On the linear convergence of circumcentered isometry methods

Heinz H. Bauschke[*], Hui Ouyang[†] and Xianfu Wang[‡]

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Abstract

The circumcentered Douglas–Rachford method (C–DRM), introduced by Behling, Bello Cruz and Santos, iterates by taking the circumcenter of associated successive reflections. It is an acceleration of the well-known Douglas-Rachford method (DRM) for finding the best approximation onto the intersection of finitely many affine subspaces. Inspired by the C–DRM, we introduced the more flexible circumcentered reflection method (CRM) and circumcentered isometry method (CIM). The CIM essentially chooses the closest point to the solution among all of the points in an associated affine hull as its iterate and is a generalization of the CRM. The circumcentered–reflection method introduced by Behling, Bello Cruz and Santos to generalize the C–DRM is a special class of our CRM.

We consider the CIM induced by a set of finitely many isometries for finding the best approximation onto the intersection of fixed point sets of the isometries which turns out to be an intersection of finitely many affine subspaces. We extend our previous linear convergence results on CRMs in finite-dimensional spaces from reflections to isometries. In order to better accelerate the symmetric method of alternating projections (MAP), the accelerated symmetric MAP first applies another operator to the initial point. (Similarly, to accelerate the DRM, the C–DRM first applies another operator to the initial point as well.) Motivated by these facts, we show results on the linear convergence of CIMs in Hilbert spaces with first applying another operator to the initial point. In particular, under some restrictions, our results imply that some CRMs attain the known linear convergence rate of the accelerated symmetric MAP in Hilbert spaces. We also exhibit a class of CRMs converging to the best approximation in Hilbert spaces with a convergence rate no worse than the sharp convergence rate of MAP. The fact that some CRMs attain the linear convergence rate of MAP or accelerated symmetric MAP is entirely new.

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1 Introduction

Throughout this paper, we assume that

$$\mathcal{H} \text{ is a real Hilbert space,}$$

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Denote by $\mathcal{P}(\mathcal{H})$ the set of nonempty subsets of $\mathcal{H}$ containing finitely many elements. The circumcenter operator $CC\colon \mathcal{P}(\mathcal{H}) \to \mathcal{H} \cup \{\varnothing\}$ maps every $K \in$

[*]Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: heinz.bauschke@ubc.ca.
[†]Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: hui.ouyang@alumni.ubc.ca.
[‡]Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: shawn.wang@ubc.ca.

$\mathcal{P}(\mathcal{H})$ to the *circumcenter* $CC(K)$ *of* K, where $CC(K)$ is either the empty set or the unique point $CC(K)$ such that $CC(K) \in \operatorname{aff}(K)$ and $CC(K)$ is equidistant from all points in K (see [5, Proposition 3.3]).

Throughout the paper, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $m \in \mathbb{N} \smallsetminus \{0\}$ and

$$(\forall i \in \{1, \ldots, m\}) \quad T_i : \mathcal{H} \to \mathcal{H} \text{ is affine isometry} \quad \text{with} \quad \bigcap_{j=1}^{m} \operatorname{Fix} T_j \neq \varnothing.$$

Unless stated otherwise, we set

$$\mathcal{S} := \{T_1, \ldots, T_{m-1}, T_m\}.$$

The associated set-valued operator $\mathcal{S} : \mathcal{H} \to \mathcal{P}(\mathcal{H})$ is defined by

$$(\forall x \in \mathcal{H}) \quad \mathcal{S}(x) := \{T_1 x, \ldots, T_{m-1} x, T_m x\}.$$

The *circumcenter mapping* $CC_{\mathcal{S}}$ *induced by* $\mathcal{S}$ is defined by the composition of CC and $\mathcal{S}$, that is $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}(x) := CC(\mathcal{S}(x))$. Inspired by the circumcentered Douglas-Rachford method (C–DRM) introduced by Behling, Bello Cruz and Santos [8], we proved in [7, Theorem 3.3] that the $CC_{\mathcal{S}}$ is *proper*, i.e., $(\forall x \in \mathcal{H})$, $CC_{\mathcal{S}}(x) \in \mathcal{H}$. Hence, we are able to define the *circumcenter method induced by* $\mathcal{S}$ as

$$x_k := CC_{\mathcal{S}}(x_{k-1}) = CC_{\mathcal{S}}^{k}(x_0), \text{ where } x_0 \in \mathcal{H} \text{ and } k = 1, 2, \ldots.$$

Since every element of $\mathcal{S}$ is isometry, we say that the circumcenter method induced by the $\mathcal{S}$ is the *circumcentered isometry method* (CIM). Since reflectors associated with affine subspaces are isometries, we call the circumcenter method induced by a set of reflectors the *circumcentered reflection method* (CRM).

Our goal in this paper is to study the linear convergence of CIMs in Hilbert spaces for finding the best approximation $\mathrm{P}_{\cap_{i=1}^{m} \operatorname{Fix} T_i} x$ *onto the intersection of finitely many affine subspaces, where* $x \in \mathcal{H}$ *is an arbitrary but fixed point. In particular, given affine subspaces* $U_1, U_2, \ldots, U_m$ *with* $\cap_{i=1}^{m} U_i \neq \varnothing$*, finding the best approximation* $\mathrm{P}_{\cap_{i=1}^{m} U_i} x$ *is covered by our work.*

The main results in this paper are the following.

- **R1:** Theorems 5.4 and 5.6 extend the [7, Propositions 5.15 and 5.10] respectively from reflections to isometries and establish the linear convergence of CIMs for finding the best approximation onto the intersection of the fixed point sets of finitely many isometries in finite-dimensional spaces. Moreover, [9, Theorem 3.3] is a special instance of Theorem 5.6.

- **R2:** Theorem 5.10 provides two sufficient conditions for the linear convergence of CIMs in Hilbert spaces with first applying another operator on the initial point. The applications of Theorem 5.10 can be found in Theorem 6.10, [7, Proposition 5.19] and [8, Theorem 1].

- **R3:** Theorems 6.6, 6.8 and 6.10 present sufficient conditions for the linear convergence of CRMs for finding the best approximation onto the intersection of finitely many closed linear subspaces in Hilbert spaces, by using the linear convergence of MAP and accelerated symmetric MAP.

In fact, we generalize all of results on the linear convergence of CRMs shown in [8], [9] and [7] from reflections to isometries. We prove in Theorem 4.16 that the linear convergence of any general CIM is equivalent to the linear convergence of the CIM induced by a corresponding set of linear isometries. Hence, to study the linear convergence of CIM, we are free in our proofs to assume that all of the related isometries are linear. We also prove in Theorem 3.14(ii) that given a linear isometry T, T is reflector if and only if T is self-adjoint. In fact, the linear isometries on $\mathbb{R}^n$ are precisely orthogonal matrices. But orthogonal matrices are in general not symmetric. Hence, our generalizations are indeed less restrictive.

In [4], Bauschke, Deutsch, Hundal and Park studied the acceleration scheme for linear nonexpansive operators which was considered by Gubin, Polyak, and Raik [15] and by Gearhart and Koshy [14]. It was proved that the acceleration scheme for (symmetric) MAP is indeed faster than the (symmetric) MAP. Note that Example 6.3, which is a corollary of Theorem 6.6, states that the convergence rate of some CRMs is no worse than the sharp convergence rate of MAP in Hilbert spaces. Moreover, Theorems 6.8 and 6.10 illustrate that some CRMs attain the known linear convergence rate of the accelerated symmetric MAP in Hilbert spaces. In fact, in [7, Section 6] we showed numerically the outstanding performance of some instances of those CRMs without analytical proof by comparing four CRMs with MAP and DRM. Now, Theorems 6.6 and 6.8 provide theoretical support for the results presented by the numerical experiments in [7, Section 6].

For the readers who are interested in CRMs for general convex or nonconvex feasibility problems, we recommend [11], [13] and [17].

The paper is organized as follows. In Sections 2 and 3, we collect various auxiliary results to facilitate the proofs in the sequel. Some results are interesting on their own (see Proposition 2.25, and Theorems 3.14 and 3.16). Some properties of CRMs shown in [7] are generalized to CIMs in Section 4. Section 5 focuses on the linear convergence of CIMs for finding the best approximation onto intersections of fixed point sets of finitely many affine isometries. More precisely, in Section 5, the linear convergence of CIMs in $\mathbb{R}^n$ is presented, and two sufficient conditions for the linear convergence of CIMs in Hilbert spaces with first applying another operator to the initial point are provided. In Section 6, we use the linear convergence of MAP to deduce sufficient conditions for the linear convergence of CRMs in Hilbert spaces. We also provide examples of CRMs with convergence rate no worse than the sharp convergence rate of MAP. In addition, we prove that some CRMs attain the known convergence rate of the accelerated symmetric MAP.

