in I}$ be a finite family in $\mathcal{H}$, let $(\alpha_i)_{i \in I}$ be a family in $\mathbb{R}$, and set $\alpha := \sum_{i \in I} \alpha_i$. Then the following hold:*

(i) $\|x - \sum_{i \in I} \alpha_i x_i\|^2 = (1-\alpha)\|x\|^2 + \sum_{i \in I} \alpha_i \|x - x_i\|^2 + (\alpha - 1)\sum_{i \in I} \alpha_i \|x_i\|^2 - \frac{1}{2}\sum_{i \in I}\sum_{j \in I} \alpha_i\alpha_j \|x_i - x_j\|^2$.

(ii) *Suppose that $(\forall i \in I)\ \alpha_i = 1$. Then $\alpha = \operatorname{card} I$ and*

$$(\alpha - 1)\sum_{i \in I}\|x_i\|^2 - \frac{1}{2}\sum_{i \in I}\sum_{j \in I}\|x_i - x_j\|^2 = \sum_{\substack{(i,j) \in I \times I \\ i \neq j}} \langle x_i \mid x_j\rangle \tag{9}$$

and

$$\Big\| x - \sum_{i \in I} x_i \Big\|^2 = (1-\alpha)\|x\|^2 + \sum_{i \in I}\|x - x_i\|^2 + \sum_{\substack{(i,j) \in I \times I \\ i \neq j}} \langle x_i \mid x_j\rangle. \tag{10}$$

Proof. (i): Without loss of generality, assume that $I = \{1, \ldots, m\}$, where $m \in \mathbb{N} \smallsetminus \{0\}$. Let us proceed by induction on m.

Base case: When $m = 1$, by applying [5, Corollary 2.15] to $(x - x_1, x_1)$ and noticing that $\alpha = \alpha_1$, we obtain

$$\|x - \alpha_1 x_1\|^2 = \|(1-\alpha)x + \alpha(x - x_1)\|^2 \tag{11a}$$

$$= (1-\alpha)\|x\|^2 + \alpha\|x - x_1\|^2 - (1-\alpha)\alpha\|x_1\|^2 \tag{11b}$$

$$= (1-\alpha)\|x\|^2 + \alpha_1\|x - x_1\|^2 + (\alpha-1)\alpha_1\|x_1\|^2 - \tfrac{1}{2}\alpha_1\alpha_1\|x_1 - x_1\|^2. \tag{11c}$$

Inductive step: Assume that $m \geqslant 2$ and that the result holds for families containing $m-1$ or fewer elements. Moreover, set $J := \{1, \ldots, m-1\}$ and $\beta := \sum_{j\in J}\alpha_j$. Then, by the base case, we have

$$\alpha_m^2\|x_m\|^2 - 2\alpha_m\langle x_m \mid x\rangle = \|x - \alpha_m x_m\|^2 - \|x\|^2 = -\alpha_m\|x\|^2 + \alpha_m\|x - x_m\|^2 + (\alpha_m - 1)\alpha_m\|x_m\|^2. \tag{12}$$

Hence, since $\beta + \alpha_m = \alpha$, we infer from the induction hypothesis that

$$\begin{aligned}
\Big\|x - \sum_{i\in I}\alpha_i x_i\Big\|^2 &= \Big\|\Big(x - \sum_{j\in J}\alpha_j x_j\Big) - \alpha_m x_m\Big\|^2 && (13a)\\
&= \Big\|x - \sum_{j\in J}\alpha_j x_j\Big\|^2 + \alpha_m^2\|x_m\|^2 - 2\alpha_m\Big\langle x_m \,\Big|\, x - \sum_{j\in J}\alpha_j x_j\Big\rangle && (13b)\\
&= \Big\|x - \sum_{j\in J}\alpha_j x_j\Big\|^2 + \Big(\alpha_m^2\|x_m\|^2 - 2\alpha_m\langle x_m \mid x\rangle\Big) + 2\alpha_m\sum_{j\in J}\alpha_j\langle x_m \mid x_j\rangle && (13c)\\
&= \Big\|x - \sum_{j\in J}\alpha_j x_j\Big\|^2 - \alpha_m\|x\|^2 + \alpha_m\|x - x_m\|^2 + (\alpha_m - 1)\alpha_m\|x_m\|^2\\
&\quad + \alpha_m\sum_{j\in J}\alpha_j\Big(\|x_m\|^2 + \|x_j\|^2 - \|x_m - x_j\|^2\Big) && (13d)\\
&= (1-\beta)\|x\|^2 + \sum_{j\in J}\alpha_j\|x - x_j\|^2 + (\beta - 1)\sum_{j\in J}\alpha_j\|x_j\|^2 - \tfrac{1}{2}\sum_{j\in J}\sum_{k\in J}\alpha_j\alpha_k\|x_j - x_k\|^2\\
&\quad - \alpha_m\|x\|^2 + \alpha_m\|x - x_m\|^2 + \alpha_m\Big(-1 + \alpha_m + \sum_{j\in J}\alpha_j\Big)\|x_m\|^2\\
&\quad + \alpha_m\sum_{j\in J}\alpha_j\|x_j\|^2 - \sum_{j\in J}\alpha_m\alpha_j\|x_m - x_j\|^2 && (13e)\\
&= (1 - \beta - \alpha_m)\|x\|^2 + \sum_{i\in I}\alpha_i\|x - x_i\|^2 + (\beta + \alpha_m - 1)\sum_{j\in J}\alpha_j\|x_j\|^2\\
&\quad - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j\|x_i - x_j\|^2 + \alpha_m(\alpha - 1)\|x_m\|^2 && (13f)\\
&= (1-\alpha)\|x\|^2 + \sum_{i\in I}\alpha_i\|x - x_i\|^2 + (\alpha - 1)\sum_{i\in I}\alpha_i\|x_i\|^2 - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j\|x_i - x_j\|^2, && (13g)
\end{aligned}$$

which completes the induction argument.

(ii): Since $(\forall i \in I)\,\alpha_i = 1$, we have $\alpha = \operatorname{card} I$, and thus

$$\begin{aligned}
&(\alpha - 1)\sum_{i\in I}\|x_i\|^2 - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\|x_i - x_j\|^2\\
&\quad = \sum_{i\in I}\Big((\alpha - 1)\|x_i\|^2 - \tfrac{1}{2}\sum_{j\in I\smallsetminus\{i\}}\|x_i - x_j\|^2\Big) && (14a)\\
&\quad = \sum_{i\in I}\Big((\alpha - 1)\|x_i\|^2 - \tfrac{1}{2}\sum_{j\in I\smallsetminus\{i\}}\big(\|x_i\|^2 + \|x_j\|^2 - 2\langle x_i \mid x_j\rangle\big)\Big) && (14b)\\
&\quad = \sum_{i\in I}\Big((\alpha - 1)\|x_i\|^2 - \tfrac{1}{2}(\alpha - 1)\|x_i\|^2 - \tfrac{1}{2}\sum_{j\in I\smallsetminus\{i\}}\|x_j\|^2 + \sum_{j\in I\smallsetminus\{i\}}\langle x_i \mid x_j\rangle\Big) && (14c)
\end{aligned}$$

$$= \tfrac{1}{2}(\alpha-1)\sum_{i\in I}\|x_i\|^2 - \tfrac{1}{2}\sum_{i\in I}\Big(-\|x_i\|^2 + \sum_{j\in I}\|x_j\|^2\Big) + \sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle x_i \,|\, x_j\rangle \tag{14d}$$

$$= \tfrac{1}{2}(\alpha-1)\sum_{i\in I}\|x_i\|^2 + \tfrac{1}{2}\sum_{i\in I}\|x_i\|^2 - \tfrac{\alpha}{2}\sum_{i\in I}\|x_i\|^2 + \sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle x_i \,|\, x_j\rangle \tag{14e}$$

$$= \sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle x_i \,|\, x_j\rangle, \tag{14f}$$

and hence (9) holds. Consequently, (10) follows from (i) and (9). ■

We shall need the following identities involving convex cones.

Lemma 2.2 *Let K and S be nonempty closed convex cones in $\mathcal{H}$. Then the following hold:*

(i) $(\forall x\in\mathcal{H})\ \|P_K x\|^2 = \langle x \,|\, P_K x\rangle$.

(ii) $(\forall x\in\mathcal{H})\ \langle P_{K^\ominus} x \,|\, P_{S^\ominus} x\rangle + \|P_K x\|^2 + \|P_S x\|^2 = \|x\|^2 + \langle P_K x \,|\, P_S x\rangle$.

Proof. Take $x\in\mathcal{H}$. (i): We derive from [5, Theorem 6.30(i)&(ii)] that $\|P_K x\|^2 = \langle P_K x \,|\, P_K x\rangle = \langle x - P_{K^\ominus} x \,|\, P_K x\rangle = \langle x \,|\, P_K x\rangle$, as claimed. (ii): The Moreau conical decomposition ([20]) and (i) give

$$\langle P_{K^\ominus} x \,|\, P_{S^\ominus} x\rangle = \langle x - P_K x \,|\, x - P_S x\rangle \tag{15a}$$
$$= \|x\|^2 - \langle x \,|\, P_S x\rangle - \langle x \,|\, P_K x\rangle + \langle P_K x \,|\, P_S x\rangle \tag{15b}$$
$$= \|x\|^2 - \|P_S x\|^2 - \|P_K x\|^2 + \langle P_K x \,|\, P_S x\rangle, \tag{15c}$$

and the assertion follows. ■

Fact 2.3 Let C be a nonempty closed convex subset of $\mathcal{H}$. Then the following hold:

(i) P_C is maximally monotone.

(ii) P_C is 3^* monotone[2].

Proof. (i): See [5, Example 20.32]. (ii): Because P_C is firmly nonexpansive by [5, Proposition 4.16], the conclusion follows from [5, Example 25.20(ii)]. ■

In the finite-dimensional case, Proposition 2.4(ii) can also be deduced from [8, Theorem 3.15]. Furthermore, let us point out that Proposition 2.4(iii) generalizes Zarantonello's [27, Theorem 5.4].

Proposition 2.4 *Let $(C_i)_{i\in I}$ be a finite family of nonempty closed convex subsets of $\mathcal{H}$, let $(\alpha_i)_{i\in I}$ be a family in $\mathbb{R}$, and set $\alpha := \sum_{i\in I}\alpha_i$. Then the following hold:*

(i) *For every $x\in\mathcal{H}$,*

$$\mathrm{q}\Big(x - \sum_{i\in I}\alpha_i P_{C_i} x\Big) = \tfrac{1}{2}\sum_{i\in I}\alpha_i d^2_{C_i}(x) - (\alpha-1)\mathrm{q}(x) + (\alpha-1)\sum_{i\in I}\alpha_i \mathrm{q}(P_{C_i}x) - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j \mathrm{q}(P_{C_i}x - P_{C_j}x). \tag{16}$$

[2] A monotone operator $A\colon \mathcal{H}\to 2^{\mathcal{H}}$ is 3^* *monotone* if $(\forall (x,u)\in \operatorname{dom} A\times \operatorname{ran} A)\ \inf_{(y,v)\in \operatorname{gra} A}\langle x-y \,|\, u-v\rangle > -\infty$.

(ii) *Suppose that* $(\forall i \in I)\ \alpha_i \geqslant 0$. *Then* $\overline{\operatorname{ran}} \sum_{i\in I} \alpha_i P_{C_i} = \overline{\sum_{i\in I} \alpha_i C_i}$.

(iii) *Suppose that* $(\forall i \in I)\ \alpha_i \geqslant 0$ *and that there exists a closed convex set* C *such that* $\sum_{i\in I} \alpha_i P_{C_i} = P_C$. *Then* $\sum_{i\in I} \alpha_i C_i$ *is closed and* $C = \sum_{i\in I} \alpha_i C_i$.

Proof. (i): Let x be in $\mathcal{H}$. Apply Lemma 2.1(i) to $(x, (P_{C_i}x)_{i\in I}, (\alpha_i)_{i\in I})$ and notice that $(\forall i \in I)\ \|x - P_{C_i}x\| = d_{C_i}(x)$.

(ii): Because the operators $(P_{C_i})_{i\in I}$ are 3^* monotone by Fact 2.3(ii) and because $(\forall i \in I)$ $\operatorname{dom} P_{C_i} = \mathcal{H}$, we derive from [10, Lemma 3.1(ii)] that $\overline{\operatorname{ran}} \sum_{i\in I} \alpha_i P_{C_i} = \overline{\sum_{i\in I} \alpha_i \operatorname{ran} P_{C_i}} = \overline{\sum_{i\in I} \alpha_i C_i}$, as desired.

