On Dykstra's algorithm: finite convergence, stalling, and the method of alternating projections

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Abstract

A popular method for finding the projection onto the intersection of two closed convex subsets in Hilbert space is Dykstra's algorithm.

In this paper, we provide sufficient conditions for Dykstra's algorithm to converge rapidly, in finitely many steps. We also analyze the behaviour of Dykstra's algorithm applied to a line and a square. This case study reveals stark similarities to the method of alternating projections. Moreover, we show that Dykstra's algorithm may stall for an arbitrarily long time. Finally, we present some open problems.

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1 Introduction

Suppose that

$$X \text{ is a Hilbert space,} \tag{1}$$

with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Suppose that

$$A \text{ and } B \text{ are closed convex subsets of } X \text{ with } A\cap B\neq\varnothing, \text{ and } z\in X. \tag{2}$$

Our goal is to find

$$P_{A\cap B}(z), \tag{3}$$

the point in $A\cap B$ nearest to z. Even when A and B are "simple" in the sense that P_A and P_B are easily computable, there is in general no simple formula for $P_{A\cap B}(z)$. Instead, one may employ *Dykstra's algorithm* (see [5] and also [1],[7],[4]) to find this point. The algorithm proceeds as follows. Set $b_0 := z$, $p_0 := 0$, $q_0 := 0$, and generate sequences iteratively via

$$a_n = P_A(b_{n-1}+p_{n-1}), \qquad p_n = b_{n-1}+p_{n-1}-a_n, \tag{4a}$$

$$b_n = P_B(a_n+q_{n-1}), \qquad q_n = a_n+q_{n-1}-b_n, \tag{4b}$$

where $n\geq 1$. The sequences (a_n) and (b_n) are the *main sequences* while (p_n) and (q_n) are the *auxiliary sequences* of Dykstra's algorithm. The central convergence result concerning Dykstra's algorithm is the following.

Fact 1.1. (Boyle–Dykstra) *(See [5].) The main sequences (a_n) and (b_n) of Dykstra's algorithm both converge strongly to $P_{A\cap B}(z)$.*

A closely related algorithm is the *Method of Alternating Projections (MAP)*, which can be thought of as a cousin of Dykstra's algorithm with $p_n\equiv q_n\equiv 0$: We define $c_0 := z$ and proceed via

$$c_{2n-1} = P_A(c_{2n-2}) \quad\text{and}\quad c_{2n} = P_B(c_{2n-1}) \tag{5}$$

for $n\geq 1$.

Fact 1.2. (Bregman) *(See [6].) The MAP sequence (c_n) converges weakly to some point in $A\cap B$.*

Note that MAP is simpler than Dykstra's algorithm, but the conclusion is also markedly weaker: the convergence is only weak (and this indeed can happen, see [8]) and the limit may not be $P_{A\cap B}(z)$ (see the next example).

Example 1.3. (MAP does not produce the projection) *Suppose that* $X = \mathbb{R}^2$, $A = \mathbb{R} \times \mathbb{R}_-$, $B = \{x = (x(1), x(2)) \in \mathbb{R}^2 \mid x(1) + x(2) \leq 0\}$, *and* $z = (\zeta, \zeta)$, *where* $\mathbb{R}_- = \,]-\infty, 0]$ *and* $\zeta > 0$. *Then* $P_{A\cap B}(z) = (0,0)$ *while* $P_A(z) = (\zeta, 0)$ *and* $P_B P_A z = \frac{1}{2}(\zeta, -\zeta) \in A$. *Thus, MAP converges in finitely many steps to a point different from* $P_{A\cap B}(z)$ *while Dykstra's algorithm follows the infinitely many steps of MAP, with respect to the boundaries of the sets* A *and* B.

However, when A and B are affine subspaces, then $(p_n)_{n\in\mathbb{N}}$ lies in $(A-A)^\perp$ and (q_n) lies $(B-B)^\perp$; thus, the main sequences of Dykstra's algorithm coincide with the one produced by MAP in the sense that

$$(\forall n \geq 1) \quad c_{2n-1} = a_n \text{ and } c_{2n} = b_n. \tag{6}$$

We record this classical result (see Deutsch's monograph [7] for further information) next.

Fact 1.4. (von Neumann) *If* A *and* B *are closed affine subspaces with nonempty intersection, then the MAP sequence coincides with the main sequences of Dykstra's algorithm and thus converges strongly to* $P_{A\cap B}(z)$.

The goal of this paper is to highlight various behaviours of Dykstra's algorithm that have received little attention so far: (1) we discuss when Dykstra's method converges in finitely many steps; (2) we exhibit an example where the algorithm stalls for an arbitrarily long time; and (3) we provide examples where MAP produces the same limit as Dykstra, with less computational overload and in fewer steps.

