

MAXIMAL MONOTONICITY OF DENSE TYPE,
LOCAL MAXIMAL MONOTONICITY,
AND MONOTONICITY OF THE CONJUGATE
ARE ALL THE SAME FOR
CONTINUOUS LINEAR OPERATORS

HEINZ H. BAUSCHKE AND JONATHAN M. BORWEIN

The concept of a monotone operator — which covers both linear positive semi-definite operators and subdifferentials of convex functions — is fundamental in various branches of mathematics. Over the last few decades, several stronger notions of monotonicity have been introduced: Gossez's maximal monotonicity of **dense** type, Fitzpatrick and Phelps's local maximal monotonicity, and Simons's monotonicity of type **(NI)**. While these monotonicities are automatic for maximal monotone operators in reflexive Banach spaces and for subdifferentials of convex functions, their precise relationship is largely unknown.

Here, it is shown — within the beautiful framework of Convex Analysis — that for continuous linear monotone operators, *all these notions coincide and are equivalent to the monotonicity of the conjugate operator*. This condition is further analyzed and illustrated by examples.

1. Introduction.

Motivation.

A monotone operator is a (possibly set-valued) map from a Banach space to its dual satisfying a certain relation. In the simplest case, when the space is just the real line, this relation corresponds precisely to increasing (possibly set-valued) functions, hence the name.

Monotone operators appear in diverse areas such as Operator Theory, Numerical Analysis, Differentiability Theory of Convex Functions, and Partial Differential Equations, because the notion of a monotone operator is broad enough to cover two fundamental mathematical objects: *Linear positive semi-definite operators* and *subdifferentials of convex functions*. Although the former object gave rise to the field, it is the latter that has been receiving much of the recent attention. (For more on monotone operators, the reader is referred to the conference proceedings [11, 4, 31], the books

[21, 29, 32, 33], the electronic lecture notes [22], and the historical account [19]; applications are discussed in [6, 10, 20].)

The urge to extract and study the quite strong monotonicity properties of subdifferentials of convex functions has led to the introduction of several new more powerful notions of monotonicity. While these notions are automatic for maximal monotone operators on reflexive Banach spaces, the situation in nonreflexive Banach spaces is far less well understood. Surprisingly, these notions of monotonicity were largely untested even for the most natural candidates: Continuous linear monotone operators. Thus:

The aim of this paper is to study the various notions of monotonicity for continuous linear monotone operators.

Using elegant and potent tools from Convex Analysis, we show that *these notions all coincide with the monotonicity of the conjugate operator.*

In contrast to the subdifferential case, this condition is *not* automatic and we present a new derivation of two classical counter-examples. Using Banach space theory, it can be shown that monotonicity of the conjugate operator is the rule — with the notable exception of spaces containing a complemented copy of ℓ_1 .

Overview.

In Section 2, we introduce the various notions of monotonicity coined by Gossez, by Fitzpatrick and Phelps, and by Simons and then review their basic relationship. From Section 3 on, we focus on the case when the monotone operator is continuous and linear. The main result, whose proof depends crucially on Fenchel’s Duality Theorem, is presented in Section 4. It allows us to give an affirmative answer to a question posed by Gossez more than two decades ago. In the *last* section, we derive and extend classical counter-examples by Gossez and by Fitzpatrick and Phelps *systematically* from a result that can be viewed as an “instruction manual” for constructing interesting continuous linear monotone operators whose skew parts have *nonmonotone* conjugates. We conclude by remarking that such strange operators occur only in a few classical Banach spaces like ℓ_1 and $L_1[0, 1]$ and that preliminary results on *regularizations* demonstrate the close relationship between the various monotonicities even in a nonlinear context.

Notation.

The notation we employ is standard. Throughout, we assume that

X is a real Banach space with norm $\|\cdot\|$ and dual X^* .

The evaluation of a functional $x^* \in X^*$ at a point $x \in X$ is written as $\langle x^*, x \rangle$ or as $\langle x, x^* \rangle$. We often view X as a subspace in its bidual $X^{**} := (X^*)^*$. The *unit ball* $\{x \in X : \|x\| \leq 1\}$ is denoted B_X . If (x_α^*) is a net in some dual space, then we write $x_\alpha^* \xrightarrow{w^*} x^*$ (resp. $x_\alpha^* \rightarrow x^*$) to indicate *convergence* in the weak* (resp. norm) topology with limit point x^* . If $x, y \in X$, then $[x, y]$

stands for the *line segment* $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$. If U (resp. V) is a subset of X (resp. X^*), then U^\perp (resp. ${}^\perp V$) stands for the *annihilator* $\{x^* \in X^* : \langle x^*, u \rangle = 0, \forall u \in U\}$ (resp. $\{x \in X : \langle x, v \rangle = 0, \forall v \in V\} = X \cap V^\perp$). If T is a continuous linear operator from X to some other Banach space, then the *conjugate* (or *adjoint*, *transpose*) is denoted T^* , the *restriction* of T to some subset U of X is written as $T|_U$, and $\ker T$ is the *kernel* (or *null space*) of T : $\ker T := \{x \in X : Tx = 0\}$. Suppose C is a subset of X . Then $\text{span } C$ is the *span* of C (i.e. the set of all linear combinations of elements of C). Also, $\text{cl } C$ (resp. $\text{int } C$) stands for the *closure* (resp. *interior*) of C ; here, the norm topology is the “default topology”. If these operations are meant with respect to some other topology \mathcal{T} , then we indicate this by subscripts; for instance, $\text{cl}_{\mathcal{T}} C$ would be the closure of C with respect to the topology \mathcal{T} .

Suppose Y, Z are sets and T is a *set-valued map* from Y to Z , i.e., T is a map from Y to 2^Z . Then the *graph* of T is denoted $\text{gra } T$; so $z \in Ty$ if and only if $(y, z) \in \text{gra } T, \forall y \in Y, z \in Z$. The *domain* (resp. *range*) of T is given by $\text{dom } T := \{y \in Y : Ty \neq \emptyset\}$ (resp. $\text{ran } T := \{z \in Z : z \in Ty, \text{ for some } y \in Y\}$). The *inverse* of T , denoted T^{-1} , is the set-valued map from Z to Y defined by $y \in T^{-1}z$ if and only if $z \in Ty, \forall y \in Y, z \in Z$. If U is a subset of Y , then we write $T(U)$ for $\bigcup_{u \in U} Tu$.

Notation from Convex Analysis appears throughout the paper. For the reader’s convenience, we review the definitions. The *indicator function* of a subset C of X , denoted ι_C , is given by

$$\iota_C : X \rightarrow \mathbb{R} \cup \{+\infty\} : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

Suppose f is a convex lower semi-continuous function from X to $\mathbb{R} \cup \{+\infty\}$. Then the (*essential*) *domain* of f is the set $\text{dom } f := \{x \in X : f(x) < +\infty\}$. The conjugate of f , denoted f^* , is given by

$$f^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\} : x^* \mapsto \sup_{x \in X} [\langle x^*, x \rangle - f(x)].$$

Note that f^ is defined on X^* and hence $f^{**} := (f^*)^*$ is defined on X^{**} . The *subdifferential* of f at $x \in \text{dom } f$ is the set*

$$\{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \quad \forall y \in X\}.$$

If $x \in \text{int dom } f$ and $\partial f(x)$ is singleton, then the element in $\partial f(x)$ coincides with the (*Gâteaux*) *gradient* $\nabla f(x)$.

Finally, the reals (resp. strictly positive integers $\{1, 2, 3, \dots\}$) are abbreviated \mathbb{R} (resp. \mathbb{N}) and we used already \forall (resp. \exists) as a short form for “for all” (resp. “there exists”).

As general references on Functional Analysis, we recommend [5, 17, 18, 30]; for more on Convex Analysis, see [25, 1, 7, 15, 16].