(iii): It follows from (ii) and our assumption that

$$\sum_{i\in I} \alpha_i C_i \subseteq \overline{\sum_{i\in I} \alpha_i C_i} = \overline{\operatorname{ran}} \sum_{i\in I} \alpha_i P_{C_i} = \overline{\operatorname{ran}}\, P_C = C = \operatorname{ran} P_C = \operatorname{ran} \sum_{i\in I} \alpha_i P_{C_i} \subseteq \sum_{i\in I} \alpha_i C_i. \tag{17}$$

Thus, we conclude that $\sum_{i\in I} \alpha_i C_i = C$ and that $\sum_{i\in I} \alpha_i C_i$ is closed. ■

Remark 2.5 Proposition 2.4(ii)&(iii) may fail if $(\exists i \in I)\ \alpha_i < 0$. Indeed, in the setting of Proposition 2.4, suppose that $I = \{1,2\}$, that $C_1 = C_2$, and that $\alpha_1 = -\alpha_2 = 1$. Then $\alpha_1 P_{C_1} + \alpha_2 P_{C_2} = 0 = P_{\{0\}}$, but $\alpha_1 C_1 + \alpha_2 C_2 = C_1 - C_1 \neq \{0\}$ if C_1 is not a singleton.

The proofs of the following two results are omitted but can be found in [3, Lemmata 2.6 and 2.7].

Lemma 2.6 *Let* $T\colon \mathcal{H} \to \mathcal{H}$ *be monotone and positively homogeneous*[3]. *Then* $T0 = 0$.

The following is a variant of [28, Lemma 6.1].

Lemma 2.7 *Let* $f\colon \mathcal{H} \to \mathbb{R}$ *be Gâteaux differentiable on* $\mathcal{H}$, *and suppose that* ∇f *is monotone and positively homogeneous. Then*

$$(\forall x \in \mathcal{H}) \quad f(x) = \tfrac{1}{2}\langle x \mid \nabla f(x)\rangle + f(0). \tag{18}$$

Recall from [19, pp. 89–90] that, if $f\colon \mathcal{H} \to \left]-\infty, +\infty\right]$, then the *Fréchet subdifferential* of f at $x \in \operatorname{dom} f$ is

$$\hat{\partial} f(x) = \left\{ u \in \mathcal{H} \;\middle|\; \varliminf_{x\neq y\to x} \frac{f(y) - f(x) - \langle u \mid y - x\rangle}{\|y - x\|} \geqslant 0 \right\}. \tag{19}$$

Lemma 2.8 *Let* $f\colon \mathcal{H} \to \mathbb{R}$ *and* $z \in \mathcal{H}$. *Suppose that* f *is Fréchet differentiable on* $\mathcal{H}$. *Then*

$$(\forall \varepsilon \in \mathbb{R}_{++}) \quad \hat{\partial}(f + \varepsilon d_{\{z\}})(z) = \nabla f(z) + \varepsilon\, \mathrm{B}(0;1)\,. \tag{20}$$

Proof. Fix $\varepsilon \in \mathbb{R}_{++}$. Since f is Fréchet differentiable and $d_{\{z\}}$ is convex, we derive from [19, Proposition 1.107 and Theorem 1.93] that $\hat{\partial}(g + \varepsilon d_{\{z\}})(z) = \nabla f(z) + \hat{\partial}(\varepsilon d_{\{z\}})(z) = \nabla f(z) + \partial(\varepsilon d_{\{z\}})(z) = \nabla f(z) + \varepsilon \partial d_{\{z\}}(z)$. Hence, in view of [5, Example 16.62] (applied to $C = \{z\}$), (20) follows. ■

3 Main results

Theorem 3.1 (Characterization theorem for proximity operators) *Let* $\varphi \in \Gamma_0(\mathcal{H})$, *let* $T\colon \mathcal{H} \to \mathcal{H}$, *and set* $f := \varphi \circ T + \mathrm{q} \circ (\mathrm{Id} - T)$. *Then the following are equivalent:*

(i) $T = \operatorname{Prox}_\varphi$.

[3] A mapping $F\colon \mathcal{H} \to \mathcal{H}$ is *positively homogeneous* if $(\forall x \in \mathcal{H})(\forall \lambda \in \mathbb{R}_{++})\ F(\lambda x) = \lambda F x$. Note that we do not require $F0 = 0$.

(ii) *T is monotone, $\operatorname{gra}(\varphi + \iota_{\operatorname{ran} T})$ is a dense subset of $\operatorname{gra} \varphi$, and f is Gâteaux differentiable on $\mathcal{H}$ with $\nabla f = \operatorname{Id} - T$.*

Furthermore, if (i) or (ii) holds, then $f = \operatorname{env} \varphi$ and f is Fréchet differentiable on $\mathcal{H}$.

Proof. "(i)⇒(ii)": First, by [5, Example 20.30], $T = \operatorname{Prox}_\varphi$ is monotone. Next, since $\varphi \in \Gamma_0(\mathcal{H})$ and $T = \operatorname{Prox}_\varphi$, [5, Eq. (24.3)] gives $\operatorname{ran} T = \operatorname{dom} \partial\varphi$, and hence, according to [5, Proposition 16.38], it follows that $\operatorname{gra}(\varphi + \iota_{\operatorname{ran} T})$ is a dense subset of $\operatorname{gra} \varphi$. Finally, in view of [5, Remark 12.24], we see that $f = \varphi \circ T + \mathrm{q} \circ (\operatorname{Id} - T) = \varphi \circ \operatorname{Prox}_\varphi + \mathrm{q} \circ (\operatorname{Id} - \operatorname{Prox}_\varphi) = \operatorname{env} \varphi$, and [5, Proposition 12.30] thus entails that f is Fréchet (thus Gâteaux) differentiable on $\mathcal{H}$ with $\nabla f = \operatorname{Id} - \operatorname{Prox}_\varphi = \operatorname{Id} - T$.

"(i)⇐(ii)": Set $g := \mathrm{q} - f$. Then, on the one hand, because q and f are Gâteaux differentiable, so is g. On the other hand, since $\nabla \mathrm{q} = \operatorname{Id}$ and $\nabla f = \operatorname{Id} - T$, we infer that $\nabla g = \nabla(\mathrm{q} - f) = \nabla \mathrm{q} - \nabla f = T$, which is monotone by assumption. Altogether, [5, Proposition 17.7] yields the convexity of g. Therefore, since g is Gâteaux differentiable on $\mathcal{H}$, it follows from [5, Proposition 17.48(i)] that g is lower semicontinuous on $\mathcal{H}$. To sum up, we have shown that

$$g = \mathrm{q} - f \text{ belongs to } \Gamma_0(\mathcal{H}) \text{ and is Gâteaux differentiable on } \mathcal{H} \text{ with } \nabla g = T. \tag{21}$$

Moreover, (21) and [5, Corollary 13.38] yield

$$g^* \in \Gamma_0(\mathcal{H}). \tag{22}$$

In turn, set $h := g^* - \mathrm{q}$. Let us now establish that

$$h = \varphi \text{ on } \operatorname{ran} T. \tag{23}$$

Towards this goal, fix $u \in \operatorname{ran} T$, say $u = Tx \overset{(21)}{=} \nabla g(x)$, where $x \in \mathcal{H}$. Then (21), [5, Proposition 17.35], and the very definitions of g and f assert that

$$\begin{aligned}
h(u) = g^*(u) - \mathrm{q}(u) &= g^*(\nabla g(x)) - \mathrm{q}(Tx) && \text{(24a)}\\
&= \langle x \mid \nabla g(x) \rangle - g(x) - \mathrm{q}(Tx) && \text{(24b)}\\
&= \langle x \mid Tx \rangle - \mathrm{q}(x) + f(x) - \mathrm{q}(Tx) && \text{(24c)}\\
&= \langle x \mid Tx \rangle - \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x - Tx\|^2 + \varphi(Tx) - \tfrac{1}{2}\|Tx\|^2 && \text{(24d)}\\
&= \varphi(Tx) && \text{(24e)}\\
&= \varphi(u). && \text{(24f)}
\end{aligned}$$

Hence, (23) holds. Next, fix $v \in \operatorname{dom} h$, and we shall prove that $\varphi(v) \leqslant h(v)$. Indeed, on the one hand, because $h = g^* - \mathrm{q}$ and $\operatorname{dom} \mathrm{q} = \mathcal{H}$, we have $\operatorname{dom} h = \operatorname{dom} g^*$. On the other hand, due to (21) and [5, Corollary 16.30], $\operatorname{dom} \partial g^* = \operatorname{dom}(\partial g)^{-1} = \operatorname{ran} \partial g$, and since $\operatorname{ran} \partial g = \operatorname{ran} \nabla g = \operatorname{ran} T$ thanks to (21) and [5, Proposition 17.31(i)], we deduce that $\operatorname{dom} \partial g^* = \operatorname{ran} T$. Altogether, because $v \in \operatorname{dom} h = \operatorname{dom} g^*$, (22) and [5, Proposition 16.38] ensures the existence of a sequence $(v_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} \partial g^* = \operatorname{ran} T$ such that $v_n \to v$ and $g^*(v_n) \to g^*(v)$. Therefore, by the definition of h, we get $h(v_n) = g^*(v_n) - \mathrm{q}(v_n) \to g^*(v) - \mathrm{q}(v) = h(v)$. However, because $\{v_n\}_{n\in\mathbb{N}} \subseteq \operatorname{ran} T$ and $v_n \to v$, the lower semicontinuity of φ and (23) imply that $h(v) = \lim h(v_n) = \lim \varphi(v_n) \geqslant \varphi(v)$. Hence, we have established that

$$(\forall v \in \operatorname{dom} h) \quad h(v) \geqslant \varphi(v). \tag{25}$$

Next, let us show that

$$(\forall w \in \operatorname{dom} \varphi) \quad h(w) \leqslant \varphi(w). \tag{26}$$

To this end, let $w \in \operatorname{dom} \varphi$. Then, since $\operatorname{gra}(\varphi + \iota_{\operatorname{ran} T})$ is a dense subset of $\operatorname{gra} \varphi$ by assumption, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $\operatorname{ran} T$ such that $w_n \to w$ and $\varphi(w_n) \to \varphi(w)$. In turn, since h is lower semicontinuous by (22), we infer from (23) that $\varphi(w) = \lim \varphi(w_n) = \lim h(w_n) \geqslant h(w)$, from which (26) follows. Consequently, combining (25) and (26) yields $h = \varphi$. Finally, since $\varphi \in \Gamma_0(\mathcal{H})$, it follows from (21), the definition of h, the Fenchel–Moreau theorem, and [5, Proposition 24.4] that $\operatorname{Prox}_\varphi = \nabla(\varphi + \mathrm{q})^* = \nabla(h+\mathrm{q})^* = \nabla g^{**} = \nabla g = T$, as desired. ■

In [27], Zarantonello provided a necessary and sufficient condition in terms of a differential equation for an operator on $\mathcal{H}$ to be a projector. The proof there, however, is not within the scope of Convex Analysis. He also conjectured (see the paragraph after [27, Corollary 2, p. 306]) that the Fréchet differentiability of the operator P in [27, Theorem 4.1] can be replaced by the Gâteaux one. By assuming the monotonicity of P instead of the Lipschitz continuity, we provide below an affirmative answer. The next result, which plays a crucial role in determining whether a sum of projectors is a projector (see Theorem 3.12 below), is a variant of [27, Theorem 4.1] with a proof rooted in Convex Analysis.

Theorem 3.2 (Characterization theorem for projectors) *Let $T\colon \mathcal{H} \to \mathcal{H}$, and set $f := \mathrm{q} \circ (\mathrm{Id} - T)$. Then the following are equivalent:*

(i) $T \in \mathtt{Proj}(\mathcal{H})$.

(ii) *T is monotone, f is Gâteaux differentiable on $\mathcal{H}$, and $\nabla f = \mathrm{Id} - T$.*

If (i) or (ii) holds, then $\operatorname{ran} T$ is closed and convex, $T = P_{\operatorname{ran} T}$, and $f = (1/2)d^2_{\operatorname{ran} T}$ is Fréchet differentiable on $\mathcal{H}$.

Proof. Set $\varphi := \iota_{\overline{\operatorname{ran}}\, T}$.

"(i)⇒(ii)": Suppose that $T = P_C$, where C is convex, closed, and nonempty. Then clearly $\operatorname{ran} T = \operatorname{ran} P_C = C$ is closed and convex. This implies that $\varphi = \iota_{\operatorname{ran} T} \in \Gamma_0(\mathcal{H})$ and that $T = P_{\operatorname{ran} T} = \operatorname{Prox}_{\iota_{\operatorname{ran} T}} = \operatorname{Prox}_\varphi$. In turn, because $f = \varphi \circ T + \mathrm{q} \circ (\mathrm{Id} - T)$ by the definition of φ, we infer from Theorem 3.1 that (ii) holds and, moreover, $f = \operatorname{env} \varphi = \operatorname{env} \iota_{\operatorname{ran} T} = (1/2)d^2_{\operatorname{ran} T}$ is Fréchet differentiable on $\mathcal{H}$.