We now turn to the notation used in this paper. Let C be a nonempty subset of $\mathcal{H}$. C is an *affine subspace* of $\mathcal{H}$ if $C \neq \varnothing$ and $(\forall \rho \in \mathbb{R})\ \rho C + (1-\rho)C = C$. The smallest affine subspace of $\mathcal{H}$ containing C is the denoted by $\operatorname{aff} C$ and called the *affine hull* of C. The *orthogonal complement of* C is the set $C^\perp := \{x \in \mathcal{H} \mid \langle x, y\rangle = 0 \text{ for all } y \in C\}$. The *best approximation operator* (or *projector*) onto C is denoted by P_C. $\mathrm{R}_C := 2\,\mathrm{P}_C - \mathrm{Id}$ is the *reflector associated with* C.

Let $T : \mathcal{H} \to \mathcal{H}$ be an operator. Let $\ker T := \{x \in \mathcal{H} \mid Tx = 0\}$ be the *kernel of* T. The *set of fixed points of the operator* T is denoted by $\operatorname{Fix} T$, i.e., $\operatorname{Fix} T := \{x \in \mathcal{H} \mid Tx = x\}$. The *range of* T is defined as $\operatorname{ran} T := \{Tx \ :\ x \in \mathcal{H}\}$; moreover, $\overline{\operatorname{ran}}\, T$ is the closure of $\operatorname{ran} T$. Denote by $\mathcal{B}(\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} \ :\ T \text{ is bounded and linear}\}$. For every $T \in \mathcal{B}(\mathcal{H})$, the *operator norm* $\|T\|$ *of* T is defined by $\|T\| := \sup_{\|x\| \leq 1} \|Tx\|$. Let m, n be in $\mathbb{N} \smallsetminus \{0\}$ and let $A \in \mathbb{R}^{n \times m}$. The *matrix 2-norm induced by the Euclidean vector norm* is $\|A\|_2 := \max_{\|x\|_2 \leq 1} \|Ax\|_2$. For other notation not explicitly defined here, we refer the reader to [3].

2 Auxiliary results

To facilitate the proofs in our main results in the sequel, we collect and prove some useful results in this section.

Projections and Friedrichs angles

Fact 2.1 [3, Proposition 3.19] *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let $x \in \mathcal{H}$. Set $D := z + C$, where $z \in \mathcal{H}$. Then $\mathrm{P}_D\, x = z + \mathrm{P}_C(x - z)$.*

Fact 2.2 [12, Theorem 4.9] *Let M be a linear subspace in $\mathcal{H}$, $x \in \mathcal{H}$, and $p \in M$. Then $p = \mathrm{P}_M\, x$ if and only if $x - p \in M^\perp$; that is, $(\forall y \in M)\ \langle x - p, y\rangle = 0$.*

Fact 2.3 [12, Theorems 3.5 and 5.5] *Let C be a nonempty closed convex set of $\mathcal{H}$. Then the following assertions hold:*

(i) P_C *is idempotent:* $P_C^2 = P_C$.

(ii) P_C *is firmly nonexpansive:* $(\forall x \in \mathcal{H})\ (\forall y \in \mathcal{H})\ \langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$.

(iii) P_C *is monotone:* $(\forall x \in \mathcal{H})\ (\forall y \in \mathcal{H})\ \langle x - y, P_C x - P_C y\rangle \geq 0$.

Fact 2.4 [12, Theorems 5.8 and 5.13] *Let M be a closed linear subspace of $\mathcal{H}$. Then the following statements hold:*

(i) *$M^\perp$ is a closed linear subspace.*

(ii) $\mathrm{Id} = P_M + P_{M^\perp}$.

(iii) P_M *is a bounded linear operator and* $\|P_M\| = 1$ *(unless $M = \{0\}$, in which case $\|P_M\| = 0$).*

(iv) P_M *is self-adjoint:* $\langle P_M x, y\rangle = \langle x, P_M y\rangle$ *for all x, y in $\mathcal{H}$.*

Fact 2.5 [12, Theorem 6.24] *Let M be a closed linear subspace of $\mathcal{H}$ and $x \in \mathcal{H} \smallsetminus M$. Then there exists a point $z \in M^\perp$ with $\|z\| = 1$ and $\langle z, x\rangle > 0$.*

Fact 2.6 [12, Lemma 9.2] *Let M and N be closed linear subspaces of $\mathcal{H}$. Assume $M \subseteq N$ or $N \subseteq M$. Then* $P_M P_N = P_N P_M = P_{M\cap N}$.

Definition 2.7 [12, Definition 9.4] The *Friedrichs angle* between two linear subspaces U and V is the angle $\alpha(U, V)$ between 0 and $\frac{\pi}{2}$ whose cosine, $c(U, V) := \cos\alpha(U, V)$, is defined by the expression

$$c(U, V) := \sup\{|\langle u, v\rangle| \mid u \in U \cap (U \cap V)^\perp, v \in V \cap (U \cap V)^\perp, \|u\| \leq 1, \|v\| \leq 1\}.$$

Fact 2.8 [12, Lemma 9.5] *Let U and V be closed linear subspaces of $\mathcal{H}$. Then* $c(U, V) = \|P_V P_U - P_{U\cap V}\| = \|P_V P_U P_{(U\cap V)^\perp}\|$.

Fact 2.9 [12, Theorem 9.35] *Let U and V be closed linear subspaces of $\mathcal{H}$. Then $c(U, V) < 1$ if and only if $U + V$ is closed.*

Definition 2.10 [2, Definition 3.7.5] Let $L_1, \ldots, L_m$ be closed linear subspaces of $\mathcal{H}$. Define the *angle* $\beta := \beta(L_1, \ldots, L_m) \in [0, \frac{\pi}{2}]$ *of the m-tuple* $(L_1, \ldots, L_m)$ by

$$\cos\beta := \|P_{L_m} \cdots P_{L_1} P_{(\cap_{i=1}^m L_i)^\perp}\|.$$

Fact 2.11 [2, Proposition 3.7.7] *Let $L_1, \ldots, L_m$ be closed linear subspaces of $\mathcal{H}$. The angle of the m-tuple $(L_1, \ldots, L_m)$ is positive if and only if the sum $L_1^\perp + \cdots + L_m^\perp$ is closed.*

Corollary 2.12 *Let $L_1, \ldots, L_m$ be closed linear subspaces of $\mathcal{H}$. Then $L_1^\perp + \cdots + L_m^\perp$ is closed if and only if* $\|P_{L_m} \cdots P_{L_1} P_{(\cap_{i=1}^m L_i)^\perp}\| < 1$.

Proof. Combine Definition 2.10 and Fact 2.11. ■

Averaged nonexpansive operators

Definition 2.13 [3, Definition 4.1] Let D be a nonempty subset of $\mathcal{H}$ and let $T : D \to \mathcal{H}$. Then T is

(i) *firmly nonexpansive* if

$$(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2; \tag{2.1}$$

(ii) *nonexpansive* if it is Lipschitz continuous with constant 1, i.e.,

$$(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\| \leq \|x - y\|; \tag{2.2}$$

(iii) *firmly quasinonexpansive* if

$$(\forall x \in D)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\|^2 + \|Tx - x\|^2 \le \|x - y\|^2; \tag{2.3}$$

(iv) *quasinonexpansive* if

$$(\forall x \in D)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\| \le \|x - y\|; \tag{2.4}$$

(v) and *strictly quasinonexpansive* if

$$(\forall x \in D \smallsetminus \operatorname{Fix} T)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\| < \|x - y\|. \tag{2.5}$$

Remark 2.14 [3, page 70] Concerning Definition 2.13, by definitions we have the implications:

$$(2.1) \Rightarrow (2.2) \Rightarrow (2.4) \quad \text{and} \quad (2.1) \Rightarrow (2.3) \Rightarrow (2.5) \Rightarrow (2.4)\,.$$

Definition 2.15 [3, Definition 4.33] Let D be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$ be nonexpansive, and let $\alpha \in \left]0,1\right[$. Then T is *averaged with constant* α, or α*-averaged* for short, if there exists a nonexpansive operator $F : D \to \mathcal{H}$ such that $T = (1-\alpha)\operatorname{Id} + \alpha F$.

