Maximally monotone operators with ranges whose closures are not convex and an answer to a recent question by Stephen Simons

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Abstract

In his recent *Proceedings of the AMS* paper "Gossez's skew linear map and its pathological maximally monotone multifunctions", Stephen Simons proved that the closure of the range of the sum of the Gossez operator and a multiple of the duality map is not convex whenever the scalar is between 0 and 4. The problem of the convexity of that range when the scalar is equal to 4 was explicitly stated. In this paper, we answer this question in the negative for any scalar greater than or equal to 4. We derive this result from an abstract framework that allows us to also obtain a corresponding result for the Fitzpatrick-Phelps integral operator.

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1 Introduction

Throughout, we assume that

$$X \text{ is a real Banach space with dual pairing } \langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}, \tag{1}$$

where X^* is the dual space of X. The *duality mapping* J of X is the subdifferential operator of the function $\frac{1}{2}\|\cdot\|^2 \colon X \to \mathbb{R}$; it satisfies

$$(\forall (x,x^*) \in X \times X^*) \quad x^* \in \mathrm{J}x \;\Leftrightarrow\; \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2. \tag{2}$$

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Now let $A\colon X \to X^*$ be a bounded linear monotone operator, i.e., $(\forall x \in X)(\forall y \in X)$ $\langle x-y, Ax-Ay\rangle \geq 0$. Then both A and J are maximally monotone, and so is their sum $A+\mathrm{J}$ thanks to a result by Heisler (see, e.g., [8, Theorem 40.4]). If X is a Hilbert space, then $\mathrm{J} = \mathrm{Id}$ and it is well known that the range $\operatorname{ran}(A+\lambda\mathrm{J})$ is equal to X. In striking contrast, it was shown very recently by Simons that for the so-called Gossez operator G which acts on ℓ_1, we have that $\overline{\operatorname{ran}}\,(G+\lambda\mathrm{J})$ is *not convex* for $0<\lambda<4$ (see Section 4 below.) It is also known that for the so-called Fitzpatrick-Phelps operator F, which acts on $L_1[0,1]$, the set $\overline{\operatorname{ran}}\,(F+1\,\mathrm{J})$ is *not convex*.

In this paper, we unify these results by providing an abstract result that allows us to deduce that no matter how $\lambda>0$ is chosen, neither $\overline{\operatorname{ran}}\,(F+\lambda\mathrm{J})$ nor $\overline{\operatorname{ran}}\,(G+\lambda\mathrm{J})$ is convex. This provides a negative answer to Simons's [7, Problem 3.6].

Let us note here that neither F nor G is a subdifferential operator of a convex function; indeed, if $f\colon X\to\,]-\infty,+\infty]$ is convex, lower semicontinuous, and proper, then $f+\lambda\frac{1}{2}\|\cdot\|^2$ is supercoercive[1] and hence $\overline{\operatorname{ran}}\,\partial(f+\lambda\frac{1}{2}\|\cdot\|^2) = \overline{\operatorname{ran}}\,(\partial f+\lambda\mathrm{J}) = X^*$ by Gossez's [4, Corollaire 8.2].

The remainder of this note is organized as follows. In Section 2 we collect some auxiliary results for later use. Section 3 contains our main abstract results. Section 4 discusses the Gossez operator while Section 5 deals with the Fitzpatrick-Phelps operator.

2 Auxiliary results

In this section, we cover some technical results that will ease the proofs in subsequent sections.

Rugged Banach spaces

Definition 2.1. *(See also [1].) We say that X is* rugged *if*

$$\overline{\operatorname{span}}\operatorname{ran}(\mathrm{J}-\mathrm{J}) = X^*. \tag{3}$$

Example 2.2. *(See [2, Remark 5.6] and also [1, Section 2.3.4].) The following Banach spaces are rugged:* (i) ℓ_1 *and* (ii) $L_1[0,1]$.

Proof. First, we define $(\forall\xi\in\mathbb{R})$ $\operatorname{Sign}(\xi) = \{\operatorname{sign}(\xi)\}$, if $\xi\neq 0$; and $\operatorname{Sign}(\xi) = [-1,1] = \partial|\cdot|(0)$, otherwise.