"(i)⇐(ii)": We first show that $\overline{\operatorname{ran}}\, T$ is convex. Indeed, by our assumption, $\mathrm{q} - f$ is Gâteaux differentiable on $\mathcal{H}$ with

$$\nabla(\mathrm{q} - f) = \nabla \mathrm{q} - \nabla f = \mathrm{Id} - (\mathrm{Id} - T) = T. \tag{27}$$

Thus, since T is monotone, [5, Proposition 17.7] ensures that $\mathrm{q} - f$ is convex, and thus, the Gâteaux differentiability of $\mathrm{q} - f$ and [5, Proposition 17.48(i)] imply that $\mathrm{q} - f \in \Gamma_0(\mathcal{H})$. Hence, due to (27) and [5, Proposition 17.31(i)], Moreau's theorem [23] (see also [5, Theorem 20.25]) asserts that $T = \nabla(\mathrm{q} - f)$ is maximally monotone. Consequently, [5, Corollary 21.14] yields the convexity of $\overline{\operatorname{ran}}\, T$, as claimed. In turn, on the one hand, this implies that $\varphi = \iota_{\overline{\operatorname{ran}}\, T} \in \Gamma_0(\mathcal{H})$. On the other hand, we deduce from the definition of φ that $f = \varphi \circ T + \mathrm{q} \circ (\mathrm{Id} - T)$ and $\operatorname{gra}(\varphi + \iota_{\operatorname{ran} T}) = \operatorname{gra}(\iota_{\overline{\operatorname{ran}}\, T \cap \operatorname{ran} T}) = \operatorname{gra} \iota_{\operatorname{ran} T} = \operatorname{ran} T \times \{0\}$ is a dense subset of $\overline{\operatorname{ran}}\, T \times \{0\} = \operatorname{gra} \iota_{\overline{\operatorname{ran}}\, T} = \operatorname{gra} \varphi$. Thus, the implication "(ii)⇒(i)" of Theorem 3.1 and our assumption guarantee that $T = \operatorname{Prox}_\varphi = \operatorname{Prox}_{\iota_{\overline{\operatorname{ran}}\, T}} = P_{\overline{\operatorname{ran}}\, T}$, which completes the proof. ■

Remark 3.3 Consider the implication "(ii)⇒(i)" of Theorem 3.2. If we merely assume that T is defined on a proper open subset D of $\mathcal{H}$, then, although there may exist a closed set C such that T is the restriction to D of the projector onto C, the set C may fail to be convex. An example can be constructed as follows. Suppose that $\mathcal{H} \neq \{0\}$, and set

$$T\colon \mathcal{H} \smallsetminus \{0\} \to \mathcal{H} : x \mapsto \frac{x}{\|x\|}, \quad f := \mathrm{q} \circ (\mathrm{Id} - T), \quad \text{and } C := \{x \in \mathcal{H} \mid \|x\| = 1\}, \tag{28}$$

i.e., C is the unit sphere of $\mathcal{H}$. Then clearly C is a closed nonconvex set and T is the restriction to $\mathcal{H} \smallsetminus \{0\}$ of the set-valued projector P_C. Thus, in the light of [5, Example 20.12], T is monotone. Next, since

$(\forall x \in \mathcal{H} \smallsetminus \{0\})\, f(x) = (1/2)\|(1-1/\|x\|)x\|^2 = (1/2)(\|x\|-1)^2 = \mathrm{q}(x) - \|x\| + 1/2$, we infer that f is Fréchet differentiable on $\mathcal{H} \smallsetminus \{0\}$ and

$$(\forall x \in \mathcal{H} \smallsetminus \{0\}) \quad \nabla f(x) = x - \frac{x}{\|x\|} = x - Tx. \tag{29}$$

Open Problem 3.4 We do not know whether the monotonicity of T can be omitted in Theorem 3.2. Nevertheless, on the one hand, the following remark might be useful in finding counterexamples if one thinks the answer is negative; on the other hand, Proposition 3.6 provides information on the set $\operatorname{Fix} T$ in the absence of monotonicity.

Remark 3.5 ([13]) Consider the setting of Theorem 3.2 and suppose that $\mathcal{H} = \mathbb{R}^2$. Set $F := \mathrm{Id} - T$ and $(\forall (x,y) \in \mathcal{H}^2)\ F(x,y) := (F_1(x,y), F_2(x,y))$. Now assume that f is Fréchet differentiable on $\mathcal{H}$ with $\nabla f = \mathrm{Id} - T = F$; in addition, suppose that F_1 and F_2 are continuously differentiable. Then, since $(F_1, F_2) = \nabla f$, it follows that $\partial f/\partial x = F_1$ and that $\partial f/\partial y = F_2$. Hence, due to Schwarz's theorem (see, e.g., [11, Theorem 4.1]),

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x}. \tag{30}$$

However, because $\nabla f = F$, a direct computation gives

$$\begin{cases} F_1(x,y) = F_1(x,y)\dfrac{\partial F_1}{\partial x}(x,y) + F_2(x,y)\dfrac{\partial F_2}{\partial x}(x,y) = F_1(x,y)\dfrac{\partial F_1}{\partial x}(x,y) + F_2(x,y)\dfrac{\partial F_1}{\partial y}(x,y) \\[2ex] F_2(x,y) = F_1(x,y)\dfrac{\partial F_1}{\partial y}(x,y) + F_2(x,y)\dfrac{\partial F_2}{\partial y}(x,y) = F_1(x,y)\dfrac{\partial F_2}{\partial x}(x,y) + F_2(x,y)\dfrac{\partial F_2}{\partial y}(x,y). \end{cases} \tag{31}$$

In the first equation of (31), one can try to solve for F_1 in term of F_2, and vise versa by using the second one. This approach recovers projectors onto linear subspaces of $\mathbb{R}^2$ and might suggest a nonmonotone solution of the equation $\nabla f = \mathrm{Id} - T$. In addition, it is worth noticing that the function $g \colon x \mapsto \|x - Tx\|$ satisfies the eikonal equation (see, e.g., [1]), i.e., $(\forall x \in \mathcal{H} \smallsetminus \overline{\operatorname{Fix} T})\ \|\nabla g(x)\| = 1$. This might give us some insights into Open Problem 3.4.

Proposition 3.6 *Let $T \colon \mathcal{H} \to \mathcal{H}$, and set $f := \mathrm{q} \circ (\mathrm{Id} - T)$. Suppose that f is Fréchet differentiable on $\mathcal{H}$ with $\nabla f = \mathrm{Id} - T$. Then $\operatorname{Fix} T \neq \varnothing$.*

Proof. Let us proceed by contradiction and therefore assume that $\operatorname{Fix} T = \varnothing$. Then clearly $(\forall x \in \mathcal{H})\, f(x) > 0$. Hence, because $f \colon \mathcal{H} \to \mathbb{R}_{++}$ is Fréchet differentiable with $\nabla f = \mathrm{Id} - T$ and $\sqrt{\cdot} \colon \mathbb{R}_{++} \to \mathbb{R}$ is Fréchet differentiable, we deduce from [14, Theorem 5.1.11(b)] that $g := \sqrt{\cdot} \circ (2f)$ is Fréchet differentiable on $\mathcal{H}$ (thus continuous) and

$$(\forall x \in \mathcal{H}) \quad \nabla g(x) = \frac{2\nabla f(x)}{2\sqrt{2f(x)}} = \frac{x - Tx}{\|x - Tx\|}. \tag{32}$$

Now let $\varepsilon \in \left]0,1\right[$. Since g is bounded below and continuous, Ekeland's variational principle (see, e.g., [5, Theorem 1.46(iii)]) applied to g and $(\alpha, \beta) = (\varepsilon^2, \varepsilon)$ yields the existence of $z \in \mathcal{H}$ such that $(\forall x \in \mathcal{H} \smallsetminus \{z\})\, g(z) + \varepsilon d_{\{z\}}(z) = g(z) < g(x) + \varepsilon d_{\{z\}}(x)$. This guarantees that z is the unique minimizer of $g + \varepsilon d_{\{z\}}$. Thus, [19, Proposition 1.114], Lemma 2.8, and (32) imply that

$$0 \in \hat{\partial}(g + \varepsilon d_{\{z\}})(z) = \nabla g(z) + \varepsilon\, \mathrm{B}(0;1) = \frac{z - Tz}{\|z - Tz\|} + \varepsilon\, \mathrm{B}(0;1)\,, \tag{33}$$

which is absurd since $\varepsilon \in \left]0,1\right[$ and $\|(z - Tz)/(\|z - Tz\|)\| = 1$. ■

Remark 3.7 Consider the setting and the assumption of Proposition 3.6.

(i) Zarantonello established in the proof of [27, Theorem 4.1] that, if (in addition to our assumption) T is Lipschitz continuous, then $\operatorname{Fix} T \neq \varnothing$. However, we do not need the Lipschitz continuity of T in our proof.

(ii) Suppose, in addition, that ∇f is continuous. Then we obtain an alternative proof as follows. Assume to the contrary that $\operatorname{Fix} T = \varnothing$. Then $g := \sqrt{\cdot} \circ (2f)$ is continuously Fréchet differentiable on $\mathcal{H}$ (hence continuous) with

$$(\forall x \in \mathcal{H}) \quad \nabla g(x) = \frac{x - Tx}{\|x - Tx\|}. \tag{34}$$

Fix $\varepsilon \in \left]0,1\right[$. Since g is bounded below and continuous, Ekeland's variational principle implies that there exists $z \in \mathcal{H}$ such that $(\forall x \in \mathcal{H} \smallsetminus \{z\})\, g(z) + \varepsilon d_{\{z\}}(z) = g(z) < g(x) + \varepsilon d_{\{z\}}(x)$. Thus, z is a minimizer of $g + \varepsilon d_{\{z\}}(z)$. Therefore, because $d_{\{z\}}$ is convex, in view of [26, Theorem 3.2.4(iii)&(vi)&(ii)] and [5, Example 16.62], we see that

$$0 \in \nabla g(z) + \varepsilon \partial d_{\{z\}}(z) = \nabla g(z) + \varepsilon\, \mathrm{B}(0;1) = \frac{z - Tz}{\|z - Tz\|} + \varepsilon\, \mathrm{B}(0;1)\,, \tag{35}$$

which contradicts the fact that $\varepsilon \in \left]0,1\right[$.

By specializing Theorem 3.2 to positively homogeneous operators on $\mathcal{H}$, we obtain a characterization for projectors onto closed convex cones.

Corollary 3.8 *Let $T\colon \mathcal{H} \to \mathcal{H}$ and set $f := \mathrm{q} \circ T$. Then the following are equivalent:*

(i) *There exists a nonempty closed convex cone K such that $T = P_K$.*

(ii) *T is monotone and positively homogeneous, f is Gâteaux differentiable on $\mathcal{H}$, and $\nabla f = T$.*

If (i) or (ii) holds, then $K = \operatorname{ran} T$.

Proof. "(i)⇒(ii)": Clearly $\operatorname{ran} T = \operatorname{ran} P_K = K$. Now, it follows from [5, Example 20.32] that $T = P_K$ is monotone. Next, because K is a nonempty closed convex cone, [5, Proposition 29.29] guarantees that T is positively homogeneous. In turn, since $f = \mathrm{q} \circ T = \mathrm{q} \circ P_K$, [5, Proposition 12.32 and Lemma 2.61(i)] yield the Gâteaux differentiability of f and, moreover, $\nabla f = \nabla(\mathrm{q} \circ P_K) = P_K = T$, as desired.

"(i)⇐(ii)": First, since T is positively homogeneous,

$$\operatorname{ran} T \text{ is a cone in } \mathcal{H}. \tag{36}$$

Now set $g\colon \mathcal{H} \to \mathbb{R} : x \mapsto (1/2)\langle x \mid Tx\rangle = (1/2)\langle x \mid \nabla f(x)\rangle$ and $h := \mathrm{q} \circ (\mathrm{Id} - T)$. Since $\nabla f = T$ is monotone and positively homogeneous by assumption, Lemma 2.7 ensures that g is Gâteaux differentiable on $\mathcal{H}$ and $\nabla g = \nabla f = T$. Thus, because $h = \mathrm{q} - 2g + f$, it follows that h is Gâteaux differentiable on $\mathcal{H}$ with gradient $\nabla h = \nabla \mathrm{q} - 2\nabla g + \nabla f = \mathrm{Id} - 2T + T = \mathrm{Id} - T$. Consequently, since T is monotone, we conclude via Theorem 3.2 (applied to h) and (36) that $\operatorname{ran} T$ is a closed convex cone in $\mathcal{H}$ and that $T = P_{\operatorname{ran} T}$. ■

In Corollary 3.8, if T is a bounded linear operator, then we recover the following well-known characterization of orthogonal projectors. (For proofs, see, e.g., [3, Corollary 3.9] or [25, Theorem 4.29].)