The paper is organized as follows. Section 2 provides necessary conditions for rapid finite convergence. In Section 3, we develop auxiliary results for the case of a line and a square. Convergence results are presented in Section 4. The final Section 5 contains concluding remarks and some open problems.

The notation employed is standard and follows, e.g., [4] and [7]. A word on notation is in order. As in Example 1.3 and also later on, we shall encounter vectors and sequences in $\mathbb{R}^2$. If x is such a vector and (x_n) is such a sequence, then we write $x = (x(1), x(2))$ and $x_n = (x_n(1), x_n(2))$ provided we have a need to refer to their coordinates.

2 Finite convergence of Dykstra's algorithm

Lemma 2.1. *Let* $n \geq 1$. *Then the following hold:*

(i) *If* $b_n = a_n$ $(\Leftrightarrow q_n = q_{n-1})$, *then* $a_{n+1} = b_n$ $(\Leftrightarrow p_{n+1} = p_n)$.
(ii) *If* $a_{n+1} = b_n$ $(\Leftrightarrow p_{n+1} = p_n)$, *then* $b_{n+1} = a_{n+1}$ $(\Leftrightarrow q_{n+1} = q_n)$.

Proof. All equivalences follow from (4). (i): Suppose $b_n = a_n$. Then, using also (4), $b_n + p_n = a_n + p_n = b_{n-1} + p_{n-1}$. Thus $a_{n+1} = P_A(b_n + p_n) = P_A(b_{n-1} + p_{n-1}) = a_n = b_n$. (ii): The proof is analogous to that of (i). ■

Remark 2.2. *If $a_1 = b_0$, then it does not necessarily follow that $b_1 = a_1$. Indeed, consider any setting where A is not a subset of B, and $z \in A \smallsetminus B$. Then $z = b_0 = a_1$ and $b_1 = P_B a_1 \neq a_1$.*

Corollary 2.3. *Let $n \geq 1$.*

(i) *If $b_n = a_n$, then $a_n = b_n = a_{n+1} = b_{n+1} = \cdots = P_{A\cap B}(z)$.*
(ii) *If $a_{n+1} = b_n$, then $b_n = a_{n+1} = b_{n+1} = a_{n+2} = \cdots = P_{A\cap B}(z)$.*

Proof. Combine Fact 1.1 with Lemma 2.1. ■

The next result provides conditions under which Dykstra's method converges almost immediately and where it behaves exactly like MAP.

Theorem 2.4. (finite convergence) *Suppose that one of the following holds:*

(i) $z - P_A(z) \in N_A(P_B P_A z)$
(ii) *A is affine and $P_B P_A z \in A$.*

Then $P_{A\cap B}(z) = P_B P_A z = b_1 = a_2 = b_2 = a_3 = b_3 = \cdots$; moreover, the main sequences of MAP and Dykstra's algorithm fully coincide.

Proof. Clearly, $a_1 = P_A z$, $p_1 = b_0 + p_0 - a_1 = b_0 - a_1 = z - P_A z \in N_A(a_1)$, $b_1 = P_B(a_1 + q_0) = P_B a_1 = P_B P_A z$, and $q_1 = a_1 + q_0 - b_1 = a_1 - b_1 \in N_B(b_1)$. Recall that $a_2 = P_A(b_1 + p_1)$.

(i): We have

$$z - P_A z \in N_A(P_B P_A z) \Leftrightarrow p_1 \in N_A(b_1) \tag{7a}$$
$$\Leftrightarrow b_1 + p_1 \in b_1 + N_A(b_1) \tag{7b}$$
$$\Leftrightarrow b_1 = P_A(b_1 + p_1) \tag{7c}$$
$$\Leftrightarrow b_1 = a_2. \tag{7d}$$

Now apply Corollary 2.3(ii) with $n = 1$.

(ii): Because A is affine, we have $(\forall a \in A)\ N_A(a) = (A - A)^\perp = \operatorname{ran}(\operatorname{Id} - P_A)$. Hence if $P_B P_A z \in A$, then $z - P_A z \in (A - A)^\perp = N_A(P_B P_A z)$ and we are done by (i). ■

Let us present an example of Theorem 2.4 that was obtained differently in [2].