(i): Let $x=(x_n)_{n\in\mathbb{N}}\in\ell_1$, where $\mathbb{N}=\{1,2,\ldots\}$. We start by proving the following claim:

$$(\forall n\in\mathbb{N})\quad (\mathrm{J}x)_n = \|x\|_1\operatorname{Sign}(x_n). \tag{4}$$

[1] Recall that $g\colon X\to\,]-\infty,+\infty]$ is supercoercive if $\lim_{\|x\|\to+\infty} g(x)/\|x\| = +\infty$.

Indeed, let $y = (y_n)_{n\in\mathbb{N}} \in \ell_\infty$. If $(\forall n \in \mathbb{N})\ y_n \in \|x\|_1 \operatorname{Sign}(x_n)$, then $\|x\|_1^2 = \|y\|_\infty^2 = \langle x, y\rangle$ and hence $y \in \mathrm{J}x$. Conversely, suppose that $y \in \mathrm{J}x$. Then

$$(\forall n \in \mathbb{N}) \quad |y_n| \leq \|y\|_\infty = \|x\|_1 \tag{5}$$

and

$$\|x\|_1^2 = \|y\|_\infty^2 = \|x\|_1\|y\|_\infty = \langle x, y\rangle = \sum_{n\in\mathbb{N}} x_n y_n \tag{6a}$$

$$\leq \sum_{n\in\mathbb{N}} |x_n||y_n| \leq \sum_{n\in\mathbb{N}} |x_n|\|y\|_\infty = \|x\|_1\|y\|_\infty. \tag{6b}$$

Hence $\sum_{n\in\mathbb{N}} |x_n|(\|y\|_\infty - |y_n|) = 0$; equivalently, $(\forall n \in \mathbb{N})\ |x_n|(\|y\|_\infty - |y_n|) = 0$. Now, if $x_n = 0$ then (5) implies that $y_n \in \|x\|_1 \operatorname{Sign}(x_n)$. Alternatively, if $x_n \neq 0$ we must have $|y_n| = \|y\|_\infty$ which, in view of (6), implies that $y_n = \operatorname{sign}(x_n)|y_n| = \operatorname{sign}(x_n)\|y\|_\infty \in \operatorname{Sign}(x_n)\|x\|_1$. Next, we show that ℓ_1 satisfies (3). Clearly, $\{1, 2\} \subseteq \mathbb{N}$, and we denote the corresponding canonical unit vectors in ℓ_1 by e_1 and e_2, respectively. If $i \in \{1, 2\}$ and $n \in \mathbb{N} \smallsetminus \{i\}$, then (4) yields $(\mathrm{J}e_i)_i = \{1\}$ and $(\mathrm{J}e_i)_n = [-1, 1]$; consequently, $(\mathrm{J}e_i - \mathrm{J}e_i)_i = \{0\}$ and $(\mathrm{J}e_i - \mathrm{J}e_i)_n = [-2, 2]$. Therefore, if $i \in \{1, 2\}$ and $n \in \mathbb{N} \smallsetminus \{1, 2\}$, then $(\mathrm{J}e_1 - \mathrm{J}e_1 + \mathrm{J}e_2 - \mathrm{J}e_2)_i = [-2, 2]$ while $(\mathrm{J}e_1 - \mathrm{J}e_1 + \mathrm{J}e_2 - \mathrm{J}e_2)_n = [-4, 4]$. It follows that $(\forall n \in \mathbb{N})\ [-2, 2] \subseteq (\mathrm{J}e_1 - \mathrm{J}e_1 + \mathrm{J}e_2 - \mathrm{J}e_2)_n$, which in turn implies (3).

(ii): The proof parallels that of item (i); for completeness, we provide the details. Let $x \in L_1[0, 1]$. We first claim that

$$(\forall t \in [0, 1]) \quad (\mathrm{J}x)(t) = \|x\|_1 \operatorname{Sign}(x(t)) \ \text{ almost everywhere (a.e.).} \tag{7}$$

Indeed, let $y = L_\infty[0, 1]$. If $(\forall t \in [0, 1])\ y(t) \in \|x\|_1 \operatorname{Sign}(x(t))$ a.e., then $\|x\|_1^2 = \|y\|_\infty^2 = \langle x, y\rangle$ and hence $y \in \mathrm{J}x$. Conversely, suppose that $y \in \mathrm{J}x$. Then

$$(\forall t \in [0, 1]) \quad |y(t)| \leq \|y\|_\infty = \|x\|_1 \ \text{ a.e.} \tag{8}$$

and

$$\|x\|_1^2 = \|y\|_\infty^2 = \|x\|_1\|y\|_\infty = \langle x, y\rangle = \int_0^1 x(t)y(t)\,dt \tag{9a}$$

$$\leq \int_0^1 |x(t)||y(t)|\,dt \leq \int_0^1 |x(t)|\|y\|_\infty\,dt = \|x\|_1\|y\|_\infty. \tag{9b}$$