Corollary 3.9 *Let $L\colon \mathcal{H} \to \mathcal{H}$. Then the following are equivalent:*

(i) *There exists a closed linear subspace V of $\mathcal{H}$ such that $L = P_V$.*

(ii) $L \in \mathcal{B}(\mathcal{H})$ *and* $L = L^* = L^2$.

(iii) $L \in \mathcal{B}(\mathcal{H})$ *and* $L = L^*L$.

If one of (i)–(iii) holds, then $V = \operatorname{ran} L$.

Theorem 3.10 (Linear combination of projectors) *Let* $(C_i)_{i\in I}$ *be a finite family of nonempty closed convex subsets of* $\mathcal{H}$, *let* $(\alpha_i)_{i\in I}$ *be a family in* $\mathbb{R}$, *and set* $\alpha := \sum_{i\in I}\alpha_i$. *Then, there exists a closed convex set* C *such that* $\sum_{i\in I}\alpha_i P_{C_i} = P_C$ *if and only if* $\sum_{i\in I}\alpha_i P_{C_i}$ *is monotone and*

$$(\exists \gamma\in\mathbb{R})(\forall x\in\mathcal{H})\quad (\alpha-1)\sum_{i\in I}\alpha_i \mathrm{q}(P_{C_i}x) - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j \mathrm{q}(P_{C_i}x - P_{C_j}x) = \gamma; \tag{37}$$

in which case,

$$d_C^2 = \sum_{i\in I}\alpha_i d_{C_i}^2 - 2(\alpha-1)\mathrm{q} + 2\gamma. \tag{38}$$

Proof. Set $T := \sum_{i\in I}\alpha_i P_{C_i}$, set $f := \mathrm{q}\circ(\mathrm{Id}-T)$, and define

$$g\colon \mathcal{H}\to\mathbb{R} : x\mapsto (\alpha-1)\sum_{i\in I}\alpha_i \mathrm{q}(P_{C_i}x) - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j \mathrm{q}(P_{C_i}x - P_{C_j}x). \tag{39}$$

In view of Proposition 2.4(i), we have

$$(\forall x\in\mathcal{H})\quad f(x) = \tfrac{1}{2}\sum_{i\in I}\alpha_i d_{C_i}^2(x) - (\alpha-1)\mathrm{q}(x) + g(x). \tag{40}$$

Now assume that there exists a nonempty closed convex subset C of $\mathcal{H}$ such that $T = P_C$. Then, due to Fact 2.3(i), we see that T is monotone. Next, on the one hand, since $T = P_C$, it follows from Theorem 3.2 that f is Fréchet differentiable on $\mathcal{H}$ and $\nabla f = \mathrm{Id} - T = \mathrm{Id} - \sum_{i\in I}\alpha_i P_{C_i}$. On the other hand, for every $i\in I$, since C_i is convex, closed, and nonempty, we infer that $d_{C_i}^2 = 2\mathrm{q}\circ(\mathrm{Id}-P_{C_i})$ is Fréchet differentiable on $\mathcal{H}$ with $\nabla d_{C_i}^2 = 2(\mathrm{Id}-P_{C_i})$. Altogether, since $\alpha = \sum_{i\in I}\alpha_i$ by definition, it follows from (40) that g is Fréchet differentiable on $\mathcal{H}$ and that

$$\nabla g = \nabla f - \nabla\Big(\tfrac{1}{2}\sum_{i\in I}\alpha_i d_{C_i}^2 - (\alpha-1)\mathrm{q}\Big) = \Big(\mathrm{Id} - \sum_{i\in I}\alpha_i P_{C_i}\Big) - \sum_{i\in I}\alpha_i(\mathrm{Id}-P_{C_i}) + (\alpha-1)\mathrm{Id} = 0. \tag{41}$$

Consequently, there exists $\gamma\in\mathbb{R}$ such that $(\forall x\in\mathcal{H})\ g(x)=\gamma$. Conversely, assume that T is monotone and that (37) holds. Then, we derive from (40) that

$$f = \tfrac{1}{2}\sum_{i\in I}\alpha_i d_{C_i}^2 - (\alpha-1)\mathrm{q} + \gamma, \tag{42}$$

and it thus follows that f is Fréchet differentiable on $\mathcal{H}$ and, since $\alpha = \sum_{i\in I}\alpha_i$, $\nabla f = \sum_{i\in I}\alpha_i(\mathrm{Id}-P_{C_i}) - (\alpha-1)\mathrm{Id} = \mathrm{Id} - \sum_{i\in I}\alpha_i P_{C_i} = \mathrm{Id} - T$. Hence, since T is monotone by our assumption, Theorem 3.2 ensures the existence of a nonempty closed convex set C such that $T = P_C$. Therefore, $f = \mathrm{q}\circ(\mathrm{Id}-P_C) = (1/2)d_C^2$ and (38) follows from (42). ■

Remark 3.11 As we have seen in Remark 2.5, the set C in Theorem 3.10 need not be $\sum_{i\in I}\alpha_i C_i$.

We now establish a necessary and sufficient condition under which a finite sum of projectors is a projector.

Theorem 3.12 (Sum of projectors) *Let $(C_i)_{i\in I}$ be a finite family of nonempty closed convex subsets of $\mathcal{H}$, and set $\alpha := \operatorname{card} I$. Then $\sum_{i\in I} P_{C_i} \in \mathtt{Proj}(\mathcal{H})$ if and only if*

$$(\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H}) \qquad \sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle P_{C_i}x \mid P_{C_j}x\rangle = \gamma; \tag{43}$$

in which case, $\sum_{i\in I} C_i$ is a closed convex set,

$$\sum_{i\in I} P_{C_i} = P_{\sum_{i\in I} C_i}, \tag{44}$$

and

$$d^2_{\sum_{i\in I} C_i} = \sum_{i\in I} d^2_{C_i} - 2(\alpha-1)\mathrm{q} + \gamma. \tag{45}$$

Proof. Since it is clear that $\sum_{i\in I} P_{C_i}$ is monotone, we derive from Theorem 3.10 (applied to $(C_i)_{i\in I}$, $(\alpha_i)_{i\in I} = (1)_{i\in I}$, and $\alpha = \operatorname{card} I = \sum_{i\in I} 1$) and (9) that

$$\begin{aligned}
\sum_{i\in I} P_{C_i} \in \mathtt{Proj}(\mathcal{H}) &\Leftrightarrow (\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H})\ (\alpha-1)\sum_{i\in I}\mathrm{q}(P_{C_i}x) - \tfrac{1}{2}\sum_{i\in I}\sum_{j\in I}\mathrm{q}(P_{C_i}x - P_{C_j}x) = \gamma && \text{(46a)}\\
&\Leftrightarrow (\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H})\ \tfrac{1}{2}\sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle P_{C_i}x \mid P_{C_j}x\rangle = \gamma && \text{(46b)}\\
&\Leftrightarrow \text{(43)}, && \text{(46c)}
\end{aligned}$$

as desired. Next, suppose that $\sum_{i\in I} P_{C_i} \in \mathtt{Proj}(\mathcal{H})$. Then, there exists a closed convex set C such that

$$\sum_{i\in I} P_{C_i} = P_C; \tag{47}$$

therefore, as we have shown above, there exists $\gamma \in \mathbb{R}$ such that

$$(\forall x \in \mathcal{H}) \qquad \sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle P_{C_i}x \mid P_{C_j}x\rangle = \gamma. \tag{48}$$

According to Proposition 2.4(iii) and (47), we see that $\sum_{i\in I} C_i = C$ is a closed convex set, from which and (47) we get (44). Furthermore, it follows from (10) and (48) that

$$\begin{aligned}
(\forall x \in \mathcal{H}) \quad d^2_C(x) = \|x - P_Cx\|^2 &= \Big\|x - \sum_{i\in I} P_{C_i}x\Big\|^2 && \text{(49a)}\\
&= (1-\alpha)\|x\|^2 + \sum_{i\in I}\|x - P_{C_i}x\|^2 + \sum_{\substack{(i,j)\in I\times I\\ i\neq j}} \langle P_{C_i}x \mid P_{C_j}x\rangle && \text{(49b)}\\
&= \sum_{i\in I} d^2_{C_i}(x) - 2(\alpha-1)\mathrm{q}(x) + \gamma, && \text{(49c)}
\end{aligned}$$

and (45) follows. ■

Corollary 3.13 *Let C and D be nonempty closed convex subsets of $\mathcal{H}$. Then the following are equivalent:*

(i) $P_C + P_D \in \mathtt{Proj}(\mathcal{H})$.

(ii) $(\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H})\ \langle P_Cx \mid P_Dx\rangle = \gamma$.

If (i) *or* (ii) *holds, then $C + D$ is a closed convex set,*

$$P_C + P_D = P_{C+D}, \tag{50}$$

and

$$d^2_{C+D} = d^2_C + d^2_D - 2\mathrm{q} + 2\gamma. \tag{51}$$

Remark 3.14 Consider the setting of Corollary 3.13. In view of [5, Example 12.3], we see that (51) is equivalent to $(\iota_C \,\square\, \iota_D) \,\square\, \mathrm{q} = \iota_C \,\square\, \mathrm{q} + \iota_D \,\square\, \mathrm{q} - \mathrm{q} + \gamma$. Hence, using [5, Proposition 13.24(i)] and Moreau's decomposition [22] (see also [5, Remark 14.4]), we infer that

$$\begin{aligned}
(51) &\Leftrightarrow \mathrm{q} - (\iota_C \,\square\, \iota_D)^* \,\square\, \mathrm{q} = -(\iota_C^* \,\square\, \mathrm{q}) - (\iota_D^* \,\square\, \mathrm{q}) + \mathrm{q} + \gamma && \text{(52a)}\\
&\Leftrightarrow \mathrm{q} - (\iota_C^* + \iota_D^*) \,\square\, \mathrm{q} = -(\iota_C^* \,\square\, \mathrm{q}) - (\iota_D^* \,\square\, \mathrm{q}) + \mathrm{q} + \gamma && \text{(52b)}\\
&\Leftrightarrow (\iota_C^* + \iota_D^*) \,\square\, \mathrm{q} = \iota_C^* \,\square\, \mathrm{q} + \iota_D^* \,\square\, \mathrm{q} - \gamma. && \text{(52c)}
\end{aligned}$$

This type of relationship is used in [12, Proposition 3.16] to establish a condition for the sum of two proximity operators to be a proximity operator.

The following simple example shows that the constant γ in Corollary 3.13 can take on any value.

Example 3.15 Let u and v be in $\mathcal{H}$, set $C := \{u\}$, and set $D := \{v\}$. Then clearly $P_C + P_D = P_{\{u+v\}} = P_{C+D}$ and $(\forall x \in \mathcal{H})\ \langle P_C x \mid P_D x\rangle = \langle u \mid v\rangle$.

As a consequence of Corollary 3.13, a sum of projectors onto orthogonal sets is a projector; see [7, Proposition 2.6] for a different derivation.

Corollary 3.16 *Let C and D be nonempty closed convex subsets of $\mathcal{H}$ such that $C \perp D$. Then the following hold:*

(i) *$C + D$ is a nonempty closed convex set.*

(ii) $P_C + P_D = P_{C+D}$.

(iii) $d^2_{C+D} = d^2_C + d^2_D - 2\mathrm{q}$.

Proof. Since $(\forall x \in \mathcal{H})\ \langle P_C x \mid P_D x\rangle = 0$, the conclusions readily follow from Corollary 3.13. ■

We now provide an instance where item (ii) of Corollary 3.13 holds, $C \not\subseteq D^\perp$ in general, and neither C nor D is a cone.