Example 2.5. (cone and ball) *(See also [2, Corollary 7.3].) Suppose K is a nonempty closed convex cone in X, and let B be a multiple of the unit ball. Then $P_{B\cap K} = P_B \circ P_K$.*

Proof. Let $z \in X$. Then there exists $\gamma \geq 0$ such that $P_B P_K z = \gamma P_K z$. By [4, Example 6 40], $N_K(P_B P_K z) = K^{\ominus} \cap \{\gamma P_K z\}^{\perp}$, where $K^{\ominus} = \{x \in X \mid \max \langle x, K\rangle = 0\}$ is the dual cone of K. On the other hand, $z - P_K z = P_{K^{\ominus}} z$ and $P_{K^{\ominus}} z \perp P_K z$; see, e.g., [4, Theorem 6.30]. Altogether,

$$z - P_K z \in N_K(P_B P_K z) \tag{8}$$

and the result follows from Theorem 2.4(i). ■

Remark 2.6. *Under the assumptions of Example 2.5, it is* not *true that $P_{B\cap K} = P_K \circ P_B$; see [2, Example 7.5] for more on this.*

Remark 2.7. (two intervals) *By discussing cases, it is straightforward to verify that for any two nonempty closed intervals A and B in $X = \mathbb{R}$, we have*

$$P_{A\cap B} = P_B P_A = P_A P_B. \tag{9}$$

Now consider

$$A = [0, +\infty[\,, \quad B = [1, +\infty[\,, \quad \text{and} \quad z \in \mathbb{R}. \tag{10}$$

If $z \geq 1$, i.e., $z \in A \cap B = B$, then $a_n \equiv b_n \equiv z = P_{A\cap B}(z)$ and $p_n \equiv q_n \equiv 0$. Now assume that $z < 1$. Then $n := -\lfloor z \rfloor \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $0 \leq n + z < 1$, where $\lfloor\cdot\rfloor$ denotes the floor function. It is tedious but straightforward to verify that

$$(\forall 1 \leq k \leq n) \quad a_k = 0, \;\; p_k = z + k - 1, \;\; b_k = 1, \;\; q_k = -k, \tag{11}$$

that

$$a_{n+1} = z + n, \;\; p_{n+1} = 0, \;\; b_{n+1} = 1, \;\; q_{n+1} = z - 1, \tag{12}$$

and that

$$(\forall k \geq n+2) \quad a_k = b_k = 1 = P_{A\cap B}(z), \;\; p_k = q_k = 0. \tag{13}$$

In particular, when $z = -1$, we have $P_A z = 0$, $z - P_A z = -1$, $P_B P_A z = 1 \in \operatorname{int}(A)$, and thus $N_A(P_B P_A z) = \{0\}$. Thus, contrasting to Theorem 2.4(i), it is possible to have $P_{A\cap B} = P_B P_A$ even though there exists some point $z \in X$ such that $z - P_A(z) \notin N_A(P_B P_A z)$.

We conclude this section with a characterization of equality of Dykstra and MAP.

Lemma 2.8. *Suppose that A is affine and that $c_0 = b_0$. Then*

$$(\forall n \geq 2) \quad b_n = P_B a_n \tag{14}$$

if and only if Dykstra and MAP coincide, i.e., $(\forall n \geq 0)$ $c_{2n} = b_n$ and $c_{2n+1} = a_{n+1}$.

Proof. We always have $c_1 = a_1$ and $c_2 = b_1$. Because A is affine, we also have $(\forall n \geq 1)$ $a_n = P_A b_{n-1}$.

"⇒": Because $c_2 = b_1$, we deduce that $c_3 = P_A c_2 = P_A b_1 = a_2$. In turn, $c_4 = P_B c_3 = P_B a_2 = b_2$ by (14). Continuing in this fashion, we obtain the conclusion.

"⇐": Let $n \geq 2$. Then $n - 1 \geq 1$ and so $b_n = c_{2n} = P_B c_{2(n-1)+1} = P_B a_n$. ■

3 Line and square: set up and auxiliary results

We assume from now on that

$$X = \mathbb{R}^2 \text{ and } z \in X, \tag{15}$$

that

$$A := \text{is a line in } X, \tag{16}$$

and that

$$B := [-1, 1] \times [-1, 1] \tag{17}$$

is a square of side length 2 (the unit ball with respect to the max-norm). Specifically, in view of symmetry, we also assume that

$$u \in A \cap B, \ \ v \in V := (A - A)^{\perp}, \ \ \|v\| = 1; \quad \text{thus,} \quad A = u + \{v\}^{\perp}. \tag{18}$$

and that

$$v \in \mathbb{R}^2_{++}, \ \ -1 < u(1), \ \text{ and } \ u(2) = 1. \tag{19}$$

(We discuss the case when $v(1)v(2) = 0$ separately later.) Then, for every $x \in X$, $P_A x = u + P_{\{v\}^{\perp}}(x - u) = u + (x - u) - \langle x - u, v \rangle\, v$; thus,

$$P_A x = x - \langle x - u, v \rangle\, v \quad \text{and} \quad P_B x = P_B\big(x(1), x(2)\big) = \big(P_{[-1,1]} x(1), P_{[-1,1]} x(2)\big). \tag{20}$$

Finally, assume that

$$(a_n) \text{ and } (b_n) \text{ are the main sequences of Dykstra's algorithm (see (4))} \tag{21}$$

while (p_n) and (q_n) are the auxiliary sequences. Because A is an affine subspace, the sequence (p_n) lies entirely in $(A - A)^{\perp}$ and thus we always have

$$a_n = P_A b_{n-1}, \tag{22}$$

where $n \geq 1$; in other words, we can simply ignore p_{n-1} when computing $a_n = P_A(b_{n-1} + p_n) = P_A b_{n-1}$.