Fact 2.16 [3, Remark 4.34(i)&(iii)] *Let D be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$.*

(i) *If T is averaged, then it is nonexpansive.*

(ii) *T is firmly nonexpansive if and only if it is $\frac{1}{2}$-averaged.*

Fact 2.17 [3, Proposition 4.35] *Let D be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$ be nonexpansive, and let $\alpha \in \left]0,1\right[$. Then the following are equivalent:*

(i) *T is α-averaged.*

(ii) $(\forall x \in D)\ (\forall y \in D)\ \|Tx - Ty\|^2 + \frac{1-\alpha}{\alpha}\|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2 \le \|x - y\|^2$.

Fact 2.18 [3, Proposition 4.42] *Let D be a nonempty subset of $\mathcal{H}$, let $(T_i)_{i\in \mathrm{I}}$ be a finite family of nonexpansive operators from D to $\mathcal{H}$, let $(\omega_i)_{i\in\mathrm{I}}$ be real numbers in $\left]0,1\right]$ such that $\sum_{i\in\mathrm{I}} \omega_i = 1$, and let $(\alpha_i)_{i\in\mathrm{I}}$ be real numbers in $\left]0,1\right[$ such that, for every $i \in \mathrm{I}$, T_i is α_i-averaged, and set $\alpha := \sum_{i\in\mathrm{I}} \omega_i\alpha_i$. Then $\sum_{i\in\mathrm{I}} \omega_i T_i$ is α-averaged.*

Fact 2.19 [3, Proposition 4.47] *Let D be a nonempty subset of $\mathcal{H}$, let $(T_i)_{i\in\mathrm{I}}$ be a finite family of quasinonexpansive operators from D to $\mathcal{H}$ such that $\cap_{i\in\mathrm{I}} \operatorname{Fix} T_i \neq \varnothing$, and let $(\omega_i)_{i\in\mathrm{I}}$ be strictly positive real numbers such that $\sum_{i\in\mathrm{I}} \omega_i = 1$. Then $\operatorname{Fix} \sum_{i\in\mathrm{I}} \omega_i T_i = \cap_{i\in\mathrm{I}} \operatorname{Fix} T_i$.*

Fact 2.20 [3, Proposition 4.49] *Let D be a nonempty subset of $\mathcal{H}$, and let T_1 and T_2 be quasinonexpansive operators from D to D. Suppose that T_1 or T_2 is strictly quasinonexpansive, and that $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \neq \varnothing$. Then the following hold:*

(i) $\operatorname{Fix} T_1T_2 = \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$.

(ii) *Suppose that T_1 and T_2 are strictly quasinonexpansive. Then T_1T_2 is strictly quasinonexpansive.*

Lemma 2.21 *Let $T : \mathcal{H} \to \mathcal{H}$ be α-averaged with $\alpha \in \left]0,1\right[$. Assume that $0 \in \operatorname{Fix} T$. Then*

$$(\forall x \in \mathcal{H} \smallsetminus \operatorname{Fix} T) \quad \|Tx\| < \|x\|. \tag{2.6}$$

Proof. Since T is α-averaged, by Fact 2.17,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|Tx - Ty\|^2 + \frac{1-\alpha}{\alpha}\|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2 \le \|x - y\|^2, \tag{2.7}$$

Applying (2.7) with $x \notin \operatorname{Fix} T$ and $y = 0$, we obtain (2.6). ■

The following result is motivated by [9, Lemma 2.1(iv)]. Moreover, Proposition 2.22(ii) was shown in [7, Proposition 2.10]

Proposition 2.22 *Suppose that* $\mathcal{H} = \mathbb{R}^n$. *Let* $T : \mathcal{H} \to \mathcal{H}$ *be linear and* α*-averaged with* $\alpha \in \left]0,1\right[$. *Then the following assertions hold:*

(i) *Let* $F : \mathcal{H} \to \mathcal{H}$ *be nonexpansive and linear. If* $\operatorname{Fix}(T) \cap \operatorname{ran}(F) = \{0\}$, *then* $\|TF\| < 1$.

(ii) $\|T \operatorname{P}_{(\operatorname{Fix} T)^\perp}\| < 1$.

Proof. (i): Lemma 2.21 implies

$$(\forall x \in \mathbb{R}^n \smallsetminus \operatorname{Fix} T) \quad \|Tx\| < \|x\|. \tag{2.8}$$

Both F and T are nonexpansive and linear, so $\|TF\| \leq \|T\|\|F\| \leq 1$. Assume to the contrary $\|TF\| = 1$, that is, $1 = \|TF\| = \max_{\|x\|=1} \|TFx\|$. Then there exists $\bar{x} \in \mathcal{H}$ with $\|\bar{x}\| = 1$ and $1 = \|TF\| = \|TF\bar{x}\|$. Denote $\hat{x} := F\bar{x}$. Then $\hat{x} \neq 0$ and $\hat{x} \in \operatorname{ran} F$. By assumption, $\operatorname{Fix}(T) \cap \operatorname{ran}(F) = \{0\}$, so $\hat{x} \notin \operatorname{Fix} T$. Substitute $x = \hat{x}$ in (2.8) to obtain that

$$1 = \|TF\bar{x}\| = \|T\hat{x}\| < \|\hat{x}\| = \|F\bar{x}\| \leq \|\bar{x}\| = 1,$$

which is absurd.

(ii): By Fact 2.4(iii), $\operatorname{P}_{(\operatorname{Fix} T)^\perp}$ is nonexpansive and linear. Moreover, $\operatorname{Fix} T \cap \operatorname{ran}(\operatorname{P}_{(\operatorname{Fix} T)^\perp}) = \operatorname{Fix} T \cap (\operatorname{Fix} T)^\perp = \{0\}$. Hence, the desired result is clear by substituting $F = \operatorname{P}_{(\operatorname{Fix} T)^\perp}$ in (i). ∎

Fact 2.23 [16, Page 111–113] *Let* $(X, \|\cdot\|_2)$ *and* $(Y, \|\cdot\|_2)$ *be finite dimensional real vector spaces. Let* E *and* B *be bases of* X *and* Y *respectively, with the elements of* E *and* B *arranged in a definite order (which is arbitrary but fixed). Let* $T : X \to Y$ *be a linear operator. Then there exists a matrix* T_{EB} *uniquely determined by the linear operator* T. *We say that the matrix* T_{EB} *represents the operator* T *with respect to those bases. Moreover,* $\|T\| = \|T_{EB}\|_2$.

Fact 2.24 [18, Page 281] *Let* $A \in \mathbb{R}^{n \times m}$. *The matrix* 2*-norm induced by the Euclidean vector norm is*

$$\|A\|_2 = \max_{\|x\|_2 \leq 1} \|Ax\|_2 = \sqrt{\lambda_{\max}},$$

where $\lambda_{\max}$ *is the largest eigenvalue of* $A^\intercal A$.

Proposition 2.25 *Suppose that* $\mathcal{H} = \mathbb{R}^n$ *with the Euclidean norm* $\|\cdot\|_2$. *Let* $T : \mathcal{H} \to \mathcal{H}$ *be linear and* α*-averaged with* $\alpha \in \left]0,1\right[$. *Assume that* A *is a matrix representing of the linear operator* $T \operatorname{P}_{(\operatorname{Fix} T)^\perp}$. *Denote the largest eigenvalue of the matrix* $A^\intercal A$ *as* $\lambda_{\max}$. *Then*

$$\lambda_{\max} = \|A\|_2^2 = \|T \operatorname{P}_{(\operatorname{Fix} T)^\perp}\|^2 < 1.$$

Proof. By Fact 2.23, the matrix A above is well-defined. Combining Proposition 2.22(ii), Facts 2.23 and 2.24, we obtain the desired results. ∎

Definition 2.26 [16, Definition 3.10-1] Let $T \in \mathcal{B}(\mathcal{H})$ with the *adjoint* T^*. T is said to be

(i) *self-adjoint* if $T^* = T$,

(ii) *unitary* if T is bijective and $T^* = T^{-1}$,

(iii) *normal* if $TT^* = T^*T$.

Fact 2.27 [3, Fact 2.25] *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then the following statements hold:*

(i) $T^{**} = T$.

(ii) $\|T\| = \|T^*\| = \sqrt{\|T^*T\|}$.

(iii) $(\ker T)^\perp = \overline{\operatorname{ran}} T^*$.

(iv) $(\operatorname{ran} T)^\perp = \ker T^*$.