Example 3.17 Let K be a nonempty closed convex cone in $\mathcal{H}$, let ρ_1 and ρ_2 be in $\mathbb{R}_{++}$, set $C := K \cap B(0;\rho_1)$, and set $D := K^\ominus \cap B(0;\rho_2)$. It then immediately follows from [4, Theorem 7.1] and [5, Theorem 6.30(ii)] that

$$(\forall x \in \mathcal{H}) \quad \langle P_C x \mid P_D x\rangle = \left\langle \frac{\rho_1}{\max\{\|P_K x\|, \rho_1\}} P_K x \,\middle|\, \frac{\rho_2}{\max\{\|P_{K^\ominus} x\|, \rho_2\}} P_{K^\ominus} x \right\rangle = 0. \tag{53}$$

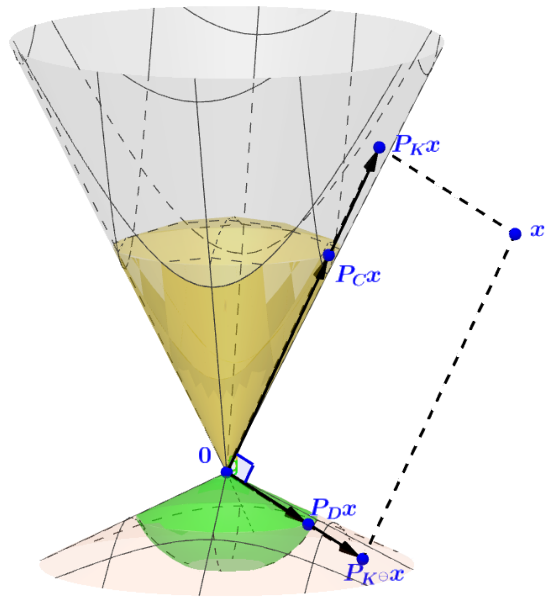

Figure 1. A `GeoGebra` [17] snapshot illustrating the sets C (yellow) and D (green) in the setting of Example 3.17.

We next establish a necessary and sufficient condition for $u + P_C$ to be a projector.

Example 3.18 Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $u \in \mathcal{H}$. Then, since $(\forall x \in \mathcal{H})\, u = P_{\{u\}}x$, we deduce from Corollary 3.13 that

$$
\begin{aligned}
u + P_C = P_{\{u\}} + P_C \in \mathtt{Proj}(\mathcal{H}) &\Leftrightarrow (\exists \gamma \in \mathbb{R})(\forall x \in \mathcal{H})\ \langle u \mid P_C x\rangle = \gamma && \text{(54a)}\\
&\Leftrightarrow (\exists \gamma \in \mathbb{R})(\forall x \in C)\ \langle u \mid x\rangle = \gamma && \text{(54b)}\\
&\Leftrightarrow (\forall x \in C)(\forall y \in C)\ \langle u \mid x\rangle = \langle u \mid y\rangle && \text{(54c)}\\
&\Leftrightarrow (\forall x \in C)(\forall y \in C)\ \langle u \mid x - y\rangle = 0 && \text{(54d)}\\
&\Leftrightarrow u \in (C - C)^{\perp}; && \text{(54e)}
\end{aligned}
$$

in which case, $u + P_C = P_{u+C}$ due to Corollary 3.13.

Remark 3.19 Consider the setting of Example 3.18. Since $u + P_C$ is monotone, nonexpansive, and a sum of proximity operators, [2, Corollary 2.5] guarantees that $u + P_C$ is a proximity operator. However, by Example 3.18, it is not a projector unless $u \in (C - C)^{\perp}$.

Here is a sufficient, but not necessary, condition for a sum of projectors to be a projector.

Corollary 3.20 *Let $m \geqslant 2$ be an integer, set $I := \{1, \ldots, m\}$, let $(C_i)_{i\in I}$ be a family of nonempty closed convex subsets of $\mathcal{H}$, and set $C := \sum_{i\in I} C_i$. Suppose that, for every $(i,j) \in I \times I$ with $i < j$, there exists $\gamma_{i,j} \in \mathbb{R}$ such that $(\forall x \in \mathcal{H})\ \langle P_{C_i}x \mid P_{C_j}x\rangle = \gamma_{i,j}$. Then C is a closed convex set and $\sum_{i\in I} P_{C_i} = P_C$.*

Proof. Set $(\forall k \in I)\ D_k := \sum_{i=1}^{k} C_i$, and let us establish that

$$
(\forall k \in I \smallsetminus \{1\}) \qquad D_k \text{ is a closed convex set and } \sum_{i=1}^{k} P_{C_i} = P_{D_k}. \tag{55}
$$

Due to Corollary 3.13, the claim holds if $k = 2$, and we therefore assume that, for some $k \in \{2, \ldots, m-1\}$, D_k is a closed convex set and that $\sum_{i=1}^{k} P_{C_i} = P_{D_k}$. Then, by our assumption, $(\forall x \in \mathcal{H})\ \langle P_{D_k}x \mid P_{C_{k+1}}x\rangle =$

$\sum_{i=1}^{k} \langle P_{C_i} x \mid P_{C_{k+1}} x\rangle = \sum_{i=1}^{k} \gamma_{i,k+1}$, from which and Corollary 3.13 (applied to D_k and C_{k+1}) we infer that $D_{k+1} = D_k + C_{k+1}$ is a closed convex set and, due to the induction hypothesis, $\sum_{i=1}^{k+1} P_{C_i} = \sum_{i=1}^{k} P_{C_i} + P_{C_{k+1}} = P_{D_k} + P_{C_{k+1}} = P_{D_k + C_{k+1}} = P_{D_{k+1}}$. Hence, letting $k = m$ in (55) yields the conclusion. ∎

We now illustrate that the assumption of Corollary 3.20 need not hold when merely $\sum_{i\in I} P_{C_i} = P_C$.

Example 3.21 Let C be a nonempty closed convex subset of $\mathcal{H}$ such that $\mathcal{H} \smallsetminus (C-C)^{\perp} \neq \varnothing$, and suppose that $u \in \mathcal{H} \smallsetminus (C-C)^{\perp}$. Then $P_{\{u\}} + P_{\{-u\}} + P_C = P_C$ is a projector. However, if $x \mapsto \langle P_{\{u\}} x \mid P_C x\rangle = \langle u \mid P_C x\rangle$ were a constant, then it would follow from Corollary 3.13 that $u + P_C = P_{\{u\}} + P_C$ is a projector, which violates Example 3.18 and the assumption that $u \notin (C-C)^{\perp}$.

We conclude this section with a result concerning the difference of two projectors.

Proposition 3.22 *Let C and D be nonempty closed convex subsets of $\mathcal{H}$. Then $P_D - P_C \in \mathtt{Proj}(\mathcal{H})$ if and only if $P_D - P_C$ is monotone and there exists $\gamma \in \mathbb{R}$ such that $(\forall x \in \mathcal{H})\ \langle P_C x \mid P_D x - P_C x\rangle = \gamma$.*

Proof. Using Theorem 3.10 with $I = \{1,2\}$, $(C_1, C_2) = (D, C)$, and $(\alpha_1, \alpha_2) = (1, -1)$, we infer that $P_D - P_C \in \mathtt{Proj}(\mathcal{H})$ if and only if $P_D - P_C$ is monotone and there exists $\gamma \in \mathbb{R}$ such that $(\forall x \in \mathcal{H})\ -\gamma = -(\mathrm{q}(P_D x) - \mathrm{q}(P_C x)) + \mathrm{q}(P_C x - P_D x) = \langle P_C x \mid P_C x - P_D x\rangle$, which is the desired conclusion. ∎

Remark 3.23 Zarantonello in [27, Lemma 5.12] showed that if C and D are nonempty closed convex *cones* in $\mathcal{H}$, then $P_D - P_C \in \mathtt{Proj}(\mathcal{H})$ if and only if $P_C P_D = P_C$, in which case $P_D - P_C = P_{D\cap C^{\ominus}}$. For instance, when $\mathcal{H} = \mathbb{R}^2$, $C = \mathbb{R}_+ \times \{0\}$, and $D = \mathbb{R}_+ \times \mathbb{R}_+$, the difference operator $P_D - P_C$ is the projector onto $\{0\} \times \mathbb{R}_+$.

4 Convex combination of projectors

The analysis of this section requires the following results.

Fact 4.1 (Zarantonello) Let $(T_i)_{i\in I}$ be a finite family of firmly nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$, let $(\alpha_i)_{i\in I}$ be real numbers in $]0,1]$ such that $\sum_{i\in I} \alpha_i = 1$, and let C be a nonempty closed convex subset of $\mathcal{H}$. Then $\sum_{i\in I} \alpha_i T_i = P_C$ if and only if there exist vectors $(u_i)_{i\in I}$ in $\mathcal{H}$ such that $(\forall i \in I)\ T_i = P_C + u_i$ and $\sum_{i\in I} \alpha_i u_i = 0$.

Proof. See [27, Theorem 1.3]. ∎

Lemma 4.2 *Let C and D be nonempty closed convex subsets of $\mathcal{H}$, and set $v := P_{\overline{D-C}} 0$. Then the following hold:*

(i) *Let $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ be sequences in C and D, respectively, and suppose that $d_n - c_n \to v$. Then $d_n - P_C d_n \to v$.*

(ii) *Suppose that there exists $u \in \mathcal{H}$ such that $P_D = P_C + u$. Then $u = v \in (C-C)^{\perp}$ and $D = C + v$.*

(iii) *Suppose that there exists $\gamma \in \mathbb{R}$ such that $(\forall x \in \mathcal{H})\ \|P_D x - P_C x\| = \gamma$. Then $v \in (C-C)^{\perp}$ and $D = C + v$.*

Proof. (i): See [6, Proposition 2.5(i)].

(ii): Since $P_D = P_C + u$, Example 3.18 guarantees that $u \in (C-C)^{\perp}$ and that $D = C + u$. Hence, it suffices to show that $u = v$. Indeed, since $v = P_{\overline{D-C}} 0 \in \overline{D-C}$, there exist sequences $(c_n)_{n\in\mathbb{N}}$ in C and $(d_n)_{n\in\mathbb{N}}$ in D such that $d_n - c_n \to v$. Thus, we deduce from (i) that

$$d_n - P_C d_n \to v. \tag{56}$$

On the other hand, since $P_D = P_C + u$ and $(d_n)_{n\in\mathbb{N}}$ is a sequence in D, it follows that $(\forall n \in \mathbb{N})\, u = P_D d_n - P_C d_n = d_n - P_C d_n$. This and (56) yield $u = v$, as claimed.

(iii): Let $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ be sequences in C and D, respectively, such that $d_n - c_n \to v$. According to (i), $d_n - P_C d_n \to v$, and therefore, $\|d_n - P_C d_n\| \to \|v\|$. However, since $(\forall n \in \mathbb{N})\, d_n \in D$, it follows from our assumption that $(\forall n \in \mathbb{N})\, \gamma = \|P_D d_n - P_C d_n\| = \|d_n - P_C d_n\|$. Hence, invoking the assumption once more, $(\forall x \in \mathcal{H})\, \|P_{\overline{D-C}}0\| = \|v\| = \gamma = \|P_D x - P_C x\|$. Consequently, since $(\forall x \in \mathcal{H})\, P_D x - P_C x \in \overline{D-C}$, we conclude via [9, Lemma 2.4] that $(\forall x \in \mathcal{H})\, P_D x - P_C x = P_{\overline{D-C}}0 = v$. Now apply (ii). ■

Here is our main result of this section.

Theorem 4.3 (Convex combination of projectors) *Let $(C_i)_{i\in I}$ be a finite family of nonempty closed convex subsets of $\mathcal{H}$, let $k \in I$, and set $(\forall i \in I)\, v_i := P_{\overline{C_i - C_k}}0$. Then the following are equivalent:*

(i) *There exists $(\alpha_i)_{i\in I}$ in $]0,1]^I$ such that $\sum_{i\in I} \alpha_i = 1$ and $\sum_{i\in I} \alpha_i P_{C_i} \in \mathtt{Proj}(\mathcal{H})$.*

(ii) *For every $(i,j) \in I \times I$, there exists $\alpha \in \mathbb{R} \smallsetminus \{0,1\}$ such that $(1-\alpha)P_{C_i} + \alpha P_{C_j} \in \mathtt{Proj}(\mathcal{H})$.*

(iii) *For every $i \in I$, we have $v_i \in (C_k - C_k)^\perp$ and $C_i = C_k + v_i$.*

(iv) *$\left\{\sum_{i\in I} \alpha_i P_{C_i} \,\middle|\, (\alpha_i)_{i\in I} \in \mathbb{R}^I \text{ and } \sum_{i\in I} \alpha_i = 1\right\} \subseteq \mathtt{Proj}(\mathcal{H})$.*

Furthermore, each of the above implies that, for every $(\alpha_i)_{i\in I} \in \mathbb{R}^I$ such that $\sum_{i\in I} \alpha_i = 1$, we have

$$\sum_{i\in I} \alpha_i P_{C_i} = P_{C_k + \sum_{i\in I} \alpha_i v_i}. \tag{57}$$

Proof. "(i)⇒(iii)": Suppose that there exist $(\alpha_i)_{i\in I} \in\,]0,1]^I$ and a nonempty closed convex subset C of $\mathcal{H}$ such that $\sum_{i\in I} \alpha_i = 1$ and $\sum_{i\in I} \alpha_i P_{C_i} = P_C$. Then, since $(P_{C_i})_{i\in I}$ are firmly nonexpansive by [5, Proposition 4.16], Fact 4.1 guarantees the existence of vectors $(u_i)_{i\in I}$ in $\mathcal{H}$ such that

$$(\forall i \in I) \quad P_{C_i} = P_C + u_i. \tag{58}$$

Now fix $i \in I$. We then derive from (58) that $P_{C_i} = (P_{C_k} - u_k) + u_i = P_{C_k} + u_i - u_k$, and it thus follows from Lemma 4.2(ii) (applied to $(C_k, C_i, u_i - u_k)$) that $v_i \in (C_k - C_k)^\perp$ and $C_i = C_k + v_i$, as required.