The single most important tool from Convex Analysis is the celebrated *Fenchel Duality Theorem*:

Fact 1.1. (Fenchel Duality; see, e.g., [1, Theorem 4.6.1].) Suppose A is a continuous linear operator from a Banach space X to a Banach space Y , f is a convex lower semi-continuous function on X and g is a convex lower semi-continuous function on Y . Consider the convex programs

$$(P) \quad p := \inf_{x \in X} [f(x) + g(Ax)]$$

and

$$(D) \quad d := - \inf_{y^* \in Y^*} [f^*(-A^*y^*) + g^*(y^*)].$$

Then $p \geq d$. If $A(\text{dom } f) \cap \text{int dom } g \neq \emptyset$ and p is finite, then $p = d$ and d is attained.

2. General tools.

Recall that a set-valued map from X to X^* is a *monotone operator*, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in X.$$

If T is monotone and $\text{gra } T$ is a maximal subset in $X \times X^*$, then T is called *maximal monotone*. Zorn's Lemma guarantees the existence of maximal monotone extensions for any given monotone operator. Analogously, one can speak of (maximal) monotone operators from X^* to X or from X^{**} to X^* or of monotone operators whose graphs are maximal monotone with respect to some subsets and so forth.

The following extensions have turned out to be useful when studying the nonreflexive case.

Definition 2.1. Suppose T is a set-valued map from X to X^* . Define set-valued maps T_1, T_0, \overline{T} from X^{**} to X^* via their graphs as follows:

- (i) $(x^{**}, x^*) \in \text{gra } T_1$, if there exists a bounded net (x_α, x_α^*) in $\text{gra } T$ with $x_\alpha \xrightarrow{w^*} x^{**}$ and $x_\alpha^* \rightarrow x^*$.
- (ii) $(x^{**}, x^*) \in \text{gra } T_0$, if $\inf_{(y, y^*) \in \text{gra } T} \langle y^* - x^*, y - x^{**} \rangle = 0$.
- (iii) $(x^{**}, x^*) \in \text{gra } \overline{T}$, if $\inf_{(y, y^*) \in \text{gra } T} \langle y^* - x^*, y - x^{**} \rangle \geq 0$.

Proposition 2.2. Suppose T is a monotone operator from X to X^* . Then the following inclusions hold in $X^{**} \times X^*$:

$$\text{gra } T \subseteq \text{gra } T_1 \subseteq \text{gra } T_0 \subseteq \text{gra } \overline{T} = \text{gra } \overline{T}_1 \cap (X^{**} \times X^*).$$

Proof. The inclusions $\text{gra } T \subseteq \text{gra } T_1$ and $\text{gra } T_0 \subseteq \text{gra } \overline{T} \supseteq \text{gra } \overline{T}_1 \cap (X^{**} \times X^*)$ are obvious (even without monotonicity). Fix an arbitrary $(x^{**}, x^*) \in \text{gra } T_1$ and obtain a bounded net (x_α, x_α^*) in $\text{gra } T$ with $x_\alpha \xrightarrow{w^*} x^{**}$ and $x_\alpha^* \rightarrow x^*$. Then $\langle x_\alpha - y, x_\alpha^* - y^* \rangle \geq 0, \forall \alpha, (y, y^*) \in \text{gra } T$; taking limits yields $\langle x^{**} -$

$y, x^* - y^*) \geq 0$. On the other hand, $\langle x^{**} - x_\alpha, x^* - x_\alpha^* \rangle \rightarrow 0$; altogether, $0 = \inf_{(y, y^*) \in \text{gra } T} \langle x^{**} - y, x^* - y^* \rangle$, i.e., $(x^{**}, x^*) \in \text{gra } T_0$. Hence $\text{gra } T_1 \subseteq \text{gra } T_0$. Finally, pick $(z^{**}, z^*) \in \text{gra } \overline{T}$. Then $0 \leq \inf_{(y, y^*) \in \text{gra } T} \langle y^* - z^*, y - z^{**} \rangle \leq \lim_\alpha \langle x_\alpha^* - z^*, x_\alpha - z^{**} \rangle = \langle x^* - z^*, x^{**} - z^{**} \rangle$; so (z^{**}, z^*) is monotonically related to $\text{gra } T_1$, hence $\text{gra } \overline{T} \subseteq \text{gra } \overline{T}_1 \cap (X^{**} \times X^*)$. \square

Definition 2.3. Suppose T is a monotone operator from X to X^* . Then:

- (i) (Gossez [14]) T is of dense type or of type (D), if $T_1 = \overline{T}$.
- (ii) (Simons [28, Definition 14]) T is of range-dense type or of type (WD), if for every $x^* \in \text{ran } \overline{T}$, there exists a bounded net $(x_\alpha, x_\alpha^*) \in \text{gra } T$ with $x_\alpha^* \rightarrow x^*$.
- (iii) (Simons [28, Definition 10]) T is of type (NI), if $\inf_{(y, y^*) \in \text{gra } T} \langle y^* - x^*, y - x^{**} \rangle \leq 0$, for all $(x^{**}, x^*) \in X^{**} \times X^*$. If this holds only on some subset of $X^{**} \times X^*$, then we say that T is of type (NI) with respect to this subset.
- (iv) (Fitzpatrick and Phelps [8, Section 3]) T is locally maximal monotone, if $(\text{gra } T^{-1}) \cap (V \times X)$ is maximal monotone in $V \times X$, for every convex open set V in X^* with $V \cap \text{ran } T \neq \emptyset$.
- (v) T is unique, if all maximal monotone extensions of T in $X^{**} \times X^*$ coincide.

A monotone operator which is either maximal monotone and of dense type or locally maximal monotone is certainly maximal monotone; the converse is true in reflexive spaces:

Fact 2.4. (See, e.g., Phelps's [22, Example 3.2.(b) and Proposition 4.4].) Suppose X is reflexive and T is a monotone operator from X to X^* . Then TFAE: (i) T is maximal monotone; (ii) T is maximal monotone and of dense type; (iii) T is locally maximal monotone.

It is known and very useful that subdifferentials of convex functions are “everything”: Maximal monotone, of dense type, and locally maximal monotone.

Fact 2.5. Suppose f is convex lower semi-continuous proper function on X . Then:

- (i) (Rockafellar [26]) ∂f is maximal monotone.
- (ii) (Gossez [12, Théorème 3.1]) ∂f is of dense type and $(\partial f)_1 = (\partial f^*)^{-1}$.
- (iii) (Simons [27]) ∂f is locally maximal monotone.

In general, the following is known to be true.

Fact 2.6. (Simons's [28, Lemma 15 and Theorem 19].) For any monotone operator T from X to X^* , the following implications hold:

$$\text{dense type} \Rightarrow \text{range-dense type} \Rightarrow \text{type (NI)} \Rightarrow \text{unique.}$$

Moreover, TFAE:

- (i) T is **unique**.
- (ii) \overline{T} is the unique maximal monotone extension of T in $X^{**} \times X^*$.
- (iii) \overline{T} is maximal monotone.
- (iv) \overline{T} is monotone.

It is sometimes more handy to work with the following reformulations of the various monotonicities.

Proposition 2.7. *Suppose T is a monotone operator from X to X^* . Then:*

- (i) T is of **dense** type if and only if T_1 is maximal monotone.
- (ii) T is of **range-dense** type if and only if $\text{ran } T_1 = \text{ran } \overline{T}$.
- (iii) T is of type **(NI)** if and only if $T_0 = \overline{T}$.
- (iv) (Phelps's [22, Proposition 4.3]) T is **locally maximal monotone** if and only if for every weak* closed convex bounded subset C of X^* with $\text{ran } T \cap \text{int } C \neq \emptyset$, and for every $x_0 \in X$, $x_0^* \in (\text{int } C) \setminus Tx_0$, there exist $(z, z^*) \in \text{gra } T \cap (X \times C)$ with $\langle z^* - x_0^*, z - x_0 \rangle < 0$.

Proof. (i): “ \Rightarrow ”: T is of **dense** type $\Leftrightarrow T_1 = \overline{T}$. Now T_1 is monotone (because T is), hence so is \overline{T} . By Fact 2.6, $\overline{T} = T_1$ is maximal monotone. “ \Leftarrow ”: Pick $(x^{**}, x^*) \in \text{gra } \overline{T}$. Then (by Proposition 2.2) $(x^{**}, x^*) \in \text{gra } \overline{T}_1$, i.e., this point is monotonically related to $\text{gra } T_1$. Now T_1 is maximal monotone, hence $(x^{**}, x^*) \in \text{gra } T_1$.