Fact 2.28 *[4, Lemma 2.1] Let T be a nonexpansive linear operator on $\mathcal{H}$. Then*

$$\operatorname{Fix} T = \operatorname{Fix} T^*.$$

Lemma 2.29 *Let $T : \mathcal{H} \to \mathcal{H}$ be linear, and nonexpansive. Then*

$$\operatorname{Fix} T = (\operatorname{ran}(\operatorname{Id} - T))^\perp, \quad \text{and} \quad \overline{\operatorname{ran}}\,(\operatorname{Id} - T) = (\operatorname{Fix} T)^\perp.$$

Proof. Let $x \in \mathcal{H}$. Clearly, for every operator $F : \mathcal{H} \to \mathcal{H}$, $x \in \operatorname{Fix} F \Leftrightarrow x = Fx \Leftrightarrow (\operatorname{Id} - F)x = 0 \Leftrightarrow x \in \ker(\operatorname{Id} - F)$, which implies that

$$\operatorname{Fix} F = \ker(\operatorname{Id} - F). \tag{2.9}$$

Because T is nonexpansive and linear and $\operatorname{Id} - T$ is bounded and linear, by Fact 2.28 and Fact 2.27(iv), we obtain that

$$\operatorname{Fix} T = \operatorname{Fix} T^* \overset{(2.9)}{=} \ker(\operatorname{Id} - T^*) = \ker\left((\operatorname{Id} - T)^*\right) = (\operatorname{ran}(\operatorname{Id} - T))^\perp.$$

Similarly, by Fact 2.28 and Fact 2.27(iii)&(i), we have that

$$(\operatorname{Fix} T)^\perp = (\operatorname{Fix} T^*)^\perp \overset{(2.9)}{=} (\ker(\operatorname{Id} - T^*))^\perp = (\ker\left((\operatorname{Id} - T)^*\right))^\perp = \overline{\operatorname{ran}}\,(\operatorname{Id} - T).$$

Therefore, the proof is complete. ■

Fact 2.30 [4, Lemma 2.4] *Let $U_1, \ldots, U_m$ be closed linear subspaces of $\mathcal{H}$, and let $T := \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1}$. Then T is nonexpansive and*

$$\operatorname{Fix} T = \operatorname{Fix} T^* = \operatorname{Fix}(TT^*) = \operatorname{Fix}(T^*T) = \cap_{i=1}^m U_i.$$

Fact 2.31 [4, Lemmas 3.14 and 3.15] *Let $T : \mathcal{H} \to \mathcal{H}$ be linear and nonexpansive. Then the following statements hold:*

(i) $(\forall k \in \mathbb{N})\ \|T^k - \mathrm{P}_{\operatorname{Fix} T}\| = \|(T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp})^k\|$. *In particular,*

$$(\forall k \in \mathbb{N})(\forall x \in \mathcal{H}) \quad \|T^k x - \mathrm{P}_{\operatorname{Fix} T}\, x\| \leq \|(T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp})^k\| \|x - \mathrm{P}_{\operatorname{Fix} T}\, x\|. \tag{2.10}$$

and $\|(T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp})^k\|$ is the smallest constant independent of x for which (2.10) is valid.

(ii) $\|T^*T\,\mathrm{P}_{(\operatorname{Fix} T^*T)^\perp}\| \leq \|T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp}\|^2$ *and* $\|T^*T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp}\| = \|T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp}\|^2$ *if* $\operatorname{Fix}(T^*T) = \operatorname{Fix} T$.

(iii) *If T is normal, then* $(\forall k \in \mathbb{N})\ \|T^k - \mathrm{P}_{\operatorname{Fix} T}\| = \|(T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp})^k\| = \|T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp}\|^k$.

(iv) *Let $U_1, \ldots, U_m$ be closed linear subspaces of $\mathcal{H}$, and let $T := \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1}$. Then*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|(T^*T)^k x - \mathrm{P}_{\cap_{i=1}^m U_i}\, x\| \leq \|T\,\mathrm{P}_{(\cap_{i=1}^m U_i)^\perp}\|^{2k} \|x - \mathrm{P}_{\cap_{i=1}^m U_i}\, x\|.$$

Proposition 2.32 *Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $T : \mathcal{H} \to \mathcal{H}$ be linear and α-averaged with $\alpha \in \,]0,1[$. Then $\|T\,\mathrm{P}_{(\mathrm{Fix}\,T)^\perp}\| < 1$ and*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|T^k x - \mathrm{P}_{\mathrm{Fix}\,T}\, x\| \leq \|T\,\mathrm{P}_{(\mathrm{Fix}\,T)^\perp}\|^k \|x - \mathrm{P}_{\mathrm{Fix}\,T}\, x\|.$$

Consequently, $(\forall x \in \mathcal{H})$ $(T^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\mathrm{Fix}\,T}\, x$ with a linear rate $\|T\,\mathrm{P}_{(\mathrm{Fix}\,T)^\perp}\| < 1$.

Proof. T is α-averaged implies that T is nonexpansive, so the required result follows from Fact 2.31(i) and Proposition 2.22(ii). ■

Proposition 2.33 *Let $F : \mathcal{H} \to \mathcal{H}$ be nonexpansive, linear and normal. Let $\alpha \in \,]0,1[$. Denote $T := (1-\alpha)\,\mathrm{Id} + \alpha F$. Then the following assertions hold:*

(i) *T is α-averaged, linear, and normal. Moreover, we have that $\mathrm{Fix}\,T = \mathrm{Fix}\,F$, and that*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|T^k x - \mathrm{P}_{\mathrm{Fix}\,F}\, x\| \leq \|T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp}\|^k \|x - \mathrm{P}_{\mathrm{Fix}\,F}\, x\|. \tag{2.11}$$

(ii) $(\forall k \in \mathbb{N})$ $\|(T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp})^k\| = \|T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp}\|^k$.

(iii) *Assume that $\mathcal{H} = \mathbb{R}^n$. Then $(T^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\mathrm{Fix}\,F}\, x$ with a sharp linear rate $\|T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp}\| < 1$.*

Proof. (i): It is clear that T is α-averaged, linear, and $\mathrm{Fix}\,T = \mathrm{Fix}\,F$. The inequality (2.11) follows from Fact 2.31(i). Because the normal operators form a vector space which contains F and Id, it is clear that T is normal.

(ii): Combine (i) with Fact 2.31(iii).

(iii): Combine (i) with Proposition 2.22(ii) to obtain that $\|T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp}\| < 1$. Apply Fact 2.31(i) with (ii) above to the linear and nonexpansive operator $T = (1-\alpha)\,\mathrm{Id} + \alpha F$, we know that $(\forall k \in \mathbb{N})$ $\|T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp}\|^k = \|(T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp})^k\|$ is the smallest constant independent of x for which (2.11) is valid. Therefore, $(T^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\mathrm{Fix}\,F}\, x$ with a sharp linear rate $\|T\,\mathrm{P}_{(\mathrm{Fix}\,F)^\perp}\| < 1$. ■

Definition 2.34 [4, Definition 3.1] Let $T : \mathcal{H} \to \mathcal{H}$ be linear and nonexpansive. The *accelerated mapping A_T of T* is defined on $\mathcal{H}$ by

$$(\forall x \in \mathcal{H}) \quad A_T(x) := t_x Tx + (1-t_x)x,$$

where

$$t_x := t_{x,T} := \begin{cases} \frac{\langle x, x-Tx\rangle}{\|x-Tx\|^2}, & \text{if } Tx \neq x; \\ 1, & \text{if } Tx = x. \end{cases}$$

Fact 2.35 [4, Lemmas 3.27 and 3.8(3)] *Let $T : \mathcal{H} \to \mathcal{H}$ be linear, nonexpansive, and self-adjoint. Set*

$$c_1 := \inf\{\langle Tx, x\rangle \mid x \in (\mathrm{Fix}\,T)^\perp, \|x\| = 1\}, \tag{2.12}$$

and

$$c_2 := \sup\{\langle Tx, x\rangle \mid x \in (\mathrm{Fix}\,T)^\perp, \|x\| = 1\}, \tag{2.13}$$

where both c_1 and c_2 are defined to be 0 if $(\mathrm{Fix}\,T)^\perp = \{0\}$, i.e., if $\mathrm{Fix}\,T = \mathcal{H}$. Then

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|A_T^k x - \mathrm{P}_{\mathrm{Fix}\,T}\, x\| \leq \left(\frac{c_2 - c_1}{2 - c_1 - c_2}\right)^k \|x - \mathrm{P}_{\mathrm{Fix}\,T}\, x\|.$$

Lemma 2.36 *Let $T : \mathcal{H} \to \mathcal{H}$ be linear, nonexpansive, self-adjoint and monotone. Let c_1 and c_2 be defined as in (2.12) and (2.13). Set $c(T) := \|T\,\mathrm{P}_{(\mathrm{Fix}\,T)^\perp}\|$. Then*

$$\frac{c_2 - c_1}{2 - c_1 - c_2} = \frac{c(T) - c_1}{2 - c_1 - c(T)} \leq \frac{c(T)}{2 - c(T)}.$$

Proof. This is inside the proof of [4, Theorem 3.29]. ■

3 Isometries

In this section, we show some important properties of isometries. Some of them will be used in our main linear convergence results later.