"(iii)⇒(iv)": Let $(\alpha_i)_{i\in I} \in \mathbb{R}^I$ be such that $\sum_{i\in I} \alpha_i = 1$. Then, since $(\forall i \in I)\, v_i \in (C_k - C_k)^\perp$, it follows that $\sum_{i\in I} \alpha_i v_i \in (C_k - C_k)^\perp$. In turn, because $\sum_{i\in I} \alpha_i = 1$, our assumption and Example 3.18 yield $\sum_{i\in I} \alpha_i P_{C_i} = \sum_{i\in I} \alpha_i P_{C_k + v_i} = \sum_{i\in I} \alpha_i (P_{C_k} + v_i) = P_{C_k} + \sum_{i\in I} \alpha_i v_i = P_{C_k + \sum_{i\in I} \alpha_i v_i}$, which establishes (iv) and (57).

"(iv)⇒(i)": Clear.

At this point, we have shown that

$$\text{(i)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Rightarrow (57). \tag{59}$$

To complete the proof, we shall show that (ii)⇔(iii).

"(ii)⇒(iii)": Fix $i \in I$. Then, by assumption, there exists $\alpha \in \mathbb{R} \smallsetminus \{0,1\}$ such that $(1-\alpha)P_{C_i} + \alpha P_{C_k} \in \mathtt{Proj}(\mathcal{H})$. Therefore, applying Theorem 3.10 to (C_i, C_k) and the corresponding coefficients $(1-\alpha, \alpha)$, we deduce the existence of $\gamma \in \mathbb{R}$ such that $(\forall x \in \mathcal{H})\, (1-\alpha)\alpha\|P_{C_i}x - P_{C_k}x\|^2 = \gamma$. Thus, because $(1-\alpha)\alpha \neq 0$ due to the fact that $\alpha \in \mathbb{R} \smallsetminus \{0,1\}$, it follows that $(\forall x \in \mathcal{H})\, \|P_{C_i}x - P_{C_k}x\|^2 = \gamma\alpha^{-1}(1-\alpha)^{-1}$. This and Lemma 4.2(iii) yield (iii).

"(iii)⇒(ii)": Suppose that (iii) holds. Then, due to (59), (iv) holds, from which (ii) follows. ■

The following example shows that the conclusion of Theorem 4.3 fails if we replace "convex combination" by "affine combination" in item (i).

Example 4.4 Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $u \in \mathcal{H}$. Then the affine combination of $(P_C, P_C, P_{\{u\}})$ with weights $(1/4, -1/4, 1)$ is a projector since $(1/4)P_C - (1/4)P_C + P_{\{u\}} = P_{\{u\}}$. However, Theorem 4.3(iii) fails when C is not a singleton.

Here are some direct consequences of Theorem 4.3.

Corollary 4.5 *Let $(C_i)_{i\in I}$ be a finite family of nonempty closed convex subsets of $\mathcal{H}$. Suppose that $\cap_{i\in I} C_i \neq \varnothing$ and that there exists $(\alpha_i)_{i\in I} \in \left]0,1\right]^I$ such that $\sum_{i\in I} \alpha_i = 1$ and $\sum_{i\in I} \alpha_i P_{C_i} \in \mathtt{Proj}(\mathcal{H})$. Then $(\forall i \in I)(\forall j \in I)\, C_i = C_j$.*

Proof. Let $k \in I$ and let $i \in I$. Since $C_k \cap C_i \neq \varnothing$ by assumption, we see that $P_{\overline{C_i - C_k}} 0 = 0$, and thus, due to our assumption, the implication "(i)⇒(iii)" of Theorem 4.3 yields $C_i = C_k$, as desired. ■

Corollary 4.6 *Let C and D be nonempty closed convex subsets of $\mathcal{H}$. Then the following are equivalent:*

(i) $(\exists \alpha \in \mathbb{R} \smallsetminus \{0,1\})\, (1-\alpha)P_C + \alpha P_D \in \mathtt{Proj}(\mathcal{H})$.

(ii) $(\forall \alpha \in \mathbb{R})\, (1-\alpha)P_C + \alpha P_D \in \mathtt{Proj}(\mathcal{H})$.

(iii) $P_{\overline{D-C}} 0 \in (C-C)^\perp$ *and* $D = C + P_{\overline{D-C}} 0$.

Proof. This follows from the equivalences "(ii)⇔(iii)⇔(iv)" of Theorem 4.3. ■

We now specialize Corollary 4.6 to get a result on scalar multiples of projectors.

Corollary 4.7 *Let C be a nonempty closed convex set in $\mathcal{H}$, and let $\alpha \in \mathbb{R} \smallsetminus \{0,1\}$. Then $\alpha P_C \in \mathtt{Proj}(\mathcal{H})$ if and only if C is a singleton.*

Proof. Let $D = \{0\}$ in Corollary 4.6. ■

5 The partial sum property of projectors onto convex cones

In this section, we shall discuss the partial sum property and the connections between our work, Zarantonello's [27, Theorems 5.5 and 5.3], and the recent work [2]. We shall need the following two results. Let us provide an instance where the *star-difference* of two sets (see [16]) can be explicitly determined. Lemma 5.1 was mentioned in [2, Footnote 5] and was also stated implicitly in the proof of [27, Theorem 5.2].

Lemma 5.1 (Star-difference of cones) *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$, and set*

$$K := \{u \in \mathcal{H} \mid u + K_2 \subseteq K_1\}. \tag{60}$$

Then the following are equivalent:

(i) $K = K_1$.

(ii) $0 \in K$.

(iii) $K \neq \varnothing$.

(iv) $K_2 \subseteq K_1$.

Proof. The chain of implications "(i)⇒(ii)⇒(iii)" is clear.

"(iii)⇒(iv)": Fix $u \in K$. Then, since K_1 and K_2 are cones, we infer that $(\forall \varepsilon \in \mathbb{R}_{++})\, \varepsilon u + K_2 = \varepsilon(u + K_2) \subseteq \varepsilon K_1 = K_1$. In turn, letting $\varepsilon \downarrow 0$ and using the closedness of K_1, we obtain $K_2 \subseteq K_1$.

"(iv)⇒(i)": First, take $u \in K_1$. Since $K_2 \subseteq K_1$ and K_1 is a convex cone by assumption, it follows that $u + K_2 \subseteq K_1 + K_1 \subseteq K_1$, and therefore $u \in K$. Conversely, fix $u \in K$. Because $u + K_2 \subseteq K_1$ and $0 \in K_2$, we deduce that $u \in K_1$, which completes the proof. ■

Proposition 5.2 *Let C and D be nonempty closed convex subsets of $\mathcal{H}$, and set*

$$f := \tfrac{1}{2}d_C^2 + \tfrac{1}{2}d_D^2 - \mathrm{q} \quad \textit{and} \quad h := f^* - \mathrm{q}. \tag{61}$$

Then the following hold:

(i) $(\forall u \in \mathcal{H})\, h(u) = \sup_{v \in D} \big(\sigma_C(u + v) + \langle u \mid v \rangle\big)$.

(ii) *Suppose that C and D are cones and $D \subseteq C^\ominus$. Then $h = \iota_{C^\ominus \cap D^\ominus}$.*

Proof. (i): Since D is convex, closed, and nonempty, we see that $\iota_D \in \Gamma_0(\mathcal{H})$, and so $(1/2)d_D^2 = \iota_D \,\square\, \mathrm{q} = \iota_D \,\boxdot\, \mathrm{q}$ by [5, Example 12.21 and Proposition 12.15]. In turn, Moreau's decomposition asserts that $\mathrm{q} - (1/2)d_D^2 = \mathrm{q} - \iota_D \,\boxdot\, \mathrm{q} = \iota_D^* \,\boxdot\, \mathrm{q}$. Thus, (61) yields

$$f = \tfrac{1}{2}d_C^2 - \iota_D^* \,\boxdot\, \mathrm{q}. \tag{62}$$

Moreover, since $\iota_D \in \Gamma_0(\mathcal{H})$ and $\mathrm{q}^* = \mathrm{q}$, [5, Proposition 13.24(i)] and the Fenchel–Moreau theorem guarantee that $(\iota_D^* \,\boxdot\, \mathrm{q})^* = \iota_D^{**} + \mathrm{q}^* = \iota_D + \mathrm{q}$, which implies that $\operatorname{dom}(\iota_D^* \,\boxdot\, \mathrm{q})^* = D$. Consequently, because $\iota_D^* \,\boxdot\, \mathrm{q} \in \Gamma_0(\mathcal{H})$, [5, Proposition 14.19 and Example 13.27(iii)] imply that

$$(\forall u \in \mathcal{H}) \quad f^*(u) - \mathrm{q}(u) = \big(\tfrac{1}{2}d_C^2 - \iota_D^* \,\boxdot\, \mathrm{q}\big)^*(u) - \mathrm{q}(u) \tag{63a}$$

$$= \sup_{v \in \operatorname{dom}(\iota_D^* \boxdot \mathrm{q})^*} \Big(\big(\tfrac{1}{2}d_C^2\big)^*(u + v) - (\iota_D^* \,\boxdot\, \mathrm{q})^*(v)\Big) - \mathrm{q}(u) \tag{63b}$$

$$= \sup_{v \in D} \Big(\sigma_C(u + v) + \mathrm{q}(u + v) - \mathrm{q}(v)\Big) - \mathrm{q}(u) \tag{63c}$$

$$= \sup_{v \in D} \Big(\sigma_C(u + v) + \langle u \mid v \rangle\Big), \tag{63d}$$

as announced.

(ii): First, because $D \subseteq C^\ominus$, Lemma 5.1 (applied to the pair of closed convex cones $(C^\ominus, D)$) yields

$$C^\ominus = \{u \in \mathcal{H} \mid u + D \subseteq C^\ominus\}. \tag{64}$$

Next, we derive from (i) and [5, Example 13.3(ii)] that

$$(\forall u \in \mathcal{H}) \quad h(u) = \sup_{v \in D} \Big(\sigma_C(u + v) + \langle u \mid v \rangle\Big) = \sup_{v \in D} \Big(\iota_{C^\ominus}(u + v) + \langle u \mid v \rangle\Big). \tag{65}$$

Now fix $u \in \mathcal{H}$, and let us consider two alternatives.

(a) $u \in \mathcal{H} \smallsetminus C^\ominus$: In view of (64), there exists $v \in D$ such that $u + v \in \mathcal{H} \smallsetminus C^\ominus$, and therefore, by (65), $h(u) \geqslant \iota_{C^\ominus}(u + v) + \langle u \mid v \rangle = +\infty$.

(b) $u \in C^\ominus$: Then, by (64), $u + D \subseteq C^\ominus$. Hence, since D is a nonempty cone, it follows from (65) and [5, Example 13.3(ii)] that $h(u) = \sup_{v \in D} \langle u \mid v \rangle = \sigma_D(u) = \iota_{D^\ominus}(u) = \iota_{C^\ominus \cap D^\ominus}(u)$.

Altogether, we obtain the desired conclusion. ■

Here is the first main result of this section. The proof of the implication "(v)⇒(i)" was inspired by [2, Lemma 5.3].

Theorem 5.3 *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Then the following are equivalent:*

(i) $K_1 + K_2$ *is closed and* $P_{K_1} + P_{K_2} = P_{K_1+K_2}$.

(ii) *There exists a nonempty closed convex cone K such that* $P_{K_1} + P_{K_2} = P_K$.

(iii) $P_{K_1} + P_{K_2}$ *is a proximity operator of a function in* $\Gamma_0(\mathcal{H})$.

(iv) $P_{K_1} + P_{K_2}$ *is nonexpansive.*

(v) $\mathrm{Id} - P_{K_1} - P_{K_2}$ *is monotone.*

(vi) $(\forall x \in \mathcal{H})\ \langle P_{K_1}x \mid P_{K_2}x\rangle = 0$.