(ii): “ \Rightarrow ”: Pick $x^* \in \text{ran } \overline{T}$. By assumption, there exists a bounded net (x_α, x_α^*) in $\text{gra } T$ such that $x_\alpha^* \rightarrow x^*$. Without loss, we can assume that $x_\alpha \xrightarrow{w^*} x^{**}$. Then $(x^{**}, x^*) \in \text{gra } T_1$ and in particular $x^* \in \text{ran } T_1$. “ \Leftarrow ” is even simpler.

(iii): Let us abbreviate $\inf_{(y, y^*) \in \text{gra } T} \langle x^{**} - y, x^* - y^* \rangle$ by I . “ \Rightarrow ”: If $(x^{**}, x^*) \in \text{gra } \overline{T}$, then $I \geq 0$. Now T is of type **(NI)**, hence $I \leq 0$. Thus $I = 0$. “ \Leftarrow ”: Fix $(x^{**}, x^*) \in X^{**} \times X^*$. If $(x^{**}, x^*) \notin \text{gra } \overline{T}$, then $I < 0$. Otherwise, $(x^{**}, x^*) \in \overline{T} = T_0$ and hence $I = 0$. \square

3. Linear tools.

For a continuous linear operator T from X to X^* , the extension T_1 has the following explicit description:

Fact 3.1. (Gossez's [13, End of Section 2].) Suppose T is a continuous linear operator from X to X^* . Then:

$$\text{gra } T_1 = \text{cl}_{\text{weak}^* \times \|\cdot\|} \text{gra } T = (\text{gra } T^{**}) \cap (X^{**} \times X^*).$$

Recall that a continuous linear operator T from X to X^* is *weakly compact*, if $\text{ran } T^{**}|_{X^{**} \setminus X} \subseteq X^*$; equivalently, if $\text{cl } T(B_X)$ is weakly compact or if T^* is weakly compact. Fact 3.1 yields immediately:

Corollary 3.2. *Suppose T is a continuous linear operator from X to X^* . Then T is weakly compact if and only if $T_1 = T^{**}$.*

Further recall that if T is a continuous linear operator from X to X^* with $\langle Tx, x \rangle \geq 0, \forall x \in X$, then T is called *positive* or *positive semi-definite*. The following result is part of the folklore.

Proposition 3.3. *Suppose T is a continuous linear operator from X to X^* . Then TFAE: (i) T is positive; (ii) T is monotone; (iii) T is maximal monotone.*

Proof. By linearity of T , (i) and (ii) are equivalent; also, (iii) implies (ii). For “(ii) \Rightarrow (iii)” see, e.g., [22, Proof of Example 1.5.(b)]. \square

Monotonicity of type (NI) relates to monotonicity of the conjugate operator:

Proposition 3.4. *Suppose T is a continuous linear operator from X to X^* . Then T is monotone and of type (NI) with respect to $\text{gra}(-T^*)$ if and only if T^* is monotone.*

Proof. Clearly, if T^* is monotone, then so is T . So suppose T is monotone. Fix $x^{**} \in X^{**}$ and $x \in X$. Then $\langle Tx + T^*x^{**}, x - x^{**} \rangle = \langle Tx, x \rangle - \langle Tx, x^{**} \rangle + \langle T^*x^{**}, x \rangle - \langle T^*x^{**}, x^{**} \rangle \geq -\langle T^*x^{**}, x^{**} \rangle$. Hence $-\langle T^*x^{**}, x^{**} \rangle \leq \inf_{x \in X} \langle Tx + T^*x^{**}, x - x^{**} \rangle \leq \langle T0 + T^*x^{**}, 0 - x^{**} \rangle = -\langle T^*x^{**}, x^{**} \rangle$ and thus:

$$\begin{aligned} \inf_{(y, y^*) \in \text{gra } T} \langle y^* - (-T^*x^{**}), y - x^{**} \rangle &= \inf_{x \in X} \langle Tx + T^*x^{**}, x - x^{**} \rangle \\ &= -\langle T^*x^{**}, x^{**} \rangle. \end{aligned}$$

The result follows. \square

Recall also that a continuous linear operator from X to X^* is *symmetric* (resp. *skew*), if $T^*|_X = T$ (resp. $T^*|_X = -T$); equivalently, if $\langle Tx, y \rangle = \langle x, Ty \rangle$ (resp. $\langle Tx, y \rangle = -\langle x, Ty \rangle$), $\forall x, y \in X$.

Our study of continuous linear monotone operators relies also on the following easy-to-prove yet immensely useful decomposition principle.

Proposition 3.5. *Suppose T is a continuous linear operator from X to X^* . Then T can be written as the sum of two continuous linear operators, $T = P + S$, where P is symmetric and S is skew. This decomposition is unique; in fact:*

$$Px = \frac{1}{2}Tx + \frac{1}{2}T^*x \quad \text{and} \quad Sx = \frac{1}{2}Tx - \frac{1}{2}T^*x, \quad \forall x \in X.$$

We refer to P (resp. S) as the symmetric part (resp. skew part) of T .

Symmetric operators.

Theorem 3.6. *Suppose T is a continuous linear operator from X to X^* with symmetric part P . Let $q(x) := \frac{1}{2}\langle x, Tx \rangle$, $\forall x \in X$. Then:*

$$q \text{ is convex} \Leftrightarrow T \text{ is monotone} \Leftrightarrow P \text{ is monotone.}$$

Assume in addition that T is monotone. Then:

- (i) $\nabla q = P$ and $q^* \circ P = q$.
- (ii) $P_1 = P_0 = \bar{P} = P^* = P^{**}$. Hence: P is maximal monotone of *dense* type, weakly compact, and *locally* maximal monotone; P^* is monotone and symmetric.
- (iii) For every $x^{**} \in X^{**}$, there exists a bounded net (x_α) in X such that $x_\alpha \xrightarrow{w^*} x^{**}$ and $Px_\alpha \rightarrow P^*x^{**} = P^{**}x^{**}$.
- (iv) $q^*(P^*x^{**}) = \frac{1}{2}\langle x^{**}, P^{**}x^{**} \rangle = q^{**}(x^{**})$, for every $x^{**} \in X^{**}$ and $\nabla q^{**} = P^{**} = P^*$.

Proof. Since q is continuous, it suffices to check midpoint convexity; fixing two arbitrary points $x, y \in X$, we have $q(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}q(x) + \frac{1}{2}q(y) \Leftrightarrow 0 \leq \langle x - y, T(x - y) \rangle \Leftrightarrow 0 \leq \langle x - y, P(x - y) \rangle$. The displayed equivalences follow.

(i): q is continuous, convex, and finite on X ; hence q is subdifferentiable everywhere. So fix an arbitrary $x_0 \in X$ and pick $x^* \in \partial q(x_0)$. Then $t\langle x^*, h \rangle \leq q(x_0 + th) - q(x_0)$, $\forall h \in X, t > 0$; this simplifies to $\langle x^*, h \rangle \leq \langle \frac{1}{2}Tx_0 + \frac{1}{2}T^*x_0, h \rangle + \frac{1}{2}t\langle h, Th \rangle$. Letting t tend to 0 yields $x^* = \frac{1}{2}Tx_0 + \frac{1}{2}T^*x_0 = Px_0$. Now

$$q^*(Px_0) = \sup_{x \in X} \langle Px_0, x \rangle - q(x) = - \inf_{x \in X} \{q(x) + \langle -Px_0, x \rangle\};$$

this last infimum can be viewed as a little optimization problem which is easy to solve: Indeed, after taking gradients, we learn that the set of minimizers equals $x_0 + \ker P$. It follows that $q^*(Px_0) = q(x_0)$.