Definition 3.1 [16, Definition 1.6-1] A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be *isometric* or an *isometry* if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|Tx - Ty\| = \|x - y\|. \tag{3.1}$$

Note that in some references, the definition of isometry is the linear operator satisfying (3.1). In this paper, the definition of isometry follows from [16, Definition 1.6-1] where the linearity is not required.

We show some common isometries in the following fact.

Fact 3.2 [7, Lemmas 2.23 and 2.24]

(i) *Let C be a closed affine subspace of $\mathcal{H}$. Then the reflector $\mathrm{R}_C = 2\,\mathrm{P}_C - \mathrm{Id}$ is isometric with $\operatorname{Fix} \mathrm{R}_C = C$.*

(ii) *Let $a \in \mathcal{H}$. The translation operator $(\forall x \in \mathcal{H})\ T_a x = x + a$ is isometric.*

(iii) *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and let T^* be the adjoint of T. Then T is isometric if and only if $T^*T = \mathrm{Id}$.*

(iv) *The identity operator is isometric.*

(v) *The composition of finitely many isometries is an isometry.*

Clearly, the reflector associated with an affine subspace is affine but not necessarily linear. The translation operator T_a defined in Fact 3.2(ii) is not linear and $\operatorname{Fix} T_a = \varnothing$ whenever $a \neq 0$.

Fact 3.3 [19, Lemma 1.7][3, Proposition 4.8(iii)] *Let D be a nonempty convex subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$, let $(x_i)_{i\in \mathrm{I}}$ be a finite family in D, let $(\alpha_i)_{i\in \mathrm{I}}$ be a finite family in $\mathbb{R}$ s.t. $\sum_{i\in \mathrm{I}} \alpha_i = 1$, and set $y := \sum_{i\in \mathrm{I}} \alpha_i x_i$. Then $\|Ty - \sum_{i\in \mathrm{I}} \alpha_i T x_i\|^2 + \sum_{i\in \mathrm{I}} \alpha_i \left(\|y - x_i\|^2 - \|Ty - Tx_i\|^2\right) = \frac{1}{2} \sum_{i\in \mathrm{I}} \sum_{j\in \mathrm{I}} \alpha_i\alpha_j \left(\|x_i - x_j\|^2 - \|Tx_i - Tx_j\|^2\right)$.*

Proposition 3.4 *Let $T : \mathcal{H} \to \mathcal{H}$ be isometric. Then T is affine.*

Proof. The desired result is directly from Definition 3.1 and Fact 3.3. ■

Corollary 3.5 *Let $T : \mathcal{H} \to \mathcal{H}$ be isometric. If $\operatorname{Fix} T$ is nonempty, then $\operatorname{Fix} T$ is an affine closed subspace.*

Consequently, the intersection of the fixed point sets of finitely many isometries is either empty or an affine closed subspace.

Proof. The desired result is easily from the related definitions and Proposition 3.4. ■

Fact 3.6 [18, Page 321] *The linear isometries on $\mathbb{R}^n$ are precisely the orthogonal matrices.*

Lemma 3.7 *Let $T : \mathcal{H} \to \mathcal{H}$ be isometric and let W be a nonempty closed convex set such that $W \subseteq \operatorname{Fix} T$. Then*

$$T\,\mathrm{P}_W = \mathrm{P}_W = \mathrm{P}_W\, T.$$

Proof. Let $x \in \mathcal{H}$. Because $\mathrm{P}_W x \in W \subseteq \operatorname{Fix} T$ and $\mathrm{P}_W Tx \in W \subseteq \operatorname{Fix} T$, $T\,\mathrm{P}_W x = \mathrm{P}_W x$ and $T\,\mathrm{P}_W Tx = \mathrm{P}_W Tx$. Hence, $T\,\mathrm{P}_W = \mathrm{P}_W$. Moreover, by definitions of projection and isometry, we have that $\|Tx - \mathrm{P}_W Tx\| \leq \|Tx - \mathrm{P}_W x\| = \|Tx - T\,\mathrm{P}_W x\| = \|x - \mathrm{P}_W x\| \leq \|x - \mathrm{P}_W Tx\| = \|Tx - T\,\mathrm{P}_W Tx\| = \|Tx - \mathrm{P}_W Tx\|$, which implies that $\|Tx - \mathrm{P}_W Tx\| = \|Tx - \mathrm{P}_W x\|$. By the uniqueness of projection on the nonempty closed convex set W, we obtain that $\mathrm{P}_W x = \mathrm{P}_W Tx$. Hence, $\mathrm{P}_W = \mathrm{P}_W T$. ■

Lemma 3.8 *Let $z \in \mathcal{H}$ and let $T : \mathcal{H} \to \mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ such that $(\forall x \in \mathcal{H})\ Fx = T(x + z) - z$. Then the following statements hold:*

(i) *If F is affine, then T is affine.*

(ii) *Suppose that $z \in \operatorname{Fix} T$. If T is affine, then F is linear.*

(iii) $\operatorname{Fix} F = \operatorname{Fix} T - z$.

(iv) *T is isometric if and only if F is isometric.*

Proof. Let x, y be in $\mathcal{H}$.
(i): Let λ be in $\mathbb{R}$. Because F is affine, we have

$$\begin{aligned} T(\lambda x + (1-\lambda)y) &= z + F(\lambda x + (1-\lambda)y - z) \\ &= z + F(\lambda(x-z) + (1-\lambda)(y-z)) \\ &= \lambda(z + F(x-z)) + (1-\lambda)(z + F(y-z)) \\ &= \lambda Tx + (1-\lambda)Ty. \end{aligned}$$

(ii): Let α, β be in $\mathbb{R}$. Because T is affine,

$$\begin{aligned} F(\alpha x + \beta y) &= T(\alpha x + \beta y + z) - z \\ &= T\big(\frac{1}{2}(2\alpha x + z) + \frac{1}{2}(2\beta y + z)\big) - z \\ &= \frac{1}{2}T(2\alpha x + z) + \frac{1}{2}T(2\beta y + z) - z \\ &= \frac{1}{2}T(2\alpha x + z) + \frac{1}{2}Tz + \frac{1}{2}T(2\beta y + z) + \frac{1}{2}Tz - 2z \quad (\text{by } z \in \operatorname{Fix} T) \\ &= T(\alpha x + z) + T(\beta y + z) - 2z \\ &= T\left(\alpha(x+z) + (1-\alpha)z\right) + T\left(\beta(y+z) + (1-\beta)z\right) - 2z \\ &= \alpha T(x+z) + (1-\alpha)Tz + \beta T(y+z) + (1-\beta)Tz - 2z \\ &= \alpha\left(T(x+z) - z\right) + \beta(T(y+z) - z) \\ &= \alpha Fx + \beta Fy. \end{aligned}$$

Hence, F is linear.
(iii): Clearly, $x \in \operatorname{Fix} F \Leftrightarrow x = Fx = T(x+z) - z \Leftrightarrow x + z = T(x+z) \Leftrightarrow x + z \in \operatorname{Fix} T$.
(iv): This is clear from $\|Tx - Ty\| = \|z + F(x-z) - (z + F(y-z))\| = \|F(x-z) - F(y-z)\|$. ■

Properties of surjective or self-adjoint linear isometries

Lemma 3.9 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry. Then T is unitary and normal.*

Proof. By Fact 3.6, without loss of generality, we assume that $T \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. Hence, by [18, Page 321], T has orthonormal columns and orthonormal rows, which implies that $T^\intercal T = \mathrm{Id} = TT^\intercal$. Therefore, T is unitary and normal. ■

The last result states that linear isometries on $\mathbb{R}^n$ must be normal; however, this fails in infinite-dimensional Hilbert space.

Example 3.10 Suppose that $\mathcal{H} = \ell^2 = \{(x_i)_{i\in\mathbb{N}} \ :\ \sum_{i\in\mathbb{N}} x_i^2 < \infty \text{ and } (\forall i \in \mathbb{N})\ x_i \in \mathbb{R}\}$ with the inner product $\langle x, y\rangle = \sum_{i\in\mathbb{N}} x_i y_i$ for every $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{N}}$ in ℓ^2. Define the right shift operator T_R and left shift operator T_L by

$$(\forall x = (x_i)_{i\in\mathbb{N}} \in \ell^2) \quad T_R x := (0, x_0, x_1, x_2, \cdots),$$

and

$$(\forall x = (x_i)_{i\in\mathbb{N}} \in \ell^2) \quad T_L x := (x_1, x_2, x_3, x_4, \cdots).$$

Then the following assertions hold:

(i) T_L and T_R are linear.