Hence $\int_0^1 |x(t)|(\|y\|_\infty - |y(t)|)\,dt = 0$; equivalently, $(\forall t \in [0, 1])\ |x(t)|(\|y\|_\infty - |y(t)|) = 0$ a.e. Now if $x(t) = 0$, then (8) implies that $y(t) \in \|x\|_1 \operatorname{Sign}(x(t))$. Alternatively, if $x(t) \neq 0$, then we must have $|y(t)| = \|y\|_\infty$ which, in view of (9), implies that $y(t) = \operatorname{sign}(x(t))|y(t)| = \operatorname{sign}(x(t))\|y\|_\infty \in \operatorname{Sign}(x(t))\|x\|_1$ a.e. It remains to show that $L_1[0, 1]$ satisfies (3). Set $A_1 = [0, 1/3]$ and $A_2 = [2/3, 1]$ and also $e_1 = 3\chi_{A_1}$ and $e_2 = 3\chi_{A_2}$, where χ_B denotes the characteristic function of a subset B of $[0, 1]$. If $i \in \{1, 2\}$, $s \in A_i$ and $t \in [0, 1] \smallsetminus A_i$, then $\|e_i\|_1 = 1$ and (7) yields $(\mathrm{J}e_i)(s) = \{1\}$ and $(\mathrm{J}e_i)(t) = [-1, 1]$; consequently, $(\mathrm{J}e_i - \mathrm{J}e_i)(s) = \{0\}$ and $(\mathrm{J}e_i - \mathrm{J}e_i)(t) = [-2, 2]$. Therefore, if $s \in A := A_1 \cup A_2$ and $t \in [0, 1] \smallsetminus A$, then $(\mathrm{J}e_1 - \mathrm{J}e_1 + \mathrm{J}e_2 - \mathrm{J}e_2)(s) = [-2, 2]$ while $(\mathrm{J}e_1 - \mathrm{J}e_1 + \mathrm{J}e_2 - \mathrm{J}e_2)(t) = [-4, 4]$. It follows that $(\forall t \in [0, 1])\ [-2, 2] \subseteq (\mathrm{J}e_1 - \mathrm{J}e_1 + \mathrm{J}e_2 - \mathrm{J}e_2)(t)$ a.e., which in turn implies (3). ∎

Proposition 2.3. *Let X be rugged, let $A\colon X \to X^*$ be a linear operator, and let $\lambda > 0$. Then* $\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}) = X^*$.

Proof. We prove this by contradiction and thus assume that there exists $x^* \in X^* \smallsetminus (\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}))$. The separation theorem yields $x^{**} \in X^{**} \smallsetminus \{0\}$ such that $\langle x^*, x^{**}\rangle > \sup \langle \overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}), x^{**}\rangle \geq \sup \langle \operatorname{ran}\,(A+\lambda\,\mathrm{J}), x^{**}\rangle$. Because $\operatorname{ran}\,(A+\lambda\,\mathrm{J})$ is a balanced cone (i.e., closed under scalar multiplication), we deduce that $(\forall x \in X)\ \langle Ax+\lambda\,\mathrm{J}x, x^{**}\rangle = 0$. Because A is single-valued and $\lambda \neq 0$, it follows that $(\forall x \in X)\ \langle \mathrm{J}x - \mathrm{J}x, x^{**}\rangle = 0$. Therefore, $\langle X^*, x^{**}\rangle = \langle \overline{\operatorname{span}}\operatorname{ran}\,(\mathrm{J}-\mathrm{J}), x^{**}\rangle = 0$ and thus $x^{**} = 0$ which is absurd. ■

Corollary 2.4. *Let X be rugged, let $A\colon X \to X^*$ be a linear operator, and let $\lambda > 0$. Then the following are equivalent:*

(i) $\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}) = X^*$;
(ii) $\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J})$ *is a subspace;*
(iii) $\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J})$ *is a convex set.*

Proof. (See also [1, Proposition 15.3.8] for "(i)⇔(iii)".) The implications "(i)⇒(ii)" and "(ii)⇒(iii)" are clear. "(iii)⇒(i)": Assume that $\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J})$ is convex. Then $\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}) = \overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J})$. On the other hand, by Proposition 2.3, $\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}) = X^*$. Altogether, we deduce that $\overline{\operatorname{ran}}\,(A+\lambda\,\mathrm{J}) = X^*$. ■