Furthermore, if one of (i)–(vi) holds, then

$$d^2_{K_1+K_2} = d^2_{K_1} + d^2_{K_2} - 2\mathrm{q} = d^2_{K_1} - d^2_{K_2^{\ominus}} = d^2_{K_2} - d^2_{K_1^{\ominus}}. \tag{66}$$

Proof. The chain of implications "(i)⇒(ii)⇒(iii)⇒(iv)" is clear, and the implication "(iv)⇒(v)" follows from [5, Example 20.7]. We now assume that (v) holds and establish (i). Towards this end, set

$$f := \tfrac{1}{2}d^2_{K_1} + \tfrac{1}{2}d^2_{K_2} - \mathrm{q}. \tag{67}$$

and set

$$h := f^* - \mathrm{q}. \tag{68}$$

Let us first establish that $h = \iota_{K_1^{\ominus}\cap K_2^{\ominus}}$. To do so, we derive from the monotonicity of $\mathrm{Id} - P_{K_1} - P_{K_2}$ and Moreau's conical decomposition that

$$(\forall x \in \mathcal{H}) \quad \langle x \mid P_{K_1^{\ominus}}x - P_{K_2}x\rangle = \langle x - 0 \mid (\mathrm{Id} - P_{K_1} - P_{K_2})x - (\mathrm{Id} - P_{K_1} - P_{K_2})0\rangle \geqslant 0. \tag{69}$$

Thus, because $K_1^{\ominus}$ and K_2 are closed convex cones, [27, Lemma 5.6] guarantees that $K_2 \subseteq K_1^{\ominus}$, from which and Proposition 5.2(ii) we deduce that

$$h = \iota_{K_1^{\ominus}\cap K_2^{\ominus}}, \tag{70}$$

as claimed. Next, by Theorem 3.2 (respectively applied to P_{K_1} and P_{K_2}), f is Fréchet differentiable on $\mathcal{H}$ (hence continuous) and

$$\nabla f = (\mathrm{Id} - P_{K_1}) + (\mathrm{Id} - P_{K_2}) - \mathrm{Id} = \mathrm{Id} - P_{K_1} - P_{K_2}, \tag{71}$$

which is monotone by assumption. Therefore, in view of [5, Proposition 17.7(iii)], f is convex, and so $f \in \Gamma_0(\mathcal{H})$. In turn, because $f^* = h + \mathrm{q} = \iota_{K_1^{\ominus}\cap K_2^{\ominus}} + \mathrm{q}$ by (68) and (70), the Fenchel–Moreau theorem and [5, Example 13.5] yield $f = f^{**} = (\iota_{K_1^{\ominus}\cap K_2^{\ominus}} + \mathrm{q})^* = \mathrm{q} - (1/2)d^2_{K_1^{\ominus}\cap K_2^{\ominus}}$. Hence, by (71) and [5, Corollary 12.31], we obtain $\mathrm{Id} - P_{K_1} - P_{K_2} = \nabla f = \mathrm{Id} - (\mathrm{Id} - P_{K_1^{\ominus}\cap K_2^{\ominus}}) = P_{K_1^{\ominus}\cap K_2^{\ominus}}$. Thus, the Moreau conical decomposition and [5, Proposition 6.35 and Corollary 6.34] guarantee that $P_{K_1} + P_{K_2} = \mathrm{Id} - P_{K_1^{\ominus}\cap K_2^{\ominus}} = P_{(K_1^{\ominus}\cap K_2^{\ominus})^{\ominus}} = P_{\overline{K_1^{\ominus\ominus} + K_2^{\ominus\ominus}}} = P_{\overline{K_1+K_2}}$. Consequently, Proposition 2.4(iii) asserts that $K_1 + K_2$ is closed, and therefore, $P_{K_1} + P_{K_2} = P_{K_1+K_2}$, as desired. To summarize, we have shown the equivalences of (i)–(v).

"(i)⇔(vi)": Follows from Corollary 3.13 and the fact that $\langle P_{K_1}0 \mid P_{K_2}0\rangle = 0$. Moreover, if (vi) holds, then (66) follows from Corollary 3.13 and [5, Theorem 6.30(iii)]. ■

Replacing one cone by a general convex set may make the implication "(v)⇒(i)" of Theorem 5.3 fail, as illustrated by the following example.

Example 5.4 Let K be a nonempty closed convex cone in $\mathcal{H}$, and let $u \in \mathcal{H}$. Then, by Moreau's conical decomposition, $\mathrm{Id} - P_K - P_{\{u\}} = P_{K^{\ominus}} - u$, which is clearly monotone. However, owing to Example 3.18, $P_{\{u\}} + P_K = u + P_K$ is not a projector provided that $u \notin (K-K)^{\perp}$.

Here is an instance where the projector onto the intersection can be expressed in term of the individual projectors.

Corollary 5.5 *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Then the following are equivalent:*

(i) $P_{K_1 \cap K_2} = P_{K_1} + P_{K_2} - \mathrm{Id}$.

(ii) $P_{K_1} + P_{K_2} - \mathrm{Id} \in \mathtt{Proj}(\mathcal{H})$.

(iii) $P_{K_1} + P_{K_2} - \mathrm{Id}$ *is monotone.*

(iv) $(\forall x \in \mathcal{H})\ \|P_{K_1}x\|^2 + \|P_{K_2}x\|^2 = \|x\|^2 + \langle P_{K_1}x \mid P_{K_2}x\rangle$.

Proof. We first deduce from the Moreau conical decomposition and [5, Proposition 6.35] that

$$\begin{aligned}
P_{K_1\cap K_2} = P_{K_1} + P_{K_2} - \mathrm{Id} &\Leftrightarrow \mathrm{Id} - P_{(K_1\cap K_2)^{\ominus}} = P_{K_1} + P_{K_2} - \mathrm{Id} && \text{(72a)}\\
&\Leftrightarrow \mathrm{Id} - P_{\overline{K_1^{\ominus}+K_2^{\ominus}}} = P_{K_1} + P_{K_2} - \mathrm{Id} && \text{(72b)}\\
&\Leftrightarrow P_{\overline{K_1^{\ominus}+K_2^{\ominus}}} = (\mathrm{Id} - P_{K_1}) + (\mathrm{Id} - P_{K_2}) && \text{(72c)}\\
&\Leftrightarrow P_{\overline{K_1^{\ominus}+K_2^{\ominus}}} = P_{K_1^{\ominus}} + P_{K_2^{\ominus}}. && \text{(72d)}
\end{aligned}$$

"(i)⇔(ii)": Denote by $\mathcal{C}$ the class of nonempty closed convex cones in $\mathcal{H}$. Then, because the mapping $\mathcal{C} \to \mathcal{C} : K \mapsto K^{\ominus}$ is bijective due to [5, Corollary 6.34], we derive from (72d), the equivalence "(i)⇔(ii)" of Theorem 5.3, and the Moreau conical decomposition that

$$\begin{aligned}
\text{(i)} &\Leftrightarrow P_{\overline{K_1^{\ominus}+K_2^{\ominus}}} = P_{K_1^{\ominus}} + P_{K_2^{\ominus}} && \text{(73a)}\\
&\Leftrightarrow K_1^{\ominus} + K_2^{\ominus} \text{ is closed and } P_{K_1^{\ominus}+K_2^{\ominus}} = P_{K_1^{\ominus}} + P_{K_2^{\ominus}} && \text{(73b)}\\
&\Leftrightarrow (\exists K \in \mathcal{C})\ P_K = P_{K_1^{\ominus}} + P_{K_2^{\ominus}} && \text{(73c)}\\
&\Leftrightarrow (\exists K \in \mathcal{C})\ \mathrm{Id} - P_{K^{\ominus}} = (\mathrm{Id} - P_{K_1}) + (\mathrm{Id} - P_{K_2}) && \text{(73d)}\\
&\Leftrightarrow (\exists K \in \mathcal{C})\ P_{K^{\ominus}} = P_{K_1} + P_{K_2} - \mathrm{Id} && \text{(73e)}\\
&\Leftrightarrow (\exists K \in \mathcal{C})\ P_K = P_{K_1} + P_{K_2} - \mathrm{Id} && \text{(73f)}\\
&\Leftrightarrow \text{(ii)}, && \text{(73g)}
\end{aligned}$$

where the last equivalence follows from the fact that $P_{K_1} + P_{K_2} - \mathrm{Id}$ is positively homogeneous.

"(i)⇔(iii)": Since $\mathrm{Id} - (P_{K_1^{\ominus}} + P_{K_2^{\ominus}}) = P_{K_1} + P_{K_2} - \mathrm{Id}$ by Moreau's decomposition, this equivalence is a consequence of (72d) and the equivalence "(ii)⇔(v)" of Theorem 5.3 (applied to $(K_1^{\ominus}, K_2^{\ominus})$).

"(i)⇔(iv)": This readily follows from (72d), the equivalence "(ii)⇔(vi)" of Theorem 5.3 (applied to $(K_1^{\ominus}, K_2^{\ominus})$), and Lemma 2.2(ii). ■

By replacing (K_1, K_2) by $(K_1, K_2^{\ominus})$ in Corollary 5.5, we provide an alternative proof for [27, Theorem 5.3]. The linear case of Corollary 5.6 goes back at least to Halmos (see [15, Theorem 3, p. 48]).

Corollary 5.6 (Zarantonello) *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Then $P_{K_2}P_{K_1} = P_{K_2}$ if and only if $P_{K_1} - P_{K_2}$ is a projector onto a closed convex set; in which case, $P_{K_1} - P_{K_2} = P_{K_1 \cap K_2^{\ominus}}$.*

Proof. First, suppose that $P_{K_2}P_{K_1} = P_{K_2}$. Then, by [5, Theorem 6.30(i)&(iii)] and Lemma 2.2(i),

$$(\forall x \in \mathcal{H}) \quad \langle P_{K_1}x \mid P_{K_2^{\ominus}}x\rangle = \langle P_{K_1}x \mid x\rangle - \langle P_{K_1}x \mid P_{K_2}x\rangle = \langle P_{K_1}x \mid x\rangle - \langle P_{K_1}x \mid P_{K_2}P_{K_1}x\rangle \tag{74a}$$

$$= \|P_{K_1}x\|^2 - \|P_{K_2}P_{K_1}x\|^2 \tag{74b}$$

$$= \|P_{K_1}x\|^2 - \|P_{K_2}x\|^2 \tag{74c}$$

$$= \|P_{K_1}x\|^2 + \|P_{K_2^{\ominus}}x\|^2 - \|x\|^2. \tag{74d}$$

Hence, the equivalence "(i)⇔(iv)" of Corollary 5.5 (applied to $(K_1, K_2^{\ominus})$) yields $P_{K_1} - P_{K_2} = P_{K_1} + P_{K_2^{\ominus}} - \mathrm{Id} = P_{K_1 \cap K_2^{\ominus}}$, as desired. Conversely, assume that $P_{K_1} - P_{K_2}$ is a projector associated with a closed convex set. Since $P_{K_1} - P_{K_2} = P_{K_1} + P_{K_2^{\ominus}} - \mathrm{Id}$, it follows from the equivalence "(i)⇔(ii)" of Corollary 5.5 (applied to $(K_1, K_2^{\ominus})$) that

$$P_{K_1} - P_{K_2} = P_{K_1 \cap K_2^{\ominus}}. \tag{75}$$

Now take $x \in \mathcal{H}$. On the one hand, because $P_{K_1 \cap K_2^{\ominus}} + P_{K_2} = (P_{K_1} - P_{K_2}) + P_{K_2} = P_{K_1}$ by (75), we infer from Theorem 5.3 that $P_{K_1 \cap K_2^{\ominus}}x \perp P_{K_2}x$ or, equivalently, by (75), $(P_{K_1}x - P_{K_2}x) \perp P_{K_2}x$. On the other hand, (75) implies that $P_{K_1}x - P_{K_2}x \in K_2^{\ominus}$. Altogether, since clearly $P_{K_2}x \in K_2$, [5, Proposition 6.28] asserts that $P_{K_2}P_{K_1}x = P_{K_2}x$, and the proof is complete. ■

The so-called partial sum property, i.e., if a finite sum of proximity operators is a proximity operator, then so is every partial sum, was obtained in [2]. Somewhat surprisingly, as we shall see in the following result, this property is still valid in the class of projectors onto convex cones. The equivalence "(i)⇔(iii)" of the following result was obtained by Zarantonello with a different proof (see [27, Theorem 5.5]).

Theorem 5.7 (Partial sum property for cones) *Let $(K_i)_{i\in I}$ be a family of nonempty closed convex cones in $\mathcal{H}$. Then the following are equivalent:*

(i) *For every $(i,j) \in I \times I$ such that $i \neq j$, we have $(\forall x \in \mathcal{H})\ \langle P_{K_i}x \mid P_{K_j}x\rangle = 0$.*

(ii) *$\sum_{i\in I} K_i$ is closed and $\sum_{i\in I} P_{K_i} = P_{\sum_{i\in I} K_i}$.*

(iii) *$\sum_{i\in I} P_{K_i}$ is a projection onto a closed convex cone in $\mathcal{H}$.*

(iv) *$\sum_{i\in I} P_{K_i}$ is a proximity operator of a function in $\Gamma_0(\mathcal{H})$.*

(v) *For every nonempty subset J of I, $\sum_{j\in J} P_{K_j}$ is a proximity operator of a function in $\Gamma_0(\mathcal{H})$.*

(vi) *For every nonempty subset J of I, $\sum_{j\in J} K_j$ is closed and $\sum_{j\in J} P_{K_j} = P_{\sum_{j\in J} K_j}$.*

(vii) *For every $(i,j) \in I \times I$ such that $i \neq j$, we have $P_{K_i} + P_{K_j}$ is nonexpansive.*

(viii) *For every $(i,j) \in I \times I$ such that $i \neq j$, we have $\mathrm{Id} - P_{K_i} - P_{K_j}$ is monotone.*

Proof. "(i)⇒(ii)": A direct consequence of Corollary 3.20.