(ii) $T_R^* = T_L \neq T_R$.

(iii) $T_R^* T_R = T_L T_R = \operatorname{Id}$, but $T_L^* T_L = T_R T_L \neq \operatorname{Id}$. Hence, $T_R^* T_R = \operatorname{Id} \neq T_R T_R^*$.

(iv) T_R is isometric, but $T_L = T_R^*$ is not isometric.

(v) T_R is not normal.

(vi) T_R is not surjective. Hence, T_R is not unitary.

Remark 3.11 Recall that in the Hilbert sequence space ℓ^2, we draw from Example 3.10 the following conclusions:

(i) A linear isometry need not be self-adjoint.

(ii) A linear isometry need not be surjective; hence, a linear isometry need not be unitary.

(iii) Even if T is linear and isometric, T^* may fail to be isometric.

(iv) A linear isometry need not be normal.

Corollary 3.12 *Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $F_1, F_2, \ldots, F_m$ be linear isometries on $\mathbb{R}^n$. Set $T := (1-\alpha)\operatorname{Id} + \alpha F_m \cdots F_2 F_1$ with $\alpha \in \left]0,1\right[$. Then $\operatorname{Fix} T = \operatorname{Fix} F_m \cdots F_2 F_1$. Moreover, for every $x \in \mathbb{R}^n$, $(T^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\operatorname{Fix} F_m \cdots F_2 F_1} x$ with a sharp linear rate $\|T\,\mathrm{P}_{(\operatorname{Fix} F_m\cdots F_2 F_1)^\perp}\| < 1$.*

Proof. Because $F_m \cdots F_2 F_1$ is a linear isometry on $\mathbb{R}^n$, the result comes from Lemma 3.9 and Proposition 2.33(iii). ■

Example 3.13 Suppose that $\mathcal{H} = \mathbb{R}^n$. Let U_1, U_2 be linear subspaces. Denote by $T := \frac{1}{2}\operatorname{Id} + \frac{1}{2}\mathrm{R}_{U_2}\mathrm{R}_{U_1}$, the Douglas–Rachford operator. Then $(T^k x)_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\operatorname{Fix} T} x$ with a sharp linear rate $\|T\,\mathrm{P}_{(\operatorname{Fix} T)^\perp}\| < 1$.

More relations among isometric, normal and unitary operators can be found in [16, Section 3.10].

Theorem 3.14 *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Then the following statements hold:*

(i) *If T is isometric and self-adjoint, then $\mathrm{P}_{\operatorname{Fix} T} = \frac{1}{2}(\operatorname{Id} + T)$ and $\mathrm{P}_{(\operatorname{Fix} T)^\perp} = \frac{1}{2}(\operatorname{Id} - T)$.*

(ii) *T is isometric and self-adjoint if and only if $T = \mathrm{R}_U$, where U is a closed linear subspace of $\mathcal{H}$.*

Proof. (i): Suppose that T is isometric and self-adjoint. Then $\operatorname{Fix} T$ is a closed linear subspace of $\mathcal{H}$,

$$T = T^*, \quad \text{and} \quad T^2 = T^* T = \operatorname{Id} \tag{3.2}$$

by Fact 3.2(iii). Let $x \in \mathcal{H}$. Then

$$T\left(\tfrac{1}{2}(\operatorname{Id} + T)x\right) = \tfrac{1}{2}(Tx + TTx) \overset{(3.2)}{=} \tfrac{1}{2}(Tx + x),$$

and so $\frac{1}{2}(\operatorname{Id} + T)x \in \operatorname{Fix} T$. Moreover,

$$(\forall y \in \operatorname{Fix} T)\ \left\langle x - \tfrac{1}{2}(\operatorname{Id} + T)x, y \right\rangle = \tfrac{1}{2}\langle x - Tx, y\rangle = \tfrac{1}{2}\left(\langle x, y\rangle - \langle Tx, y\rangle\right) \overset{(3.2)}{=} \tfrac{1}{2}\left(\langle x, y\rangle - \langle x, Ty\rangle\right) = 0.$$

Hence, by Fact 2.2, we obtain that $\mathrm{P}_{\operatorname{Fix} T} x = \frac{1}{2}(\mathrm{Id} + T)x$.

Because $\operatorname{Fix} T$ is a closed linear subspace, by Fact 2.4(ii), $\mathrm{P}_{(\operatorname{Fix} T)^\perp} = \mathrm{Id} - \mathrm{P}_{\operatorname{Fix} T} = \frac{1}{2}(\mathrm{Id} - T)$.

(ii): "⇒" By (i), $T = 2\mathrm{P}_{\operatorname{Fix} T} - \mathrm{Id} = \mathrm{R}_{\operatorname{Fix} T}$ is the reflector associated with the closed linear subspace $\operatorname{Fix} T$.

"⇐" By Fact 2.3(i) and Fact 2.4(iii)&(iv), we know that $\mathrm{P}_U^2 = \mathrm{P}_U$, $\mathrm{P}_U \in \mathcal{B}(\mathcal{H})$ and $\mathrm{P}_U = \mathrm{P}_U^*$. Hence, $\mathrm{R}_U = 2\mathrm{P}_U - \mathrm{Id}$ satisfies $\mathrm{R}_U \in \mathcal{B}(\mathcal{H})$, $\mathrm{R}_U = \mathrm{R}_U^*$ and $\mathrm{R}_U^* \mathrm{R}_U = \mathrm{Id}$. By Fact 3.2(iii), the proof is complete. ■

Fact 3.15 [3, Proposition 29.6] *Let C and D be nonempty closed convex subsets of $\mathcal{H}$ such that $C \perp D$. Then $C + D$ is closed.*

The following result is essentially [1, Proposition 3.6], but our proof is different.

Theorem 3.16 *Let $T_1 : \mathcal{H} \to \mathcal{H}$ and $T_2 : \mathcal{H} \to \mathcal{H}$ be linear, self-adjoint and isometric. Then*

$$\operatorname{Fix} T_2T_1 = (\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) \oplus \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big).$$

Proof. Clearly, $(\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) \cap \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big) = \{0\}$. Hence, it remains to prove $\operatorname{Fix} T_2T_1 = (\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big)$. First note that

$$
\begin{aligned}
(\forall x \in \mathcal{H}) \quad x \in \operatorname{Fix} T_2T_1 &\Leftrightarrow x = T_2T_1x && \text{(3.3a)}\\
&\Leftrightarrow T_2x = T_1x \quad (T_2^* = T_2 \text{ and } T_2^*T_2 = \mathrm{Id}) && \text{(3.3b)}\\
&\Leftrightarrow 2\mathrm{P}_{\operatorname{Fix} T_2} x - x = 2\mathrm{P}_{\operatorname{Fix} T_1} x - x \quad (\text{by Theorem 3.14(i)}) && \text{(3.3c)}\\
&\Leftrightarrow \mathrm{P}_{\operatorname{Fix} T_2} x = \mathrm{P}_{\operatorname{Fix} T_1} x && \text{(3.3d)}\\
&\Leftrightarrow \mathrm{P}_{(\operatorname{Fix} T_2)^\perp} x = \mathrm{P}_{(\operatorname{Fix} T_1)^\perp} x. \quad (\text{by Fact 2.4(ii)}) && \text{(3.3e)}
\end{aligned}
$$

Clearly, (3.3d) and (3.3e) respectively imply that

$$\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \subseteq \operatorname{Fix} T_2T_1 \quad \text{and} \quad (\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp \subseteq \operatorname{Fix} T_2T_1.$$

Since $\operatorname{Fix} T_2T_1$ is a closed linear subspace of $\mathcal{H}$, we have

$$(\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big) \subseteq \operatorname{Fix} T_2T_1.$$

It suffices to show that $\operatorname{Fix} T_2T_1 \subseteq (\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big)$. Let $x \in \operatorname{Fix} T_2T_1$ but assume to the contrary that $x \notin (\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big)$. By Fact 3.15, we know that $(\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big)$ is a closed linear subspace of $\mathcal{H}$. Then by Fact 2.5, there exists $z \in \mathcal{H}$ with $\|z\| = 1$ such that

$$\langle z, x\rangle > 0, \tag{3.4}$$

and

$$\Big(\forall y \in (\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big)\Big) \quad \langle z, y\rangle = 0. \tag{3.5}$$