In the remainder of this section, we collect various technical results that will make the proofs of the main result much simpler.

Lemma 3.1. *Suppose that $b_n = (b_n(1), 1)$, where $b_n(1) \leq u(1)$. Then*

$$a_{n+1} = b_n + \big(u(1) - b_n(1)\big) v(1) v. \tag{23}$$

Proof. Note that $b_n - u = (b_n(1) - u(1), 1-1) = (b_n(1) - u(1), 0)$. Hence $\langle b_n - u, v\rangle = (b_n(1) - u(1))v(1) \le 0$. It follows from (20) that

$$a_{n+1} = P_A b_n = b_n - \langle b_n - u, v\rangle\, v = b_n + (u(1) - b_n(1))v(1)v \tag{24}$$

as announced. ■

Lemma 3.2. *Suppose that $b_n = (b_n(1), 1)$, where $n \ge 1$. If $b_n(1) \le u(1)$, then $a_{n+1}(2) + q_n(2) \ge 1$ and thus $b_{n+1}(2) = 1$; moreover, if the first inequality is strict, then so is the second.*

Proof. Recall that $b_n = P_B(a_n + q_{n-1})$. Thus $a_n(2) + q_{n-1}(2) \ge 1$ and hence

$$q_n(2) = a_n(2) + q_{n-1}(2) - b_n(2) = a_n(2) + q_{n-1}(2) - 1 \ge 0. \tag{25}$$

On the other hand, Lemma 3.1 yields

$$a_{n+1}(2) = 1 + \big(u(1) - b_n(1)\big)v(1)v(2) \ge 1. \tag{26}$$

Altogether,

$$a_{n+1}(2) + q_n(2) \ge 1 + 0 = 1, \tag{27}$$

and the inequality is strict when $b_n(1) < u(1)$. ■

Lemma 3.3. *Suppose that $b_1 = b_2 = \cdots = b_n = (-1, 1)$, where $n \ge 1$. Then*

$$q_n = (n-1)\big(u(1) + 1\big)v(1)v + a_1 + (1, -1). \tag{28}$$

Proof. We verify this using mathematical induction on $n \ge 1$.

Base case: If $(-1, 1) = b_1 = P_B(a_1 + q_0) = P_B(a_1)$, then $q_1 = a_1 + q_0 - b_1 = a_1 - b_1 = a_1 + (1, -1)$ as claimed.

Inductive step: Assume that the result holds for some $n \ge 1$ and that $(-1, 1) = b_1 = \cdots = b_n = b_{n+1}$. By the inductive hypothesis,

$$q_n = (n-1)\big(u(1) + 1\big)v(1)v + a_1 + (1, -1). \tag{29}$$

Hence, using also Lemma 3.1,

$$\begin{aligned}
q_{n+1} &= a_{n+1} + q_n - b_{n+1} && \text{(30a)}\\
&= b_n + (u(1) - b_n(1))v(1)v + (n-1)(u(1)+1)v(1)v + a_1 + (1,-1) - b_{n+1} && \text{(30b)}\\
&= (-1,1) + (u(1)+1)v(1)v + (n-1)(u(1)+1)v(1)v + a_1 + (1,-1) + (1,-1) && \text{(30c)}\\
&= (n+1-1)(u(1)+1)v(1)v + a_1 + (1,-1), && \text{(30d)}
\end{aligned}$$

as required. ■

Lemma 3.4. *Suppose that $b_n(1) \leq u(1)$ and $b_n = (b_n(1), 1)$, where $n \geq 1$. Then*

$$b_{n+1} = (-1, 1) \quad \Leftrightarrow \quad a_{n+1}(1) + q_n(1) \leq -1. \tag{31}$$

Proof. Recall that $b_{n+1} = P_B(a_{n+1} + q_n)$.

"$\Rightarrow$": If $b_{n+1} = (-1, 1)$, then, since $b_{n+1} = P_B(a_{n+1} + q_n)$ and $b_{n+1}(1) = -1$, we have $a_{n+1}(1) + q_n(1) \leq -1$.