(ii): By (i), P is the subdifferential of q . Consequently (Fact 2.5), P is maximal monotone, of *dense* type, and *locally* maximal monotone. In particular (Fact 2.6 and Proposition 2.7.(iii)), $P_1 = P_0 = \bar{P}$ and P is of type (NI). It follows that on the one hand, P^* is a *maximal monotone extension* of P (Proposition 3.4 and Proposition 3.3). On the other hand, \bar{P} is the *unique maximal monotone extension* of P in $X^{**} \times X^*$ (Fact 2.6). Altogether, $\bar{P} = P^*$. Now $P^* = P^{**}$, because $P_1 = P^*$ and $\text{gra } P_1 \subseteq \text{gra } P^{**}$ (Fact 3.1). Thus the weak compactness of P follows from Corollary 3.2.

(iii): By (ii), $P_1 = P^* = P^{**}$.

(iv): Fix $x^{**} \in X^{**}$ and define $g(x) := \langle -P^*x^{**}, x \rangle + \frac{1}{2}\langle x^{**}, P^*x^{**} \rangle$, $\forall x \in X$. Then, by (ii), $(x^{**}, P^*x^{**}) \in \text{gra } P_0$ and hence

$$0 = \frac{1}{2} \inf_{x \in X} \langle Px - P^*x^{**}, x - x^{**} \rangle = \inf_{x \in X} q(x) + g(x).$$

The conjugate of g is given by $g^*(x^*) = -\frac{1}{2}\langle x^{**}, P^*x^{**} \rangle + \iota_{\{-P^*x^{**}\}}(x^*)$, $\forall x^* \in X^*$. Fact 1.1 yields

$$0 = -\inf_{x^* \in X^*} \{g^*(x^*) + g^*(-x^*)\} = \frac{1}{2}\langle x^{**}, P^*x^{**} \rangle - q^*(P^*x^{**}),$$

which is the first equality. To prove the second equality, we first note that the first equality implies

$$q^{**}(x^{**}) \geq \langle x^{**}, P^*x^{**} \rangle - q^*(P^*x^{**}) = \frac{1}{2}\langle x^{**}, P^*x^{**} \rangle.$$

On the other hand, by (iii), there is a bounded net (x_α) in X such that $x_\alpha \xrightarrow{w^*} x^{**}$ and $Px_\alpha \rightarrow P^*x^{**}$. Then for every $x^* \in X^*$, we estimate

$$q^*(x^*) \geq \lim_\alpha \langle x^*, x_\alpha \rangle - \frac{1}{2}\langle x_\alpha, Px_\alpha \rangle = \langle x^*, x^* \rangle - \frac{1}{2}\langle x^{**}, P^*x^{**} \rangle.$$

This in turn implies $\frac{1}{2}\langle x^{**}, P^*x^{**} \rangle \geq \sup_{x^* \in X^*} \langle x^*, x^* \rangle - q^*(x^*) = q^{**}(x^{**})$, which yields the second equality. Applying (i) to P^{**} , which is monotone and symmetric, finally yields $\nabla q^{**} = P^{**}$. \square

Skew operators.

Theorem 3.7. *Suppose S is a continuous linear skew operator from X to X^* . If $(x^{**}, x^*) \in X^{**} \times X^*$, then:*

- (i) $x^* \in S_1x^{**} \Leftrightarrow x^* = S^{**}x^{**} = -S^*x^{**}$.
- (ii) $x^* \in S_0x^{**} \Leftrightarrow x^* = -S^*x^{**}$ and $\langle S^*x^{**}, x^{**} \rangle = 0$.
- (iii) $x^* \in \overline{S}x^{**} \Leftrightarrow x^* = -S^*x^{**}$ and $\langle S^*x^{**}, x^{**} \rangle \leq 0$.

Hence: S^ is skew if and only if S is weakly compact.*

If S^ is monotone and $x^{**} \in X^{**}$ with $\langle S^*x^{**}, x^{**} \rangle = 0$, then $S^{**}x^{**} = -S^*x^{**}$.*

Proof. First “If” part: Fix an arbitrary $y \in X$. Then, using the skewness of S , $\langle x^{**} - y, x^* - Sy \rangle = \langle x^{**}, x^* \rangle - \langle y, S^*x^{**} + x^* \rangle$. Hence

$$\inf_{(y, y^*) \in \text{gra } S} \langle x^{**} - y, x^* - y^* \rangle = \begin{cases} \langle x^{**}, x^* \rangle, & \text{if } x^* = -S^*x^{**}; \\ -\infty, & \text{otherwise.} \end{cases}$$

(ii) and (iii) follow readily. For (i), observe that: $x^* \in S_1x^{**} \Leftrightarrow (x^{**}, x^*) \in \text{gra } S_1 \cap \text{gra } S_0$ (Proposition 2.2) $\Leftrightarrow x^* = S^{**}x^{**} = -S^*x^{**} \in X^*$ ((ii) and Fact 3.1) $\Leftrightarrow x^* = S^{**}x^{**} = -S^*x^{**}$ ($\text{ran } S^* \subseteq X^*$).

“Hence” part: S^* skew $\Leftrightarrow S^{**} = -S^* \Leftrightarrow S_1 = S^{**}$ (use (i)) $\Leftrightarrow S$ weakly compact (Corollary 3.2).

Second “If” part: Fix an arbitrary $y^{**} \in X^{**}$ and $\lambda > 0$. Thus:

$$\begin{aligned} 0 &= \langle S^*x^{**}, x^{**} \rangle \\ &= \langle S^*(x^{**} + \lambda y^{**}), x^{**} + \lambda y^{**} \rangle - \lambda \langle S^*(x^{**} + \lambda y^{**}), y^{**} \rangle - \lambda \langle S^*y^{**}, x^{**} \rangle \\ &\geq -\lambda \langle S^*x^{**}, y^{**} \rangle - \lambda \langle S^*y^{**}, x^{**} \rangle - \lambda^2 \langle S^*y^{**}, y^{**} \rangle. \end{aligned}$$

Now divide by λ and then let λ tend to 0 to conclude $\langle S^*x^{**}, y^{**} \rangle \geq -\langle S^*y^{**}, x^{**} \rangle, \forall y^{**} \in X^{**}$. The result follows. \square

4. Characterizations.

We are now ready for the main result.

Theorem 4.1. *Suppose T is a continuous linear operator from X to X^* with symmetric part P and skew part S . Then TFAE:*

- (i) T is monotone and of *dense* type.
- (ii) T is monotone and of *range-dense* type.
- (iii) T is monotone and of type (NI).
- (iv) T is *locally maximal monotone*.
- (v) T^* is monotone.
- (vi) P and S^* are monotone.
- (vii) P is monotone and S is of *dense* type.
- (viii) P is monotone and S is of *range-dense* type.
- (ix) P is monotone and S is of type (NI).
- (x) P is monotone and S is *locally maximal monotone*.

Proof. Throughout, let $q(x) := \frac{1}{2}\langle Tx, x \rangle = \frac{1}{2}\langle Px, x \rangle, \forall x \in X$.

“(i) \Rightarrow (ii) \Rightarrow (iii)” : Fact 2.6.

“(iii) \Rightarrow (v)” : Proposition 3.4.

“(v) \Rightarrow (vi)” : T and P are monotone, because T^* is. Fix an arbitrary $x^{**} \in X^{**}$. By Theorem 3.6.(iii), obtain a bounded net (x_α) in X such that $x_\alpha \xrightarrow{w^*} x^{**}$ and $Px_\alpha \rightarrow P^*x^{**}$. Now

$$\begin{aligned} 0 &\leq \langle T^*(x^{**} - x_\alpha), x^{**} - x_\alpha \rangle = \langle T^*x^{**}, x^{**} - x_\alpha \rangle - \langle T^*x_\alpha, x^{**} - x_\alpha \rangle \\ &= \langle T^*x^{**}, x^{**} - x_\alpha \rangle - \langle Px_\alpha - Sx_\alpha, x^{**} - x_\alpha \rangle \\ &= \langle T^*x^{**} - Px_\alpha, x^{**} - x_\alpha \rangle + \langle x_\alpha, S^*x^{**} \rangle \rightarrow \langle x^{**}, S^*x^{**} \rangle; \end{aligned}$$

consequently, S^* is monotone and (vi) holds.