3 Main result

Lemma 3.1. *Let $A\colon X \to X^*$ be a bounded linear monotone operator. Let $\lambda > 0$ and let $r^* \in X^*$. Suppose that $x \in X$ satisfies*

$$r^* \in Ax + \lambda\,\mathrm{J}x. \tag{10}$$

Then

$$\frac{\|r^*\|}{\|A\|+\lambda} \leq \|x\| \leq \frac{\|r^*\|}{\lambda} \quad \textit{and} \quad \lambda\|x\|^2 \leq \langle x, r^*\rangle\,, \tag{11}$$

and[2]

$$A \text{ is skew} \quad \Rightarrow \quad \lambda\|x\|^2 = \langle x, r^*\rangle\,. \tag{12}$$

Proof. Let $x^* \in \mathrm{J}x$ satisfy $r^* = Ax + \lambda x^*$. Combining (2) and the triangle inequality yields $\lambda\|x\| = \lambda\|x^*\| = \|r^* - Ax\| \geq \|r^*\| - \|Ax\| \geq \|r^*\| - \|A\|\|x\|$. Hence $(\lambda + \|A\|)\|x\| \geq \|r^*\|$ which implies the first inequality in (11). On the other hand, from (2) and the monotonicity of A, we deduce that $\lambda\|x\|^2 = \lambda\langle x, x^*\rangle \leq \langle x, \lambda x^* + Ax\rangle = \langle x, r^*\rangle \leq \|x\|\|r^*\|$, which yields the remaining inequalities in (11) as well as (12). ■

[2] Recall that $A\colon X \to X^*$ is skew if any of the following equivalent conditions holds: (i) $A^*|_X = -A$; (ii) $(\forall x \in X)(\forall y \in X)\ \langle x, Ay\rangle = -\langle y, Ax\rangle$; (iii) $\pm A$ are monotone; (iv) $(\forall x \in X)\ \langle x, Ax\rangle = 0$.

Theorem 3.2. *Let $A\colon X \to X^*$ be a bounded linear monotone operator. Let $\lambda > 0$ and $\varepsilon \geq 0$. Suppose that $x \in X$ and $f^* \in X^*$ satisfy*[3]

$$\big(Ax + \lambda\,\mathrm{J}x\big) \cap \mathrm{ball}(f^*;\varepsilon) \neq \varnothing \tag{13}$$

and

$$\varepsilon \leq \frac{2\lambda\|f^*\|}{\|A\| + 3\lambda}. \tag{14}$$

Then

$$\langle x, f^*\rangle \geq \lambda\left(\frac{\|f^*\| - \varepsilon}{\|A\| + \lambda}\right)^2 - \varepsilon\left(\frac{\|f^*\| - \varepsilon}{\|A\| + \lambda}\right). \tag{15}$$

Proof. Set $\varphi = \|f^*\|$ and let $r^* \in (Ax + \lambda\,\mathrm{J}x) \cap \mathrm{ball}(f^*;\varepsilon)$. Combining this with (11), we obtain

$$\lambda\|x\|^2 \leq \langle x, r^*\rangle = \langle x, r^* - f^*\rangle + \langle x, f^*\rangle \leq \|x\|\varepsilon + \langle x, f^*\rangle\,. \tag{16}$$

Next, set $\rho = \|r^*\|$, $\alpha = \|A\|$, and $\xi = \|x\|$. Then (16) and (11) yield

$$\langle x, f^*\rangle \geq \lambda\xi^2 - \varepsilon\xi, \quad \text{where} \quad \frac{\rho}{\alpha + \lambda} \leq \xi \leq \frac{\rho}{\lambda}. \tag{17}$$