"$\Leftarrow$": Clear from Lemma 3.2. ■

Lemma 3.5. *Suppose that $b_1 = \cdots = b_n = (-1, 1)$, where $n \geq 1$. Then $q_n(1) \leq 0$,*

$$a_{n+1}(1) + q_n(1) = a_1(1) + n\big(u(1) + 1\big)v^2(1) \tag{32}$$

and

$$a_{n+1}(1) + q_n(1) \leq a_{n+1}(1) < u(1); \tag{33}$$

moreover,

$$b_{n+1} = (-1, 1) \quad \Leftrightarrow \quad n\big(u(1) + 1\big)v^2(1) + a_1(1) \leq -1 \quad \Leftrightarrow \quad n \leq \left\lfloor \frac{-1 - a_1(1)}{(u(1) + 1)v^2(1)} \right\rfloor. \tag{34}$$

If $b_{n+1} \neq (-1, 1)$, then $-1 < a_{n+1}(1) + q_n(1) \leq a_{n+1}(1) < u(1) \leq 1$, $b_{n+1}(1) = a_{n+1}(1) + q_n(1)$, $b_{n+1}(2) = 1$, and $q_{n+1}(1) = 0$.

Proof. Clearly, $q_n \in N_B(b_n) = \mathbb{R}_- \times \mathbb{R}_+$, so $q_n(1) \leq 0$. Because $b_1 = P_B(a_1 + q_0) = P_B a_1$, it is clear that $a_1(1) \leq -1$ and so

$$-1 - a_1(1) \geq 0. \tag{35}$$

From Lemma 3.3, we have

$$q_n = (n - 1)(u(1) + 1)v(1)v + a_1 + (1, -1); \tag{36}$$

in particular,

$$q_n(1) = (n - 1)(u(1) + 1)v^2(1) + a_1(1) + 1. \tag{37}$$

From Lemma 3.1, we have

$$a_{n+1}(1) = -1 + (u(1) + 1)v^2(1). \tag{38}$$

Adding the last two equations gives (32). Note that $-1 < u(1) \Leftrightarrow v^2(1) - 1 < u(1)(1 - v^2(1)) \Leftrightarrow -1 + u(1)v^2(1) + v^2(1) < u(1) \Leftrightarrow a_{n+1}(1) < u(1)$, which gives (33) because $q_n(1) \leq 0$.

On the other hand, from Lemma 3.4, we have

$$b_{n+1} = (-1, 1) \quad \Leftrightarrow \quad a_{n+1}(1) + q_n(1) \leq -1. \tag{39}$$

Therefore, using (32), we obtain

$$b_{n+1} = (-1,1) \quad \Leftrightarrow \quad n(u(1)+1)v^2(1) + a_1(1) \le -1 \tag{40a}$$

$$\Leftrightarrow \quad n \le \frac{-1-a_1(1)}{(u(1)+1)v^2(1)}, \tag{40b}$$

and (34) follows.

Now assume that $b_{n+1} \neq (-1,1)$. By Lemma 3.4,

$$-1 < a_{n+1}(1) + q_n(1). \tag{41}$$

But we know already that $a_{n+1}(1) + q_n(1) \le a_{n+1}(1) < u(1) \le 1$.

The formula for $b_{n+1}(1)$ is now clear. The statement that $b_{n+1}(2) = 1$ is a consequence of Lemma 3.2. ■

Lemma 3.6. *Suppose $n \ge 1$, $-1 < b_n(1) < u(1)$, $b_n(2) = 1$, $a_n(1) < u(1)$, and $q_{n-1}(1) \le 0$. Then $-1 < a_n(1) + q_{n-1}(1) < b_{n+1}(1) = a_{n+1}(1) < u(1)$, $b_{n+1}(2) = 1$, $q_n(1) = 0$, $q_n(2) \ge 0$, and $b_{n+1} = P_B a_{n+1}$.*

Proof. We have $q_n \in N_B(b_n)$ and so $q_n(1) = 0$ and $q_n(2) \ge 0$. Hence $b_{n+1}(1) = (P_B(a_{n+1} + q_n))(1) = (P_B a_{n+1})(1)$. We also have $b_n(1) = a_n(1) + q_{n-1}(1)$ because $-1 < b_n(1) < 1$. Now $a_{n+1} = b_n + (u(1) - b_n(1))v(1)v$ by Lemma 3.1. On the one hand,

$$a_{n+1}(1) = a_n(1) + q_{n-1}(1) + (u(1) - (a_n(1) + q_{n-1}(1)))v^2(1) \tag{42a}$$

$$= (1 - v^2(1))(a_n(1) + q_{n-1}(1)) + v^2(1)u(1) \tag{42b}$$

thus

$$-1 < a_n(1) + q_{n-1}(1) < a_{n+1}(1) = a_{n+1}(1) + q_n(1) < u(1) \le 1 \tag{43}$$

and $b_{n+1}(1) = a_{n+1}(1) \in \,]-1,1[$. On the other hand,

$$a_{n+1}(2) = b_n(2) + (u(1) - b_n(1))v(1)v(2) \tag{44a}$$

$$= 1 + (u(1) - b_n(1))v(1)v(2) \tag{44b}$$

$$> 1 \tag{44c}$$

and thus $a_{n+1}(2) + q_n(2) \ge a_{n+1}(2) > 1$ which yields $b_{n+1}(2) = (P_B a_{n+1})(2) = 1$ and $q_{n+1}(2) \ge 0$. Altogether, $b_{n+1} = P_B a_{n+1}$. ■