“(vi) \Rightarrow (i)” : We start by noting that if (x^{**}, x^*) belongs to $X^{**} \times X^*$, then

$$\frac{1}{2} \inf_{x \in X} \langle Tx - x^*, x - x^{**} \rangle = \frac{1}{2} \langle x^{**}, x^* \rangle - q^* \left(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**} \right).$$

Now fix an arbitrary $(x^{**}, x^*) \in \text{gra} \bar{T}$. Then, on the one hand, $q^*(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}) \leq \frac{1}{2}\langle x^{**}, x^* \rangle$. On the other hand, Theorem 3.6.(iii) yields a bounded net (x_α) in X such that $x_\alpha \xrightarrow{w^*} x^{**}$ and $Px_\alpha \rightarrow P^*x^{**}$. Using the monotonicity of S^* , we conclude altogether

$$\begin{aligned} \frac{1}{2} \langle x^{**}, x^* \rangle &\geq q^* \left(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**} \right) \geq \lim_{\alpha} \left\langle \frac{1}{2}x^* + \frac{1}{2}T^*x^{**}, x_\alpha \right\rangle - \frac{1}{2} \langle x_\alpha, Px_\alpha \rangle \\ &= \frac{1}{2} \langle x^*, x^{**} \rangle + \frac{1}{2} \langle S^*x^{**}, x^{**} \rangle \geq \frac{1}{2} \langle x^{**}, x^* \rangle. \end{aligned}$$

Hence $\langle S^*x^{**}, x^{**} \rangle = 0$ and $q^*(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}) = \frac{1}{2}\langle x^{**}, x^* \rangle$. This has two important consequences: firstly, by Theorem 3.7,

$$S^{**}x^{**} = -S^*x^{**}.$$

Secondly, using Theorem 3.6.(iv), $\langle \frac{1}{2}x^* + \frac{1}{2}T^*x^{**}, x^{**} \rangle = \frac{1}{2}\langle x^{**}, x^* \rangle + \frac{1}{2}\langle x^{**}, P^*x^{**} \rangle = q^*(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}) + q^{**}(x^{**})$; thus: $x^{**} \in \partial q^*(\frac{1}{2}x^* + \frac{1}{2}T^*x^{**}) \Rightarrow \frac{1}{2}x^* + \frac{1}{2}T^*x^{**} \in \partial q^{**}(x^{**}) = \{P^*x^{**}\} \Leftrightarrow$

$$x^* = P^*x^{**} - S^*x^{**}.$$

Altogether, $x^* = P^*x^{**} - S^*x^{**} = P^{**}x^{**} + S^{**}x^{**} = T^{**}x^{**} \in X^*$, so that (Fact 3.1) $(x^{**}, x^*) \in \text{gra } T_1$, as desired.

“(iv) \Rightarrow (v)” : T is maximal monotone (use $V = X^*$ in Definition 2.3.(iv)), hence so is P and the function q is convex (Theorem 3.6). Fix an arbitrary $x_0^{**} \in X^{**}$. We aim for $\langle T^*x_0^{**}, x_0^{**} \rangle \geq 0$ and can thus assume WLOG that $x_0^* := T^*x_0^{**} \neq 0$. Select $x_1 \in X$ with $\langle x_0^*, x_1 \rangle < 0$ and let $x_1^* := Tx_1$. Let $x_0 := 0$, fix an arbitrary $\epsilon > 0$, and define

$$C_\epsilon := [x_0^*, x_1^*] + \epsilon B_{X^*}.$$

Then C_ϵ is weak* closed, convex, bounded with $x_1^* \in \text{ran } T \cap \text{int } C_\epsilon$. Also, $x_0^* \in (\text{int } C_\epsilon) \setminus Tx_0$. Local maximal monotonicity (via Proposition 2.7.(iv)) and Fact 1.1 yield

$$\begin{aligned} 0 &> \frac{1}{2} \inf_{x \in X: Tx \in C_\epsilon} \langle Tx - x_0^*, x - x_0 \rangle = \inf_{x \in X} q(x) + \left\langle -\frac{1}{2}x_0^*, x \right\rangle + \iota_{C_\epsilon}(Tx) \\ &\geq - \inf_{x^{**} \in X^{**}} \left\{ q^* \left(-T^*x^{**} + \frac{1}{2}x_0^* \right) + \iota_{C_\epsilon}^*(x^{**}) \right\}. \end{aligned}$$

Now pick $x^{**} := \frac{1}{2}x_0^{**}$; then, using the fact that $q^*(0) = 0$,

$$\begin{aligned} 0 &< q^* \left(-\frac{1}{2}T^*x_0^{**} + \frac{1}{2}x_0^* \right) + \iota_{C_\epsilon}^* \left(\frac{1}{2}x_0^{**} \right) = \iota_{C_\epsilon}^* \left(\frac{1}{2}x_0^{**} \right) \\ &= \epsilon \left\| \frac{1}{2}x_0^{**} \right\| + \max \left\{ \left\langle \frac{1}{2}x_0^{**}, x_0^* \right\rangle, \left\langle \frac{1}{2}x_0^{**}, x_1^* \right\rangle \right\}. \end{aligned}$$

Multiply by 2 and let ϵ tend to 0 to obtain $0 \leq \max\{\langle x_0^{**}, T^*x_0^{**} \rangle, \langle x_0^{**}, Tx_1 \rangle\}$. Since $\langle x_0^{**}, Tx_1 \rangle = \langle T^*x_0^{**}, x_1 \rangle = \langle x_0^*, x_1 \rangle < 0$, we obtain $\langle x_0^{**}, T^*x_0^{**} \rangle \geq 0$.

“(vi) \Rightarrow (iv)” : In view of Proposition 2.7.(iv), let us fix a weak* closed convex bounded subset C of X^* with $\text{ran } T \cap \text{int } C \neq \emptyset$, $x_0 \in X$, $x_0^* \in (\text{int } C) \setminus Tx_0$. Let

$$p := \inf_{x \in X: Tx \in C} \frac{1}{2} \langle Tx - x_0^*, x - x_0 \rangle.$$

Clearly, $p < +\infty$ and our aim is $p < 0$. We thus can assume WLOG that $p > -\infty$, hence p is finite. Let $f(x) := q(x) + \frac{1}{2}\langle -x_0^* - T^*x_0, x \rangle + \frac{1}{2}\langle x_0^*, x_0 \rangle$,

$\forall x \in X$, and let $g := \iota_C$. Then, using Fact 1.1,

$$\begin{aligned} p &= \inf_{x \in X} f(x) + g(Tx) = - \inf_{x^{**} \in X^{**}} \{f^*(-T^*x^{**}) + g^*(x^{**})\} \\ &= \frac{1}{2} \langle x_0^*, x_0 \rangle - \inf_{x^{**} \in X^{**}} \left\{ q^* \left(-T^*x^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \iota_C^*(x^{**}) \right\}. \end{aligned}$$

Moreover: The last infimum is *attained* (by Fact 1.1 and $\text{ran } T \cap \text{int } C \neq \emptyset$), say at some $x_0^{**} \in X^{**}$. Thus the proof of “(vi) \Rightarrow (iv)” would be complete after reaching the following

$$\text{(Aim)} \quad \frac{1}{2} \langle x_0^*, x_0 \rangle < q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \iota_C^*(x_0^{**}).$$

By assumption, $0 \leq \langle S^*(x_0 - 2x_0^{**}), x_0 - 2x_0^{**} \rangle$, which is equivalent to

$$\begin{aligned} \left\langle -T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0, x_0 - 2x_0^{**} \right\rangle - \frac{1}{2} \langle x_0 - 2x_0^{**}, P^*(x_0 - 2x_0^{**}) \rangle \\ \geq \frac{1}{2} \langle x_0^*, x_0 \rangle - \langle x_0^{**}, x_0^* \rangle. \end{aligned}$$