Note that $\|f^*\| - \|f^* - r^*\| \leq \|r^*\| \leq \|f^*\| + \|r^* - f^*\|$ which implies

$$\varphi - \varepsilon \leq \rho \leq \varphi + \varepsilon. \tag{18}$$

Now, set $l = (\varphi - \varepsilon)/(\alpha + \lambda) \geq 0$. On the one hand, simple algebraic manipulations show that (14) is equivalent to $\varphi/(\alpha + 3\lambda) \leq l$; consequently, $2\lambda\varphi/(\alpha + 3\lambda) \leq 2\lambda l$. On the other hand, (14) states that $\varepsilon \leq 2\lambda\varphi/(\alpha + 3\lambda)$. Altogether,

$$\varepsilon \leq 2\lambda l. \tag{19}$$

Furthermore, (17) and (18) yield $\xi \geq \rho/(\alpha + \lambda) \geq (\varphi - \varepsilon)/(\alpha + \lambda) = l$; hence,

$$l \leq \xi. \tag{20}$$

Next, consider the function $g(\eta) = \lambda\eta^2 - \varepsilon\eta$. Let $\eta \geq l$ and observe that (19) implies that $g'(\eta) = 2\lambda\eta - \varepsilon \geq 2\lambda l - \varepsilon \geq 0$. Hence, g is increasing on $[l, +\infty[$. Combining (17) and (20), we learn that $\langle x, f^*\rangle \geq g(\xi) \geq g(l)$. Hence $\langle x, f^*\rangle \geq g(l)$, which is precisely (15). ■

Definition 3.3 ((whs) condition). *Let $A\colon X \to X^*$ be a bounded linear monotone operator, and let $f^* \in X^*$. We say that the* (whs) condition[4] *holds if $\|A\| = 1$, $\|f^*\| = 1$, and for every $\lambda > 0$ and every $\varepsilon > 0$, the implication*

$$(Ax + \lambda\,\mathrm{J}x) \cap \mathrm{ball}(f^*;\varepsilon) \neq \varnothing \quad \Rightarrow \quad \langle x, f^*\rangle \leq 3\varepsilon \tag{21}$$

is true.

[3] We denote the closed ball centered at f^* of radius ε by $\mathrm{ball}(f^*;\varepsilon)$.

[4] "whs" stands for *wondrous half-space*.

For every $\lambda > 0$, we define for future convenience

$$m(\lambda) = \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2}. \tag{22}$$

The function m plays a role in the statement and proofs of this section. The next result will also establish the strict positivity[5] of m.

Corollary 3.4. *Let $A\colon X \to X^*$ be a bounded linear monotone operator with $\|A\| = 1$, let $f^* \in X^*$ be such that $\|f^*\| = 1$, let $\lambda > 0$ and assume that there exist $x \in X$ and $\varepsilon > 0$ such that $\varepsilon \leq 2\lambda/(1+3\lambda)$ and $(Ax + \lambda Jx) \cap \operatorname{ball}(f^*;\varepsilon) \neq \varnothing$. Finally, assume that the (whs) condition holds. Then*

$$(2\lambda+1)\varepsilon^2 - (3\lambda^2 + 9\lambda + 4)\varepsilon + \lambda \leq 0; \tag{23}$$

consequently,

$$\varepsilon \geq \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2} = m(\lambda) > 0. \tag{24}$$

Proof. The (whs) condition implies that $\langle x, f^* \rangle \leq 3\varepsilon$. On the other hand, (15) yields $\langle x, f^* \rangle \geq \lambda(1-\varepsilon)^2/(1+\lambda)^2 - \varepsilon(1-\varepsilon)/(1+\lambda)$. Altogether,

$$\lambda\frac{(1-\varepsilon)^2}{(1+\lambda)^2} - \varepsilon\frac{1-\varepsilon}{1+\lambda} \leq 3\varepsilon. \tag{25}$$

In turn, (25) is equivalent to

$$\lambda(1-\varepsilon)^2 - \varepsilon(1-\varepsilon)(1+\lambda) \leq 3\varepsilon(1+\lambda)^2 \tag{26a}$$

$$\Leftrightarrow (\lambda + \varepsilon^2\lambda - 2\lambda\varepsilon) + (-\varepsilon - \varepsilon\lambda + \varepsilon^2 + \varepsilon^2\lambda) \leq 3\varepsilon + 3\varepsilon\lambda^2 + 6\varepsilon\lambda \tag{26b}$$

$$\Leftrightarrow \varepsilon^2(1+2\lambda) + \varepsilon(-9\lambda - 4 - 3\lambda^2) + \lambda \leq 0, \tag{26c}$$

which is equivalent to (23). Now let's view (23) as a quadratic inequality. Then ε must lie in the closed interval given by the roots of the corresponding quadratic equation