Corollary 3.7. *Suppose that $-1 < b_n(1) < u(1)$, $b_n(2) = 1$, $a_n(1) < u(1)$, and $q_{n-1}(1) \le 0$, where $n \ge 1$. Then for every $k \ge 1$, we have $b_{n+k} = P_B(a_{n+k})$, $-1 < b_{n+k-1}(1) < b_{n+k}(1) < u(1)$ and $b_{n+k-1}(2) = 1$. In other words, starting with a_{n+1}, the main sequences of Dykstra coincide with the MAP sequence (starting at a_{n+1}) and all converge to $P_{A\cap B}z$, which is u in this setting.*

Proof. This follows inductively from Lemma 3.6. Notice that $\lim_{n\to\infty} b_n = u$, because $(b_m)_m$ converges to $P_{A\cap B}z$ and $(b_m)_m$ lies eventually in $B \cap [-1,1] \times \{1\}$. So the limit lies in $A \cap B \cap [-1,1] \times \{1\} = \{u\}$. ■

4 Line and square: main results

We are now ready to describe our main results for the line-square setting. There are essentially three scenarios, *depending on the starting point z*, for Dykstra's algorithm: (1) rapid finite convergence; (2) infinite convergence with steady progress; (3) initial stalling followed by infinite convergence with steady progress. These three regions are depicted in Figure 1, and we discuss them in the subsections below. As we shall see, there is a close