On the other hand, Theorem 3.6.(iii) gives a bounded net (x_α) in X such that $x_\alpha \xrightarrow{w^*} x_0 - 2x_0^{**}$ and $Px_\alpha \rightarrow P^*(x_0 - 2x_0^{**})$; thus altogether

$$\begin{aligned} & q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) \\ & \geq \lim_{\alpha} \left\langle -T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0, x_\alpha \right\rangle - \frac{1}{2} \langle x_\alpha, Px_\alpha \rangle \\ & = \left\langle -T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0, x_0 - 2x_0^{**} \right\rangle \\ & \quad - \frac{1}{2} \langle x_0 - 2x_0^{**}, P^*(x_0 - 2x_0^{**}) \rangle \\ & \geq \frac{1}{2} \langle x_0^*, x_0 \rangle - \langle x_0^{**}, x_0^* \rangle. \end{aligned}$$

Consequently, since x_0^* is in the *interior* of C ,

$$\begin{aligned} \frac{1}{2} \langle x_0^*, x_0 \rangle &\leq q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \langle x_0^{**}, x_0^* \rangle \\ &< q^* \left(-T^*x_0^{**} + \frac{1}{2}x_0^* + \frac{1}{2}T^*x_0 \right) + \iota_C^*(x_0^{**}), \end{aligned}$$

which is what we aimed for.

We just proved that (i)-(vi) are equivalent for an *arbitrary* continuous linear operator \tilde{T} from X to X^* . If we apply this to $\tilde{T} = S$, then the remaining items are readily seen to be equivalent as well. \square

Remark 4.2. Gossez [13, End of Section 2] found the following question interesting:

Suppose that T is a closed densely defined linear monotone operator from X to X^* and that T^* is monotone. Is T_1 maximal monotone?

He then proved that the answer is “yes” if T is continuous and skew. We are now able to give an affirmative answer to this question *provided that T is merely continuous*: Indeed, this follows from Theorem 4.1. “(v) \Rightarrow (i)” and Proposition 2.7.(i).

Theorem 4.3. *Suppose T is a continuous linear operator from X to X^* with symmetric part P and skew part S . Then TFAE:*

- (i) T and $T^*|_X$ are monotone and of *dense* type.
- (ii) T and $T^*|_X$ are monotone and of type (NI).
- (iii) T and $T^*|_X$ are *locally maximal monotone*.
- (iv) T^* and $(T^*|_X)^*$ are monotone.
- (v) T is monotone and weakly compact.
- (vi) P is monotone and S is weakly compact.
- (vii) P is monotone and S^* is skew.
- (viii) P is monotone and $S, -S$ are of *dense* type.
- (ix) P is monotone and $S, -S$ are of type (NI).
- (x) P is monotone and $S, -S$ are *locally maximal monotone*.

Proof. Applying Theorem 4.1 to $T = P + S$ and $T^*|_X = P - S$ yields the equivalence of (i), (ii), (iii), (iv), (vii), (viii), (ix), and (x). Now (v) \Leftrightarrow (vi) (by weak compactness of P ; see Theorem 3.6.(ii)) \Leftrightarrow (vii) (by Theorem 3.7.(v)); so (i)-(x) are all equivalent. \square

In hindsight, we can interpret monotonicity of the conjugate of the skew part of a given continuous linear monotone operator as “one half of weak compactness”.

5. Examples and concluding remarks.

Suppose we are given a continuous linear monotone operator T from X to X^* with skew part S . In view of Theorem 4.1 and Theorem 4.3, the following three mutually exclusive alternatives are conceivable:

- T is “good”: Both S^* and $-S^*$ are monotone.
- T is “so-so”: Either S^* or $-S^*$ is monotone but not both.
- T is “bad”: Neither S^* nor $-S^*$ is monotone.

A priori, it is not clear that “so-so” or “bad” operators exist. However, this is indeed the case and we will now systematically recover two classical examples.

Theorem 5.1. *Suppose T is a continuous linear operator from X to X^* with skew part S and there exists some $e \in X^*$ such that*

$$e \notin \text{clran} T \quad \text{and} \quad \langle Tx, x \rangle = \langle e, x \rangle^2, \quad \forall x \in X.$$

Then T is monotone but S^ is not.*

If $\text{ran} T^ = \text{ran} T^*|_X$, equivalently, if $(\text{ran} T)^* \subseteq X$, then $-S^*$ is monotone.*

Proof. T is obviously monotone.

Let $Px := \langle e, x \rangle e$, $\forall x \in X$; then $\langle P^*x^{**}, x \rangle = \langle x^{**}, Px \rangle = \langle x^{**}, e \rangle \langle e, x \rangle$ and hence $P^*x^{**} = \langle x^{**}, e \rangle e$, $\forall x^{**} \in X^{**}$. So P is symmetric. Consider now $S := T - P$. Then $\langle Sx, x \rangle = \langle Tx, x \rangle - \langle Px, x \rangle = \langle Tx, x \rangle - \langle e, x \rangle^2 = 0$, $\forall x \in X$, thus S is skew. Since $T = P + S$, the symmetric (resp. skew) part of T is P (resp. S) by Proposition 3.5. Because $e \notin \text{clran} T = {}^\perp(\ker T^*)$, there exists some $x_0^{**} \in \ker T^*$ with $\langle x_0^{**}, e \rangle \neq 0$. Hence

$$\langle S^*x_0^{**}, x_0^{**} \rangle = \langle T^*x_0^{**}, x_0^{**} \rangle - \langle P^*x_0^{**}, x_0^{**} \rangle = 0 - \langle x_0^{**}, e \rangle^2 < 0;$$

so S^* is not monotone.

“If” part: First note that since $\text{ran} T \subseteq X^*$, the Hahn/Banach Theorem allows us to identify $(\text{ran} T)^*$ with $\{x^{**}|_{\text{ran} T} : x^{**} \in X^{**}\}$. We thus derive the “equivalently” part as follows.

$$\begin{aligned} \text{ran} T^* = \text{ran} (T^*|_X) &\Leftrightarrow \forall x^{**} \in X^{**} \quad \exists \hat{x} \in X : T^*x^{**} = T^*\hat{x} \\ &\Leftrightarrow \forall x^{**} \in X^{**} \quad \exists \hat{x} \in X \quad \forall x \in X : \langle x^{**} - \hat{x}, Tx \rangle = 0 \\ &\Leftrightarrow \forall x^{**} \in X^{**} \quad \exists \hat{x} \in X : x^{**}|_{\text{ran} T} = \hat{x}|_{\text{ran} T} \\ &\Leftrightarrow \forall z^* \in (\text{ran} T)^* \quad \exists \hat{x} \in X \subseteq X^{**} : z^* = \hat{x}|_{\text{ran} T} \\ &\Leftrightarrow (\text{ran} T)^* \subseteq X. \end{aligned}$$

Now fix an arbitrary $x^{**} \in X^{**}$. Then there exists $\hat{x} \in X \subseteq X^{**}$ with $T^*x^{**} = T^*\hat{x}$. Thus we have $\langle S^*x^{**}, x \rangle = \langle T^*x^{**}, x \rangle - \langle P^*x^{**}, x \rangle = \langle T^*\hat{x}, x \rangle - \langle P^*x^{**}, x \rangle$, $\forall x \in X$; hence

$$S^*x^{**} = T^*\hat{x} - P^*x^{**} = T^*\hat{x} - \langle x^{**}, e \rangle e.$$

Because $T^*|_X = P - S = 2P - T$, we further obtain

$$\begin{aligned} \langle S^*x^{**}, x^{**} \rangle &= \langle T^*\hat{x}, x^{**} \rangle - \langle x^{**}, e \rangle^2 = 2\langle P\hat{x}, x^{**} \rangle - \langle T\hat{x}, x^{**} \rangle - \langle x^{**}, e \rangle^2 \\ &= 2\langle P\hat{x}, x^{**} \rangle - \langle \hat{x}, T^*x^{**} \rangle - \langle x^{**}, e \rangle^2 \\ &= 2\langle P\hat{x}, x^{**} \rangle - \langle \hat{x}, T^*\hat{x} \rangle - \langle x^{**}, e \rangle^2 \\ &= 2\langle e, \hat{x} \rangle \langle x^{**}, e \rangle - \langle e, \hat{x} \rangle^2 - \langle x^{**}, e \rangle^2 = -\langle e, x^{**} - \hat{x} \rangle^2 \leq 0; \end{aligned}$$

consequently, $-S^*$ is monotone. \square

Here is the announced example of a “so-so” operator.