By (3.3d) and (3.3e), we have

$$\mathrm{P}_{\operatorname{Fix} T_1} x = \mathrm{P}_{\operatorname{Fix} T_2} x \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \quad \text{and} \quad \mathrm{P}_{(\operatorname{Fix} T_1)^\perp} x = \mathrm{P}_{(\operatorname{Fix} T_2)^\perp} x \in (\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp. \tag{3.6}$$

Combine Fact 2.4(ii) with (3.6) to obtain that

$$x = \mathrm{P}_{\operatorname{Fix} T_1} x + \mathrm{P}_{(\operatorname{Fix} T_1)^\perp} x \in (\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2) + \big((\operatorname{Fix} T_1)^\perp \cap (\operatorname{Fix} T_2)^\perp\big).$$

Hence, by (3.5), we have $\langle z, x\rangle = 0$ which contradicts with (3.4). ■

Corollary 3.17 *Let U_1, U_2 be closed linear subspaces of $\mathcal{H}$. Let $T := \frac{1}{2}(\mathrm{Id} + \mathrm{R}_{U_2}\mathrm{R}_{U_1})$ be the Douglas–Rachford operator. Then* $\operatorname{Fix} T = \operatorname{Fix} \mathrm{R}_{U_2}\mathrm{R}_{U_1} = (U_1 \cap U_2) \oplus \left(U_1^{\perp} \cap U_2^{\perp}\right)$.

Proof. The result follows from Theorem 3.16 and Theorem 3.14(ii). ■

The following examples show that it is not clear how to generalize Theorem 3.16 from two to finitely many isometries.

Example 3.18 Let U_1, U_2 be linear subspaces of $\mathbb{R}^n$ with $U_1^{\perp} \cap U_2^{\perp} \neq \{0\}$, and let $U_3 := \mathbb{R}^n$. Then

$$\operatorname{Fix} \mathrm{R}_{U_3}\mathrm{R}_{U_2}\mathrm{R}_{U_1} = \operatorname{Fix} \mathrm{R}_{U_2}\mathrm{R}_{U_1} = (U_1 \cap U_2) + \left(U_1^{\perp} \cap U_2^{\perp}\right) \neq (U_1 \cap U_2 \cap U_3) + \left(U_1^{\perp} \cap U_2^{\perp} \cap U_3^{\perp}\right).$$

Example 3.19 Suppose that $\mathcal{H} = \mathbb{R}^2$. Let $U_1 := \mathbb{R} \cdot (1,0)$, $U_2 := \mathbb{R} \cdot (1,1)$ and $U_3 := \mathbb{R} \cdot (0,1)$. Then

$$\operatorname{Fix} \mathrm{R}_{U_3}\mathrm{R}_{U_2}\mathrm{R}_{U_1} = \mathbb{R} \cdot (1,1).$$

Consequently,

$$\operatorname{Fix} \mathrm{R}_{U_3}\mathrm{R}_{U_2}\mathrm{R}_{U_1} \neq U_1 \cap U_2 \cap U_3 \quad \text{and} \quad \operatorname{Fix} \mathrm{R}_{U_3}\mathrm{R}_{U_2}\mathrm{R}_{U_1} \neq (U_1 \cap U_2 \cap U_3) + \left(U_1^{\perp} \cap U_2^{\perp} \cap U_3^{\perp}\right).$$

4 Circumcentered isometry methods

Circumcenter mappings

Recall that $\mathcal{P}(\mathcal{H})$ is the set of all nonempty subsets of $\mathcal{H}$ containing *finitely many* elements. By [5, Proposition 3.3], the following definition is well defined.

Definition 4.1 (circumcenter operator) [5, Definition 3.4] The *circumcenter operator* is

$$CC \colon \mathcal{P}(\mathcal{H}) \to \mathcal{H} \cup \{\varnothing\} \colon K \mapsto \begin{cases} p, & \text{if } p \in \operatorname{aff}(K) \text{ and } \{\|p - y\| \mid y \in K\} \text{ is a singleton;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

In particular, when $CC(K) \in \mathcal{H}$, that is, $CC(K) \neq \varnothing$, we say that the circumcenter of K exists and we call $CC(K)$ the *circumcenter of* K.

Fact 4.2 (scalar multiples) [5, Proposition 6.1] *Let $K \in \mathcal{P}(\mathcal{H})$ and $\lambda \in \mathbb{R} \smallsetminus \{0\}$. Then $CC(\lambda K) = \lambda CC(K)$.*

Fact 4.3 (translations) [5, Proposition 6.3] *Let $K \in \mathcal{P}(\mathcal{H})$ and $y \in \mathcal{H}$. Then $CC(K + y) = CC(K) + y$.*

Throughout this subsection, we assume that

$$G_1, \ldots, G_m \text{ are operators from } \mathcal{H} \text{ to } \mathcal{H} \text{ with } \cap_{j=1}^{m} \operatorname{Fix} G_j \neq \varnothing,$$

and that

$$\mathcal{S} = \{G_1, \ldots, G_m\} \quad \text{and} \quad (\forall x \in \mathcal{H}) \quad \mathcal{S}(x) = \{G_1 x, \ldots, G_m x\}.$$

Definition 4.4 (circumcenter mapping) [6, Definition 3.1] The *circumcenter mapping* induced by $\mathcal{S}$ is

$$CC_{\mathcal{S}} \colon \mathcal{H} \to \mathcal{H} \cup \{\varnothing\} \colon x \mapsto CC(\mathcal{S}(x)),$$

that is, for every $x \in \mathcal{H}$, if the circumcenter of the set $\mathcal{S}(x)$ defined in Definition 4.1 does not exist, then $CC_{\mathcal{S}} x = \varnothing$. Otherwise, $CC_{\mathcal{S}} x$ is the *unique* point satisfying the two conditions below:

(i) $CC_{\mathcal{S}}x \in \operatorname{aff}(\mathcal{S}(x)) = \operatorname{aff}\{G_1(x), \ldots, G_m(x)\}$, and

(ii) $\{\|CC_{\mathcal{S}}x - G_i(x)\| \mid i \in \{1, \ldots, m\}\}$ is a singleton, that is,

$$\|CC_{\mathcal{S}}x - G_1(x)\| = \cdots = \|CC_{\mathcal{S}}x - G_m(x)\|.$$

In particular, if for every $x \in \mathcal{H}$, $CC_{\mathcal{S}}x \in \mathcal{H}$, then we say the circumcenter mapping $CC_{\mathcal{S}}$ induced by $\mathcal{S}$ is *proper*. Otherwise, we call $CC_{\mathcal{S}}$ *improper*.

Assume that $CC_{\mathcal{S}}$ is proper. Recall that the *circumcenter method* induced by $\mathcal{S}$ is

$$x_k := CC_{\mathcal{S}}(x_{k-1}) = CC_{\mathcal{S}}^k x_0, \text{ where } x_0 \in \mathcal{H} \text{ and } k = 1, 2, \ldots. \tag{4.1}$$

Fact 4.5 [7, Proposition 2.33] *Assume $CC_{\mathcal{S}}$ is proper. Then there exist functions $(\forall i \in \{1, \ldots, m-1\})$ $\alpha_i : \mathcal{H} \to \mathbb{R}$ such that*

$$(\forall x \in \mathcal{H}) \quad CC_{\mathcal{S}}x = G_1 x + \sum_{i=1}^{m-1} \alpha_i(x)(G_{i+1}x - G_1 x).$$

Fact 4.6 [6, Proposition 3.10] *The following hold:*

(i) *If* $\operatorname{Fix} CC_{\mathcal{S}} \subseteq \cup_{i=1}^m \operatorname{Fix} G_i$, *then* $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} G_i$.

(ii) *If* $\operatorname{Id} \in \mathcal{S}$, *then* $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} G_i$.