"(ii)⇒(iii)" and "(iii)⇒(iv)": Clear.

"(iv)⇒(v)": Let $f \in \Gamma_0(\mathcal{H})$ be such that $\sum_{i\in I} P_{K_i} = \mathrm{Prox}_f$. Then, by Moreau's decomposition ([22]), $\sum_{i\in I} P_{K_i} + \mathrm{Prox}_{f^*} = \mathrm{Prox}_f + \mathrm{Prox}_{f^*} = \mathrm{Id}$. Therefore, since $\{P_{K_i}\}_{i\in I}$ are proximity operators, the conclusion follows from [2, Theorem 4.2].

"(v)⇒(vii)": Clear.

"(vii)⇒(viii)": See [5, Example 20.7].

"(viii)⇒(i)": This is the implication "(v)⇒(vi)" of Theorem 5.3.

To sum up, we have shown the equivalence of (i)–(viii) except for (vi).

"(v)⇔(vi)": Follows from the equivalence "(ii)⇔(iv)." ■

As we now illustrate, the partial sum property may, however, fail outside the class of projectors onto convex cones.

Example 5.8 Suppose that $\mathcal{H} \neq \{0\}$, let $w \in \mathcal{H} \smallsetminus \{0\}$, set $U := \mathbb{R}_+ w$, and set $V := \mathbb{R}_+(-w) = \mathbb{R}_- w$. Then, appealing to [5, Example 29.31], we see that $(\forall x \in \mathcal{H})\ \langle P_U x \mid P_V x\rangle = 0$. Hence, by Theorem 5.3, $P_U + P_V = P_{U+V} = P_{\mathbb{R}w}$. Now suppose that $z \in \mathcal{H} \smallsetminus (U-U)^\perp = \mathcal{H} \smallsetminus (\mathbb{R}w)^\perp$. Then clearly $P_U + P_V + P_{\{z\}} + P_{\{-z\}} = P_{\mathbb{R}w}$ is the projector associated with the line $\mathbb{R}w$. However, due to Example 3.18, $P_{\{z\}} + P_U = z + P_U$ is not a projector.

To proceed further, we require the following lemma.

Lemma 5.9 *Let u and v be in $\mathcal{H} \smallsetminus \{0\}$, and set $U := \mathbb{R}_+ u$ and $V := \mathbb{R}_+ v$. Then*

$$\big[\ (\forall x \in \mathcal{H})\ \langle P_U x \mid P_V x\rangle = 0\ \big] \quad \Leftrightarrow \quad [\ u \in \mathbb{R}_{--} v \text{ or } u \perp v\ ]. \tag{76}$$

Proof. Suppose first that

$$(\forall x \in \mathcal{H}) \quad \langle P_U x \mid P_V x\rangle = 0, \tag{77}$$

and set $w := \|v\|u + \|u\|v$. Then, by the Cauchy–Schwarz inequality, $\langle w \mid u\rangle = \|v\|\|u\|^2 + \|u\|\langle u \mid v\rangle \geqslant \|v\|\|u\|^2 - \|u\|(\|u\|\|v\|) = 0$ and $\langle w \mid v\rangle = \|v\|\langle u \mid v\rangle + \|u\|\|v\|^2 \geqslant -\|v\|(\|u\|\|v\|) + \|u\|\|v\|^2 = 0$. Hence, due to [5, Example 29.31], we obtain

$$P_U w = \frac{\langle w \mid u\rangle}{\|u\|^2} u = \frac{\|v\|\|u\|^2 + \|u\|\langle u \mid v\rangle}{\|u\|^2} u \quad \text{and} \quad P_V w = \frac{\langle w \mid v\rangle}{\|v\|^2} v = \frac{\|v\|\langle u \mid v\rangle + \|u\|\|v\|^2}{\|v\|^2} v. \tag{78}$$

In turn, it follows from (77) that

$$0 = \langle P_U w \mid P_V w\rangle = \frac{1}{\|u\|^2\|v\|^2}\big(\|v\|\|u\|^2 + \|u\|\langle u \mid v\rangle\big)\big(\|v\|\langle u \mid v\rangle + \|u\|\|v\|^2\big)\langle u \mid v\rangle, \tag{79}$$

from which we derive the following conceivable cases.

(a) $\langle u \mid v\rangle = 0$: Then $u \perp v$.

(b) $\|v\|\|u\|^2 + \|u\|\langle u \mid v\rangle = 0$: Then, since $\|u\| > 0$, we get $\|v\|\|u\| + \langle u \mid v\rangle = 0$ or, equivalently, $\langle u \mid -v\rangle = \|v\|\|u\| = \|-v\|\|u\|$. Consequently, the Cauchy–Schwarz inequality yields $u \in \mathbb{R}_{++}(-v) = \mathbb{R}_{--} v$.

(c) $\|v\|\langle u \mid v\rangle + \|u\|\|v\|^2 = 0$: Proceeding as in the case (b), we obtain $u \in \mathbb{R}_{--} v$.

Conversely, assume that $u \in \mathbb{R}_{--} v$ or that $u \perp v$. Let $x \in \mathcal{H}$. If $u \in \mathbb{R}_{--} v$, then $V = \mathbb{R}_+ v = \mathbb{R}_+(-u)$, and since $U = \mathbb{R}_+ u$, it follows from [5, Example 29.31] that $0 \in \{P_U x, P_V x\}$ and so $\langle P_U x \mid P_V x\rangle = 0$. Otherwise, we have $\langle u \mid v\rangle = 0$ and, invoking [5, Example 29.31] once more,

$$\langle P_U x \mid P_V x\rangle = \frac{1}{\|u\|^2\|v\|^2} \max\{\langle x \mid u\rangle, 0\} \max\{\langle x \mid v\rangle, 0\}\ \langle u \mid v\rangle = 0, \tag{80}$$

as required. ■

Theorem 5.7 allows us to characterize finitely generated cones of which the associated projectors are the sum of projectors onto the generating rays.

Proposition 5.10 *Let $(u_i)_{i\in I}$ be a finite family in $\mathcal{H}\smallsetminus\{0\}$, set $(\forall i\in I)\ K_i := \mathbb{R}_+u_i$, and set $K := \sum_{i\in I} K_i$. Then $P_K = \sum_{i\in I} P_{K_i}$ if and only if, for every $(i,j)\in I\times I$ with $i\neq j$, we have $u_i\perp u_j$ or $u_j\in\mathbb{R}_{--}u_i$; in which case, for every $i\in I$, $\operatorname{card}\{j\in I \mid u_j\in\mathbb{R}_{--}u_i\}\leqslant 1$.*

Proof. Assume first that $P_K = \sum_{i\in I} P_{K_i}$. Then, Theorem 5.7 ensures that,

$$\text{for every } (i,j)\in I\times I \text{ with } i\neq j\text{, we have } (\forall x\in\mathcal{H})\ \langle P_{K_i}x \mid P_{K_j}x\rangle = 0. \tag{81}$$

Hence, for every $(i,j)\in I\times I$ with $i\neq j$, because $K_i=\mathbb{R}_+u_i$ and $K_j=\mathbb{R}_+u_j$, we derive from Lemma 5.9 that $u_j\in\mathbb{R}_{--}u_i$ or $u_i\perp u_j$. Now fix $i\in I$, and let us show that $\operatorname{card}\{j\in I \mid u_j\in\mathbb{R}_{--}u_i\}\leqslant 1$ by contradiction: suppose that there exists $(j,k)\in I\times I$ such that $j\neq k$ and $\{u_j,u_k\}\subseteq\mathbb{R}_{--}u_i$. Then, clearly $u_j\in\mathbb{R}_{++}u_k$. On the other hand, (81) and Lemma 5.9 imply that $u_j\perp u_k$ or $u_j\in\mathbb{R}_{--}u_k$. Altogether, we reach a contradiction. To sum up, $\operatorname{card}\{j\in I \mid u_j\in\mathbb{R}_{--}u_i\}\leqslant 1$. To see the converse, combine Lemma 5.9 and Theorem 5.7. ■

We conclude this section by summarizing our findings (see [3, Section 6] for proofs) for the case when $\mathcal{H}=\mathbb{R}$.

Remark 5.11 Let C and D be nonempty closed intervals in $\mathbb{R}$ such that $P_C+P_D\in\mathtt{Proj}(\mathbb{R})$. Then exactly one of the following cases occurs:

(i) C and D are singletons.

(ii) Neither C nor D is a singleton, and $C\cap D=\{0\}$.

6 On a result by Halmos

In this section, we revisit and extend the classical result [15, Theorem 2, p. 46] to the nonlinear case.

Proposition 6.1 *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let K be a nonempty closed convex cone in $\mathcal{H}$. Suppose that there exits a closed convex set D such that $P_C+P_K=P_D$. Then $C\subseteq K^{\ominus}$.*

Proof. Our assumption and Corollary 3.13 guarantee the existence of $\gamma\in\mathbb{R}$ such that $(\forall x\in\mathcal{H})\ \langle P_Cx \mid P_Kx\rangle=\gamma$. However, because $P_K0=0$, we infer that $\gamma=0$. Therefore, for every $x\in C$, it follows from Lemma 2.2(i) that $\|P_Kx\|^2=\langle x \mid P_Kx\rangle=\langle P_Cx \mid P_Kx\rangle=\gamma=0$; hence, $P_Kx=0$, and [5, Theorem 6.30(i)] thus implies that $x=P_{K^{\ominus}}x\in K^{\ominus}$. Consequently, $C\subseteq K^{\ominus}$, as claimed. ■

The following example shows that the conclusion of Proposition 6.1 is merely a necessary condition for $P_C+P_K=P_{C+K}$ even when C is a cone.

Example 6.2 Suppose that $\mathcal{H}=\mathbb{R}^2$. Set $u:=(1,0)$ and $v:=(-1,1)$. Moreover, set $C:=\mathbb{R}_+v$ and $K:=\mathbb{R}_+u$. Then, because $\langle u \mid v\rangle=-1<0$, we see that $C\subseteq K^{\ominus}$. Furthermore, $C+K$ is a closed cone by [5, Proposition 6.8]. Now set $x:=(1,1)=v+2u\in C+K$. According to [5, Example 29.31], $P_Cx+P_Kx=(0,0)+(1,0)=(1,0)\neq x=P_{C+K}x$. Therefore, $P_C+P_K\neq P_{C+K}$.

We now extend the classical [15, Theorem 2, p. 46] (in the case of two subspaces) by replacing one subspace by a general convex set.

Corollary 6.3 *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let V be a closed linear subspace of $\mathcal{H}$. Then the following are equivalent:*

(i) *There exists a closed convex set D such that $P_C + P_V = P_D$.*

(ii) $C \perp V$.

Moreover, if (i) and (ii) hold, then $D = C + V$ and $P_C + P_V = P_{C+V}$.

Proof. "(i)⇒(ii)": It follows from Corollary 3.13 that $D = C + V$ and that $P_C + P_V = P_{C+V}$. Now, by Proposition 6.1 and [5, Proposition 6.23], we obtain $C \subseteq V^{\ominus} = V^{\perp}$.

"(ii)⇒(i)": Immediate from Corollary 3.16. ■

However, replacing the subspace V in Corollary 6.3 by cone might not work. The following simple example shows that the implication "(i)⇒(ii)" of Corollary 6.3 may fail even when C and V are cones.

Example 6.4 Consider the setting of Example 5.8. We have seen that $P_U + P_V = P_{\mathbb{R}w}$. Yet, U is not perpendicular to V. In fact, $\operatorname{span} U = \operatorname{span} V = \mathbb{R}w$.

Combining Theorem 5.7, Theorem 5.3, and Corollary 6.3, we obtain the following well-known result; see [15, Theorem 2, p. 46].

Corollary 6.5 *Let $(V_i)_{i\in I}$ be a finite family of closed linear subspaces of $\mathcal{H}$. Then $\sum_{i\in I} P_{V_i}$ is a projector associated with a closed linear subspace if and only if, for every $(i, j) \in I \times I$ with $i \neq j$, we have $V_i \perp V_j$.*

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