Example 5.2 (Gossez). Define the map G from ℓ_1 to ℓ_∞ by

$$(Gx)_n := - \sum_{k < n} x_k + \sum_{k > n} x_k, \quad \forall x = (x_k) \in \ell_1, n \in \mathbb{N}.$$

Then: G and $-G$ are continuous linear skew operators from ℓ_1 to $\ell_1^* = \ell_\infty$.

G^* is not monotone whereas $-G^*$ is; consequently:

G is neither of type (NI) nor **locally** maximal monotone;

$-G$ is both of **dense** type and **locally** maximal monotone.

Proof. Consider the map T from ℓ_1 to ℓ_∞ given by

$$(Tx)_n := x_n + 2 \sum_{k > n} x_k, \quad \forall x = (x_k) \in \ell_1, n \in \mathbb{N}.$$

Then T is linear, continuous (in fact, $\|T\| = 2$), and $\text{ran } T \subseteq c_0 \subseteq \ell_\infty$. Let $e := (1, 1, 1, \dots) \in \ell_\infty = \ell_1^*$. Then $e \notin \text{cl } \text{ran } T \subseteq \text{cl } c_0 = c_0$ and for every $x \in \ell_1$,

$$\begin{aligned} \langle Tx, x \rangle &= \sum_n x_n \left(x_n + \sum_{k > n} 2x_k \right) = \sum_n x_n^2 + \sum_n \sum_{k > n} 2x_n x_k \\ &= \sum_n x_n^2 + \sum_{n \neq k} x_n x_k \\ &= \sum_{n, k} x_n x_k = \left(\sum_n x_n \cdot 1 \right) \left(\sum_k x_k \cdot 1 \right) = \langle e, x \rangle^2. \end{aligned}$$

The proof of Theorem 5.1 shows that the symmetric part P of T is given by $Px = \langle e, x \rangle e$, $\forall x \in \ell_1$, and the skew part of T is $S := T - P$. Now for all $x \in \ell_1$, $n \in \mathbb{N}$:

$$\begin{aligned} (Sx)_n &= (Tx)_n - (Px)_n = x_n + 2 \sum_{k > n} x_k - \sum_k x_k \\ &= - \sum_{k < n} x_k + \sum_{k > n} x_k = (Gx)_n; \end{aligned}$$

hence $S = G$. By Theorem 5.1, G^* is not monotone. Hence (Theorem 4.1) G is neither of type (NI) nor **locally** maximal monotone. Because $(\text{ran } T)^* \subseteq c_0^* = \ell_1$, Theorem 5.1 yields that $-G^*$ is monotone. By Theorem 4.1, $-G$ is of **dense** type and **locally** maximal monotone. \square

Somewhat surprisingly, the ‘‘continuous’’ version of the (negative) Gossez operator yields a ‘‘bad’’ operator.

Example 5.3 (Fitzpatrick and Phelps). Define the map F from $L_1[0, 1]$ to $L_\infty[0, 1]$ by

$$(Fx)(t) := \int_0^t x(s) ds - \int_t^1 x(s) ds, \quad \forall x \in L_1[0, 1], t \in [0, 1].$$

Then $F, -F$ are continuous linear skew operators from $L_1[0, 1]$ to $L_1^*[0, 1] = L_\infty[0, 1]$.

Neither F^* nor $-F^*$ is monotone; consequently:

F and $-F$ are not of type (NI) nor **locally** maximal monotone.

Proof. Step 1. Define the map T from $L_1[0, 1]$ to $L_\infty[0, 1]$ by

$$(Tx)(t) := 2 \int_0^t x(s) ds, \quad \forall x \in L_1[0, 1], t \in [0, 1].$$

Then T is linear and continuous (with $\|T\| = 2$). The range of T is contained in the subspace $C_{0,0}$ of $L_\infty[0, 1]$ that consists of all equivalence classes that contain a continuous function vanishing at 0. Let e denote the equivalence class in $L_\infty[0, 1]$ that contains the constant function 1. Then the distance from e to any member in $C_{0,0}$ is at least 1; thus certainly $e \notin \text{cl ran } T$. Also, for every $x \in L_1[0, 1]$,

$$\begin{aligned} \langle Tx, x \rangle &= 2 \int_0^1 \left(\int_0^t x(s) ds \right) x(t) dt = 2 \iint_{s \leq t} x(s)x(t) ds dt \\ &= \iint_{[0,1] \times [0,1]} x(s)x(t) ds dt \\ &= \left(\int_0^1 x(s) \cdot 1 ds \right) \left(\int_0^1 x(t) \cdot 1 dt \right) = \langle x, e \rangle^2. \end{aligned}$$

Then (see again the proof of Theorem 5.1) the positive part P of T is given by $Px := \langle e, x \rangle e$, $\forall x \in L_1[0, 1]$. The skew part S of T is given by

$$\begin{aligned} (Sx)(t) &:= (Tx)(t) - (Px)(t) = 2 \int_0^t x(s) ds - \langle e, x \rangle e(t) \\ &= 2 \int_0^t x(s) ds - \int_0^1 x(s) ds \\ &= \int_0^t x(s) ds - \int_t^1 x(s) ds, \end{aligned}$$

for every $x \in L_1[0, 1]$, $t \in [0, 1]$; consequently, $S = F$. Now Theorem 5.1 and Theorem 4.1 imply that F^* is not monotone and F is neither of type (NI) nor **locally** maximal monotone.

Step 2. Here we define the map T by $(Tx)(t) := 2 \int_t^1 x(s) ds$, $\forall x \in L_1[0, 1]$, $t \in [0, 1]$. We let e be as in Step 1 and check analogously: T is continuous, linear, $e \notin \text{cl ran } T$, and $\langle Tx, x \rangle = \langle x, e \rangle^2$, $\forall x \in L_1[0, 1]$. This time, however, the skew part of T is equal to $-F$. We deduce as in Step 1 that $-F^*$ is not monotone and that $-F$ is neither of type (NI) nor **locally** maximal monotone. \square

Remark 5.4. Let G be the operator from Example 5.2. Gossez [13] proved that G is not of **dense** type whereas Phelps [22, Example 4.5] showed that G

is not **locally** maximal monotone. Let F be the operator from Example 5.3. Fitzpatrick and Phelps [9, Example 3.2] showed that F is not **locally** maximal monotone.

We observe that our discussion of G and F via Theorem 5.1 is much simpler.

We conclude by reporting on two sets of results that are closely connected to the present paper. We omit the proofs as we think the results are not in their final form; nonetheless, the interested reader is able to find the details in [3] or in [2].

Remark 5.5 (conjugate monotone spaces). We say that X is a *conjugate monotone space (cms)*, if the conjugate of every continuous linear monotone operator from X to X^* is monotone as well. Thus X is a conjugate monotone space precisely when it does not admit “so-so” or “bad” operators; equivalently, when every continuous linear monotone operator from X to X^* is weakly compact.

It is clear that reflexive spaces are (cms). Also, one can use Example 5.2 and Example 5.3 to show that none of ℓ_1 , $L_1[0, 1]$, ℓ_∞^* , $L_\infty^*[0, 1]$ is (cms).

A result relying on deeper Banach space theory states that for a Banach lattice X , TFAE:

- (i) X is (cms).
- (ii) X is *(w)*, *i.e.*, every continuous linear operator from X to X^* is weakly compact.
- (iii) X does not contain a complemented copy of ℓ_1 .

As a consequence, the classical spaces c_0 , c , ℓ_∞ , $L_\infty[0, 1]$, and $C[0, 1]$ are all (cms). It would be interesting to know whether or not (i)–(iii) are still equivalent for Banach *spaces*. (Note that (i) always implies (iii).)