$$(2\lambda+1)\xi^2 - (3\lambda^2 + 9\lambda + 4)\xi + \lambda = 0, \tag{27}$$

where the variable is ξ. The two roots of (27) are

$$\frac{3\lambda^2 + 9\lambda + 4 \pm \sqrt{(3\lambda^2 + 9\lambda + 4)^2 - 4(1+2\lambda)\lambda}}{2(1+2\lambda)} \tag{28a}$$

$$= \frac{3\lambda^2 + 9\lambda + 4 \pm (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2}, \tag{28b}$$

and both roots are positive. ■

[5]In passing, we note that $\lim_{\lambda\to 0^+} m(\lambda) = 0$, $\lim_{\lambda\to +\infty} m(\lambda) = 0$, m is strictly increasing on $[0,\bar{\lambda}]$ and strictly decreasing on $[\bar{\lambda}, +\infty[$, where $\bar{\lambda} = (3+\sqrt{15})/6 \approx 1.14550$, and $m(\bar{\lambda}) = (9 - 2\sqrt{15})/(12 + 2\sqrt{15}) \approx 0.06349$. We omit the proofs as these properties are not needed in this paper.

Theorem 3.5 (key result). *Let $A\colon X \to X^*$ be a bounded linear monotone operator with $\|A\| = 1$, let $f^* \in X^*$ be such that $\|f^*\| = 1$, and let $\lambda > 0$. Furthermore, assume the (whs) condition holds. Then either*

$$d\big(f^*, \operatorname{ran}(A + \lambda\,\mathrm{J})\big) \geq \frac{2\lambda}{1+3\lambda}, \tag{29}$$

or

$$\frac{2\lambda}{1+3\lambda} > d\big(f^*, \operatorname{ran}(A + \lambda\,\mathrm{J})\big) \geq \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2} > 0. \tag{30}$$

In any case,

$$d\big(f^*, \operatorname{ran}(A + \lambda\,\mathrm{J})\big) \geq \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2} > 0. \tag{31}$$

Proof. Assume that the "either" alternative fails, i.e., $d(f^*, \operatorname{ran}(A + \lambda\,\mathrm{J})) < 2\lambda/(1+3\lambda)$. Then there exist $x_n \in X$ and $x_n^* \in \mathrm{J}x_n$ such that $\varepsilon_n := \|f^* - (Ax_n + \lambda x_n^*)\| \leq 2\lambda/(1+3\lambda)$ and $\varepsilon_n \downarrow d(f^*, \operatorname{ran}(A + \lambda\,\mathrm{J}))$. Now (24) of Corollary 3.4 yields

$$\varepsilon_n \geq \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2} > 0. \tag{32}$$

Hence the "or" case follows by letting $n \to +\infty$. Finally, we claim that

$$\frac{2\lambda}{1+3\lambda} > \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2}. \tag{33}$$

It is straightforward but a bit tedious to verify that

$$\frac{2\lambda}{1+3\lambda} - \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2} = \frac{1+\lambda}{2(1+2\lambda)(1+3\lambda)} \cdot \tau, \tag{34}$$

where

$$\tau = (3\lambda + 1)\sqrt{9\lambda^2 + 36\lambda + 16} - \big(9\lambda^2 + 13\lambda + 4\big). \tag{35}$$

It remains to show that $\tau > 0$. Indeed,

$$\tau > 0 \Leftrightarrow (3\lambda+1)^2(9\lambda^2 + 36\lambda + 16) > (9\lambda^2 + 13\lambda + 4)^2 \tag{36a}$$

$$\Leftrightarrow (3\lambda+1)^2(9\lambda^2 + 36\lambda + 16) - (9\lambda^2 + 13\lambda + 4)^2 = 144\lambda^3 + 128\lambda^2 + 28\lambda > 0. \tag{36b}$$

■

We are now ready for our abstract main result.

Corollary 3.6 (main result). *Suppose that X is rugged, let $A\colon X \to X^*$ be a bounded linear monotone operator with $\|A\| = 1$, let $f^* \in X^*$ be such that $\|f^*\| = 1$, and assume that the (whs) condition holds. Let $\lambda > 0$, set $R = \overline{\operatorname{ran}}\,(A + \lambda\,\mathrm{J})$, and let $\alpha \in \mathbb{R} \smallsetminus \{0\}$. Then R is* not convex, *$\overline{\operatorname{conv}}\,R = X^*$, and*

$$d(\alpha f^*, R) \geq |\alpha| m(\lambda) > 0, \tag{37}$$

where

$$m(\lambda) = \frac{3\lambda^2 + 9\lambda + 4 - (\lambda+1)\sqrt{9\lambda^2 + 36\lambda + 16}}{4\lambda + 2} > 0. \tag{38}$$