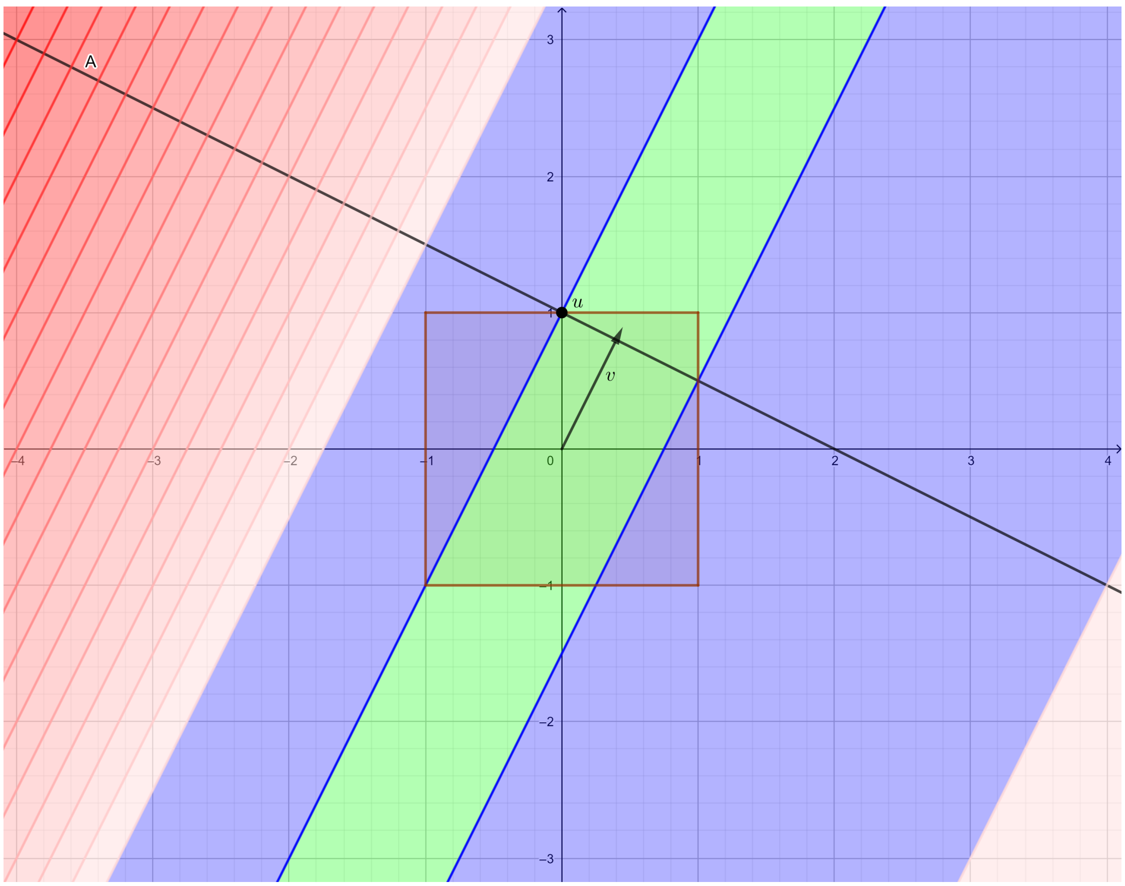

Figure 1: Three scenarios are possible, depending on the location of the starting point z. If z belongs to the green region containing the origin, then Dykstra's algorithm converges rapidly in finitely many steps. If z belongs to one of the two adjacent blue regions still intersecting the square B, then Dykstra's algorithm does not converge finitely and it coincides with MAP. Finally, if z is in the remaining red region, the stalling occurs followed by infinite progress. See the subsections in Section 4 for details.

relationship to MAP.

When Dykstra's algorithm and MAP coincide, with rapid finite convergence

We are done in two steps provided that $P_A(z) \in B$:

Theorem 4.1. **($P_A z \in B$)** *Suppose that $u(1) \leq a_1(1) \leq 1$ and $|a_1(2)| \leq 1$. Then the main sequences of Dykstra's algorithm coincides with the MAP sequence and convergence is finite and rapid: $P_{A\cap B}(z) = a_1 = b_1 = a_2 = b_2 = \cdots$.*

Proof. The hypothesis implies that $a_1 = P_A z \in B$. Hence $b_1 = P_B P_A z = P_B a_1 = a_1 = P_A z \in A$ and the result follows from Theorem 2.4(ii). ■

We now turn to the case we omitted in the previous section — the case when the line is parallel to a side of the box. It turns out that this also leads to finite convergence although two steps may be required.

Theorem 4.2. **(parallel case)** *Suppose that $A = \mathbb{R} \times \{\alpha\}$, where $|\alpha| \leq 1$. Then the main sequences of Dykstra's algorithm and the MAP sequence coincide; moreover, $P_{A\cap B}(z) = b_1 = a_2 = b_2 = a_3 = \cdots$.*

Proof. The hypothesis implies that $P_B P_A z \in A$. Now apply Theorem 2.4(ii). ■

When Dykstra's algorithm and MAP coincide with infinite convergence

Theorem 4.3. **(Dykstra $\equiv$ MAP)** *Suppose that $-1 < a_1(1) < u(1)$ and $1 < a_1(2)$. Then Dykstra's algorithm and MAP produce the exactly same main sequences, with infinite convergence.*

Proof. The hypothesis implies that $-1 < b_1(1) = a_1(1) < u(1)$ and $b_1(2) = 1$. Recall also that $q_0 = 0$. The conclusion thus follows from Corollary 3.7, with $n = 1$. ■

Remark 4.4. *We saw in Theorem 4.3 directly that MAP and Dykstra's algorithm do not converge in finitely many steps. In fact, this is a universal phenomenon of MAP because Luke, Teboulle, and Thao recently proved (see [9, Theorem 7]) that* in general *we have the dichotomy that either $P_B P_A(z) \in A$ (and MAP terminates) or MAP does not converge in finitely many steps.*

When Dykstra's algorithm stalls

Theorem 4.5. **(stalling)** *Suppose that $a_1(1) \leq -1$ and $1 < a_1(2)$. Set*

$$n := 1 + \left\lfloor \frac{-1 - a_1(1)}{(u(1)+1)v^2(1)} \right\rfloor . \tag{45}$$

Then Dykstra algorithm stalls, i.e., $b_1 = b_2 = \cdots = b_n = (-1, 1)$*, it then "breaks free" with* $b_{n+1} \neq (-1, 1)$*, and it finally acts like MAP with starting point* b_{n+1}*.*

Proof. Combine Lemma 3.5 with Corollary 3.7. ■

Remark 4.6. *Some comments regarding Theorem 4.5 are in order.*

(i) *By choosing* $a_1 = z \in A$ *with* $z(1)$ *very negative, we can arrange for* n *to be as large as we want. Thus the stalling phase for Dykstra's algorithm can be arbitrarily long!*
(ii) *The point* b_{n+1} *is not necessarily equal to* $P_B a_{n+1} = (a_{n+1}(1), 1)$*; in fact, with the help of Lemma 3.1 and Lemma 3.5, one obtains*

$$-1 < b_{n+1}(1) = a_1((1)) + n(u(1) + 1)v^2(1) \leq a_{n+1}(1). \tag{46}$$

Somewhat surprisingly, the orbits (in the sense of sets) of Dykstra's algorithm and MAP need not be identical — see Figure 2 for a visualization.