Remark 5.6 (some nonlinear results). Recall that the *duality map*, denoted J , is the subdifferential map of the function $\frac{1}{2}\|\cdot\|^2$. Suppose T is a continuous linear monotone operator from X to X^* and $\lambda > 0$. Then the operator $T + \lambda J$ is called a *regularization* of T . Regularizations of monotone operators constitute perhaps the simplest class of *nonlinear* monotone operators. Then one can show that TFAE:

- (i) T is of **dense** type.
- (ii) $\text{clran}(T + \lambda J) = X^*$, $\forall \lambda > 0$.
- (iii) $T + \lambda J$ is of **range-dense** type, $\forall \lambda \geq 0$.

If the underlying space X is *rugged* (*i.e.*, $\text{clspanran}(J - J) = X^*$), then (i)–(iii) are also equivalent to:

- (iv) $\text{clran}(T + \lambda J)$ is convex, $\forall \lambda > 0$.
- (v) $T + \lambda J$ is **locally** maximal monotone, $\forall \lambda > 0$.

The full equivalence of (i)–(v) thus holds for the following rugged spaces: c_0 , c , ℓ_1 , ℓ_∞ , $L_1[0, 1]$, $L_\infty[0, 1]$, $C[0, 1]$.

Conclusion.

Maximal monotonicities of **dense** type, **range-dense** type, or type (NI), and local maximal monotonicity all coincide: • For subdifferentials of convex functions (Fact 2.5); • in reflexive spaces; • for continuous linear monotone operators (Theorem 4.1). These monotonicities always hold for subdifferentials of convex functions. They may well be absent for continuous linear monotone operators (Example 5.2 and Example 5.3); however, in reflexive and most of the classical nonreflexive spaces, they are automatic (Remark 5.5).

The question whether or not the monotonicities all coincide for a general maximal monotone operator remains open.

Some preliminary results (Remark 5.6) seem to indicate that this may well be the case.

Acknowledgments.

It is our great pleasure to thank Tamás Erdélyi, Simon Fitzpatrick, and Jon Vanderwerff for various helpful comments and discussions. We also wish to thank Robert Phelps and Stephen Simons for kindly sending us [23, 24, 28, 29]. We would like to refer the reader especially to [24], which contains many results on linear maximal monotone operators that are not necessarily continuous.

References

- [1] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley-Interscience, New York, 1984.
- [2] H.H. Bauschke, *Projection Algorithms and Monotone Operators*, PhD thesis, Simon Fraser University, Department of Mathematics, Burnaby, British Columbia V5A 1S6, Canada, August 1996; Available at <http://www.cecm.sfu.ca/preprints/1996pp.html>.
- [3] H.H. Bauschke and J.M. Borwein, *Continuous linear monotone operators on Banach spaces*, Technical report, Centre for Experimental & Constructive Mathematics (CECM), Simon Fraser University, 1995; CECM Information Document 95-049; Available at <http://www.cecm.sfu.ca/preprints/1995pp.html>.
- [4] F.E. Browder, editor, *Nonlinear Functional Analysis*, American Mathematical Society, Providence, Rhode Island, 1970; Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society held in Chicago, Illinois, April 16-19, 1968; Proceedings of Symposia in Pure Mathematics Volume XVIII, Part I.
- [5] J.B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, **96**, Springer-Verlag, New York, second edition, 1990.
- [6] V. Dolezal, *Monotone operators and applications in Control and Network Theory*, Elsevier Scientific Publishing Company, Amsterdam, The Netherlands, 1979.
- [7] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland Publ., Amsterdam, 1976.
- [8] S.P. Fitzpatrick and R.R. Phelps, *Bounded approximations to monotone operators on Banach spaces*, Annales de l'Institut Henri Poincaré, Analyse Non Linéaire, **9(5)** (1992), 573-595.

- [9] ———, *Some properties of maximal monotone operators on nonreflexive Banach spaces*, Set-Valued Analysis, **3** (1995), 51-69.
- [10] S. Fučík and A. Kufner, *Nonlinear Differential Equations*, Studies in Applied Mechanics, **2**, Elsevier Scientific Publishing Company, Amsterdam, 1980.
- [11] A. Ghizzetti, editor. *Theory and Applications of monotone operators*, Gubbio, Italy, 1969; Edizioni ‘Oderisi’, Proceedings of a NATO Advanced Study Institute held in Venice, Italy, June 17-30, 1968.
- [12] J.-P. Gossez, *Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs*, Journal of Mathematical Analysis and Applications, **34** (1971), 371-395.
- [13] ———, *On the range of a coercive maximal monotone operator in a nonreflexive Banach space*, Proceedings of the American Mathematical Society, **35**(1) (September 1972), 88-92.
- [14] ———, *On a convexity property of the range of a maximal monotone operator*, Proceedings of the American Mathematical Society, **55**(2) (March 1976), 359-360.
- [15] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I*, Grundlehren der mathematischen Wissenschaften, **305**, Springer-Verlag, 1993.
- [16] ———, *Convex Analysis and Minimization Algorithms II*, Grundlehren der mathematischen Wissenschaften, **306**, Springer-Verlag, 1993.
- [17] R.B. Holmes, *Geometric Functional Analysis and its Applications*, Springer-Verlag, New York, 1975.
- [18] G.J.O. Jameson, *Topology and Normed Spaces*, Chapman and Hall, 1974.
- [19] G.J. Minty, *On some aspects of the theory of monotone operators*, in A. Ghizzetti, editor, Theory and Applications of monotone operators, pages 67-82, Gubbio, Italy, 1969; Edizioni ‘Oderisi’, Proceedings of a NATO Advanced Study Institute held in Venice, Italy, June 17-30, 1968.
- [20] D. Pascali and S. Sburian, *Nonlinear mappings of monotone type*, Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, The Netherlands, 1978.
- [21] R.R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics, **1364**, Springer-Verlag, second edition, 1993.
- [22] ———, *Lectures on maximal monotone operators*, 1993; TeX file: phelpsmx-monop.tex, Banach space bulletin board archive: ftp.math.okstate.edu.
- [23] ———, *Linear monotone operators as subdifferentials*, August 1994, Unpublished notes.
- [24] R.R. Phelps and S. Simons, *Unbounded linear monotone operators on nonreflexive Banach spaces*, Journal of Convex Analysis, **5**(2), 1998.
- [25] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
- [26] ———, *On the maximal monotonicity of subdifferential mappings*, Pacific Journal of Mathematics, **33** (1970), 209-216.
- [27] S. Simons, *Subdifferentials are locally maximal monotone*, Bulletin of the Australian Mathematical Society, **47** (1993), 465-471.
- [28] ———, *The range of a monotone operator*, Journal of Mathematical Analysis and Applications, **199** (1996), 176-201.
- [29] ———, *Minimax and Monotonicity*, Lecture Notes in Mathematics, **1693**, Springer-Verlag, 1998.

- [30] A. Wilansky, *Modern methods in topological vector spaces*, McGraw-Hill, 1978.
- [31] E.H. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, Academic Press, New York, 1971; University of Wisconsin, Mathematics Research Center, Publication No. 27.
- [32] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Volume 2, Part A: Linear Monotone Operators*, Springer-Verlag, New York, 1990.
- [33] ———, *Nonlinear Functional Analysis and its Applications, Volume 2 Part B: Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.

Received November 25, 1996 and revised August 28, 1998. The first author's research was supported by an NSERC Postdoctoral Fellowship. The second author's research was supported by NSERC and by the Shrum Endowment.

OKANAGAN UNIVERSITY COLLEGE
KELOWNA, B.C. V1V 1V7
CANADA
E-mail address: bauschke@cecm.sfu.ca

SIMON FRASER UNIVERSITY
BURNABY, BRITISH COLUMBIA
V5A 1S6, CANADA
E-mail address: jborwein@cecm.sfu.ca

This paper is available via <http://nyjm.albany.edu:8000/PacJ/1999/189-1-1.html>.