Proof. Because A and J are homogeneous, it follows that $\alpha R = R$. Hence $d(\alpha f^*, R) = d(\alpha f^*, \alpha R) = |\alpha| d(f^*, R) \geq |\alpha| m(\lambda)$ by Theorem 3.5. The lack of convexity of R and the fact that $\overline{\operatorname{conv}}\, R = X^*$ follow from Corollary 2.4 and Proposition 2.3, respectively. ■

4 The Gossez operator revisited

In this section, we assume that

$$X = \ell_1, \tag{39}$$

and that

$$G\colon \ell_1 \to \ell_\infty\colon x = (x_n)_{n\in\mathbb{N}} \mapsto (Gx)_{n\in\mathbb{N}}, \quad \text{where } (Gx)_n = -\sum_{k<n} x_k + \sum_{k>n} x_k \tag{40}$$

is the Gossez operator [5] and [6]. It is easy to see that G is a bounded linear operator with $\|G\| = 1$ and that G is skew, hence monotone.

We are now ready for the main result concerning the Gossez operator.

Theorem 4.1. *The Gossez operator G satisfies the (whs) condition with $f^* = -(1, 1, \ldots) \in \ell_\infty$; consequently, $(\forall \lambda > 0)$ $\overline{\operatorname{ran}}\,(G + \lambda\mathrm{J})$ is not convex yet $\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(G + \lambda\mathrm{J}) = \ell_\infty$.*

Proof. Let $\lambda > 0$, and let $\varepsilon > 0$. Assume that $(x, r^*) \in \ell_1 \times \ell_\infty$ satisfies $r^* \in (Gx + \lambda\mathrm{J}x) \cap \operatorname{ball}(f^*; \varepsilon)$. First, there exists $x^* \in \mathrm{J}x$ such that $r^* = Gx + \lambda x^*$. By definition of G, we have $(\forall n \in \mathbb{N})$ $r_n^* = -\sum_{k<n} x_k + \sum_{k>n} x_k + \lambda x_n^*$. Letting $n \to \infty$, we deduce that

$$\lim_{n\to\infty} (\lambda x_n^* - r_n^*) = \sum_{k=1}^{\infty} x_k = \langle x, -f^* \rangle =: \sigma. \tag{41}$$

Second, because $r^* \in \operatorname{ball}(f^*; \varepsilon)$, we have $\|r^* - f^*\| \leq \varepsilon$. Thus $(\forall n \in \mathbb{N})$ $|r_n^* + 1| \leq \varepsilon$ and so $r_n^* \leq -1 + \varepsilon$. Altogether, for all n *sufficiently large* we have $\lambda x_n^* - r_n^* \leq \sigma + \varepsilon$

$$\lambda x_n^* = (\lambda x_n^* - r_n^*) + r_n^* \leq (\sigma + \varepsilon) + (-1 + \varepsilon) = \sigma + 2\varepsilon - 1. \tag{42}$$

Using (2) and (11), we estimate

$$-\lambda x_n^* \leq \lambda \|x^*\| = \lambda \|x\| \leq \|r^*\| \leq \|r^* - f^*\| + \|f^*\| \leq \varepsilon + 1. \tag{43}$$

Adding (42) and (43), we obtain $0 \leq \sigma + 3\varepsilon$. Recalling the definition of σ from (41), deduce that $\langle x, f^* \rangle \leq 3\varepsilon$ and the (whs) condition holds. The conclusion now follows by applying Example 2.2(i) and Corollary 3.6. ■

Remark 4.2 (a negative answer to a question posed by Stephen Simons). *In [7], Stephen Simons proved that $\overline{\operatorname{ran}}\,(G + \lambda\mathrm{J})$ is not convex for $0 < \lambda < 4$. In [7, Problem 3.6] he asks whether $\overline{\operatorname{ran}}\,(G + 4\mathrm{J})$ is convex. Theorem 4.1 not only provides a negative answer but also establishes, for* every *$\lambda > 0$, the nonconvexity of $\overline{\operatorname{ran}}\,(G + \lambda\mathrm{J})$.*

Remark 4.3 (the negative Gossez operator). *The* negative Gossez operator, *$-G$, is much better behaved than G: indeed, combining [1, Example 14.2.2 and Theorem 15.3.7], we deduce that $(\forall \lambda > 0)$ $\overline{\operatorname{ran}}\,(-G + \lambda\mathrm{J}) = \ell_\infty$. (See also [2, Example 5.2].)*

5 The Fitzpatrick-Phelps operator revisited

In this section, we assume that

$$X = L_1[0,1] \tag{44}$$

and that

$$F\colon L_1[0,1] \to L_\infty[0,1]\colon x \mapsto Fx, \quad \text{where } (Fx)(t) = \int_0^t x(s)\,ds - \int_t^1 x(s)\,ds. \tag{45}$$

It is easy to see that F is a bounded linear operator with $\|F\| = 1$ and that F is skew, hence monotone.