5 Conclusion

The following example underlines the importance of the *order* of the sets — projecting first onto the square and then onto the line will not work!

Example 5.1. (order matters!) *Suppose that* A *is the line through the points* $(0, 1)$ *and* $(1, 0)$*, and that* $B = [-1, 1] \times [-1, 1]$ *is the square in* $\mathbb{R}^2$*. Consider* $z = (-2, -1)$*. Then* $P_B z = (-1, -1)$ *and thus* $P_A P_B z = (\frac{1}{2}, \frac{1}{2}) \in B$ *while* $P_{A\cap B} z = (0, 1)$*. Hence MAP stops right away with the limit being different from* $P_{A\cap B}(z)$*, the limit of the main sequences of Dykstra's algorithm.*

The following questions appear to be of interest and are left for future investigations.

- Can we identify more cases when it suffices to apply MAP to find $P_{A\cap B}(z)$? Reworded
- Can one prove a higher-dimensional version of the box-line scenario considered in the second half of this paper? In fact, [3] suggests that (14) holds numerically and thus that extensions may be possible.
- If MAP and Dykstra's algorithm yield the same limit, is it true that the convergence of MAP is never slower than Dykstra? All results in this paper — as well as those in [1] — suggest that this is true for some classes of problems.

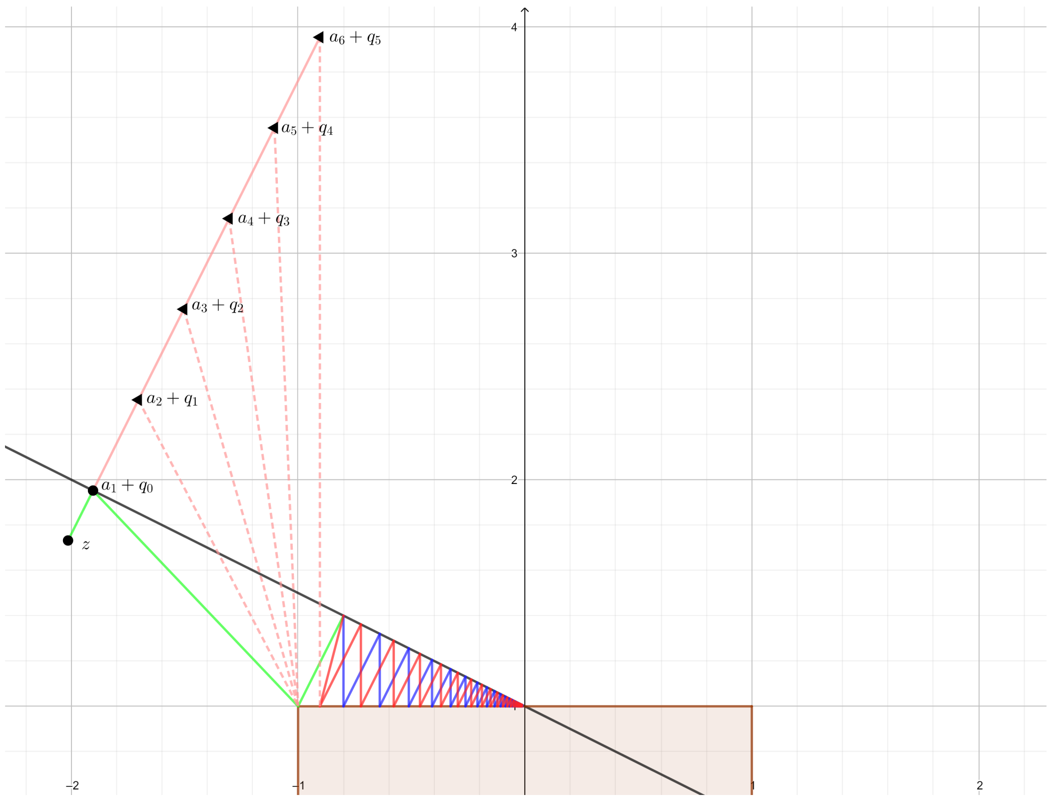

Figure 2: An illustration of Remark 4.6(ii) where the starting point z lies in the stalling region. Here $b_1 = b_2 = \cdots = b_5 = (-1, 1)$ illustrates stalling; the orbit until this point is depicted in green. (The stalling period can be made arbitrarily long by, for instance, moving the starting point z to the left.) Dykstra's algorithm then exits the stalling period; however, b_6 is *not* equal to $P_B(a_6)$! From this point onwards, Dykstra's algorithm proceeds like MAP but starting from b_6, with its orbit depicted in red. In contrast, MAP proceeds along the green and then blue orbit, without any stalling.

Acknowledgments

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