We are now ready for the main result concerning the Fitzpatrick-Phelps operator.

Theorem 5.1. *The Fitzpatrick-Phelps operator F satisfies the (whs) condition with $f^* \equiv -1 \in L_\infty[0,1]$; consequently, $(\forall \lambda > 0)$ $\overline{\operatorname{ran}}\,(F + \lambda\,\mathrm{J})$ is not convex yet $\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(F + \lambda\,\mathrm{J}) = L_\infty[0,1]$.*

Proof. Let $\lambda > 0$, and let $\varepsilon > 0$. Assume that $(x, r^*) \in L_1[0,1] \times L_\infty[0,1]$ satisfies

$$r^* \in (Fx + \lambda\,\mathrm{J}x) \cap \operatorname{ball}(f^*;\varepsilon). \tag{46}$$

First, there exists $x^* \in \mathrm{J}x$ such that $r^* = Fx + \lambda x^*$. By definition of F, $(\forall t \in [0,1])$ $r^*(t) = \int_0^t x(s)\,ds - \int_t^1 x(s)\,ds + \lambda x^*(t)$. Thus, for every $t \in [0,1]$, we have $|r^*(t) + 1| \leq \varepsilon$ and also

$$\lambda x^*(t) = r^*(t) - \int_0^t x(s)\,ds + \int_t^1 x(s)\,ds = r^*(t) + \int_0^1 x(s)\,ds - 2\int_0^t x(s)\,ds \tag{47a}$$

$$= r^*(t) - \langle x, f^*\rangle - 2\int_0^t x(s)\,ds \leq -1 + \varepsilon - \langle x, f^*\rangle - 2\int_0^t x(s)\,ds. \tag{47b}$$

In view of the (absolute) continuity of $t \mapsto \int_0^t x(s)\,ds$ (see, e.g., [9, Theorem 6.84]), we have for all $t > 0$ *sufficiently small*,

$$\lambda x^*(t) \leq -1 - \langle x, f^*\rangle + 2\varepsilon. \tag{48}$$

Now note that — using (2) and again (11) — we obtain

$$-\lambda x^*(t) \leq \lambda\|x^*\| = \lambda\|x\| \leq \|r^*\| \leq \|r^* - f^*\| + \|f^*\| \leq \varepsilon + 1. \tag{49}$$

Adding (48) and (49), we deduce that $0 \leq 3\varepsilon - \langle x, f^*\rangle$ and the (whs) condition thus holds. Finally, apply Example 2.2(ii) and Corollary 3.6. ■

Remark 5.2. *Fitzpatrick and Phelps (see [3, Example 3.2]) showed directly that $\overline{\operatorname{ran}}\,(F + 1\,\mathrm{J})$ is not convex. We extend their conclusion from $\lambda = 1$ to any $\lambda > 0$. We note that a referee pointed out to us that there is a gap in the proof of the nonconvexity of $\overline{\operatorname{ran}}\,(F + \mathrm{J})$ in the second paragraph of [3, page 64]; however, the work in this paper does not rely on their argument and thus provides an alternative (and simpler) proof.*

Remark 5.3 (the negative Fitzpatrick-Phelps operator). *The* negative Fitzpatrick-Phelps operator, *$-F$, satisfies the (whs) condition again with $f^* \equiv -1 \in L_\infty[0,1]$. (The proof is similar with the only difference being that in the derivation of the counterpart of (48), we work with t sufficiently close to 1 rather than to 0.) Consequently, $(\forall \lambda > 0)$ $\overline{\operatorname{ran}}\,(-F + \lambda\,\mathrm{J})$ is not convex yet $\overline{\operatorname{conv}}\,\overline{\operatorname{ran}}\,(-F + \lambda\,\mathrm{J}) = L_\infty[0,1]$.*

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