Lemma 4.7 *Let $z \in \mathcal{H}$ and $\alpha \in \mathbb{R}$. Define $(\forall i \in \{1, \ldots, m\})$ $(\forall x \in \mathcal{H})$ $F_i x := \alpha G_i x - z$, and set $\widehat{\mathcal{S}} := \{F_1, F_2, \ldots, F_m\}$. Then $(\forall x \in \mathcal{S})$ $CC_{\widehat{\mathcal{S}}}x = \alpha CC_{\mathcal{S}}(x) - z$.*

Proof. If $\alpha = 0$, the result is trivial. Assume $\alpha \neq 0$. By Definition 4.4 and by Facts 4.2 and 4.3 ,

$$\begin{aligned}
(\forall x \in \mathcal{H}) \quad CC_{\widehat{\mathcal{S}}}x &= CC(\widehat{\mathcal{S}}(x)) \\
&= CC(\{F_1 x, F_2 x, \ldots, F_m x\}) \\
&= CC(\{\alpha G_1 x - z, \alpha G_2 x - z, \ldots, \alpha G_m x - z\}) \\
&= CC(\alpha\{G_1 x, G_2 x, \ldots, G_m x\} - z) \\
&= \alpha CC(\{G_1 x, G_2 x, \ldots, G_m x\}) - z \\
&= \alpha CC(\mathcal{S}(x)) - z = \alpha CC_{\mathcal{S}}(x) - z.
\end{aligned}$$

Therefore, the proof is complete. ■

Lemma 4.8 *Let $z \in \mathcal{H}$ and set $(\forall i \in \{1, \ldots, m\})$ $(\forall x \in \mathcal{H})$ $F_i x := G_i(x + z) - z$ as well as $\mathcal{S}_F := \{F_1, \ldots, F_m\}$. Then the following statements hold:*

(i) $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = z + CC_{\mathcal{S}_F}(x - z)$.

(ii) $(\forall x \in \mathcal{H})$ $(\forall k \in \mathbb{N})$ $CC_{\mathcal{S}}^k x = z + CC_{\mathcal{S}_F}^k(x - z)$.

(iii) $\cap_{i=1}^m \operatorname{Fix} F_i = \cap_{i=1}^m \operatorname{Fix} G_i - z$.

(iv) $(\forall x \in \mathcal{H})$ $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} G_i} x = z + \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} F_i}(x - z)$.

Proof. (i): By Definition 4.4 and by Fact 4.3, we obtain that $(\forall x \in \mathcal{H})$, $CC_{\mathcal{S}}x = CC(\{G_1 x, \ldots, G_m x\}) = CC(\{z + F_1(x - z), \ldots, z + F_m(x - z)\}) = CC(\mathcal{S}_F(x - z)) + z = z + CC_{\mathcal{S}_F}(x - z)$.

(ii): We prove by induction. Clearly, the result holds for $k = 0$. Assume $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}^k x = z + CC_{\mathcal{S}_F}^k(x - z)$ holds for some $k \geq 0$. Let $y \in \mathcal{H}$. Now by (i) above and by inductive hypothesis, $CC_{\mathcal{S}}^{k+1} y = CC_{\mathcal{S}}(CC_{\mathcal{S}}^k y) = z + CC_{\mathcal{S}_F}(CC_{\mathcal{S}}^k(y) - z) = z + CC_{\mathcal{S}_F}(z + CC_{\mathcal{S}_F}^k(y - z) - z) = z + CC_{\mathcal{S}_F}^{k+1}(y - z)$. Hence, we proved (ii) by induction.

(iii): This is a direct result from Lemma 3.8(iii).

(iv): This follows from Fact 2.1 and (iii) above. ■

Properties of circumcentered isometry methods

Recall our global assumptions that

$$(\forall i \in \{1,\ldots,m\}) \quad T_i : \mathcal{H} \to \mathcal{H} \text{ is affine isometry,}$$

and

$$\mathcal{S} := \{T_1,\ldots,T_m\} \quad \text{with} \quad \cap_{j=1}^m \operatorname{Fix} T_j \neq \varnothing.$$

From now on, denote by

$$\Omega(T_1,\ldots,T_m) := \Big\{ T_{i_r} \cdots T_{i_2} T_{i_1} \ \Big|\ r \in \mathbb{N}, \text{ and } i_1,\ldots,i_r \in \{1,\ldots,m\} \Big\}$$

which is the set consisting of all finite compositions of operators from $\{T_1,\ldots,T_m\}$. We use the empty product convention, so for $r=0$, $T_{i_0}\cdots T_{i_2}T_{i_1} = \operatorname{Id}$.

The following Fact 4.9(i) makes the circumcentered method induced by $\mathcal{S}$ defined in (4.1) well-defined. We call the circumcentered method induced by a set of isometries the *circumcentered isometry method* (CIM).

Fact 4.9 [7, Theorem 3.3, Lemma 3.5 and Proposition 4.2] *Let $x \in \mathcal{H}$. Then the following statements hold:*

(i) *The circumcenter mapping $CC_{\mathcal{S}} : \mathcal{H} \to \mathcal{H}$ induced by $\mathcal{S}$ is proper; moreover, $CC_{\mathcal{S}}x$ is the unique point satisfying the two conditions below:*

  (a) $CC_{\mathcal{S}}x \in \operatorname{aff}(\mathcal{S}(x))$, *and*

  (b) $\{\|CC_{\mathcal{S}}x - Tx\| \mid T \in \mathcal{S}\}$ *is a singleton.*

(ii) *Let W be a nonempty closed convex set of $\cap_{i=1}^m \operatorname{Fix} T_i$. Then $CC_{\mathcal{S}}x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(\mathrm{P}_{\cap_{j=1}^m \operatorname{Fix} T_j} x) = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(\mathrm{P}_W x)$.*

(iii) *Let $F : \mathcal{H} \to \mathcal{H}$ satisfy $(\forall y \in \mathcal{H})\ F(y) \in \operatorname{aff}(\mathcal{S}(y))$. Then $(\forall z \in \cap_{i=1}^m \operatorname{Fix} T_i)\ \|z - CC_{\mathcal{S}}x\|^2 + \|CC_{\mathcal{S}}x - Fx\|^2 = \|z - Fx\|^2$.*

(iv) *If $\operatorname{Id} \in \operatorname{aff} \mathcal{S}$, then $(\forall z \in \cap_{i=1}^m \operatorname{Fix} T_i)\ \|z - CC_{\mathcal{S}}x\|^2 + \|CC_{\mathcal{S}}x - x\|^2 = \|z - x\|^2$.*

(v) *Let W be a nonempty closed affine subspace of $\cap_{i=1}^m \operatorname{Fix} T_i$. Then $(\forall k \in \mathbb{N})\ \mathrm{P}_W CC_{\mathcal{S}}^k = \mathrm{P}_W$.*

Fact 4.10 [7, Proposition 3.7] *Let $F_1,\ldots,F_t$ be isometries from $\mathcal{H}$ to $\mathcal{H}$. Let $\mathcal{S}$ be a finite subset of $\Omega(F_1,\ldots,F_t)$. Let $\{\operatorname{Id}, F_1, F_2F_1, \ldots, F_tF_{t-1}\cdots F_2F_1\} \subseteq \mathcal{S}$. Then $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{j=1}^t \operatorname{Fix} F_j = \cap_{i=1}^m \operatorname{Fix} G_i$.*

The following result is a generalization of [7, Proposition 3.8].

Proposition 4.11 *The following statements hold:*

(i) *Assume that $\operatorname{Id} \in \operatorname{aff} \mathcal{S}$ and that $\operatorname{Fix} CC_{\mathcal{S}} \subseteq \cup_{i=1}^m \operatorname{Fix} T_i$. Then $CC_{\mathcal{S}}$ is firmly quasinonexpansive.*

(ii) *If $\operatorname{Id} \in \mathcal{S}$, then $CC_{\mathcal{S}}$ is firmly quasinonexpansive.*

Proof. (i): By assumptions and by Fact 4.6(i) and Fact 4.9(iv), we obtain that $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} T_i$ and that

$$(\forall x \in \mathcal{H})(\forall y \in \cap_{i=1}^m \operatorname{Fix} T_i) \quad \|y - CC_{\mathcal{S}}x\|^2 + \|CC_{\mathcal{S}}x - x\|^2 = \|y - x\|^2.$$

Hence,

$$(\forall x \in \mathcal{H})(\forall y \in \operatorname{Fix} CC_{\mathcal{S}}) \quad \|CC_{\mathcal{S}}x - y\|^2 + \|CC_{\mathcal{S}}x - x\|^2 \leq \|x - y\|^2,$$

which, by Definition 2.13(iii), means that $CC_{\mathcal{S}}$ is firmly quasinonexpansive.

(ii): Clearly, $\operatorname{Id} \in \mathcal{S}$ implies that $\operatorname{Fix} CC_{\mathcal{S}} \subseteq \mathcal{H} = \operatorname{Fix} \operatorname{Id} = \cup_{i=1}^m \operatorname{Fix} T_i$ and that $\operatorname{Id} \in \operatorname{aff} \mathcal{S}$. Hence, (ii) comes from (i). ■

Proposition 4.12 *Let $p \in \mathbb{N} \smallsetminus \{0\}$. Denote by $\mathrm{I} := \{1,\ldots,m\}$ and $\mathrm{J} := \{1,\ldots,p\}$. Let $(\forall j \in \mathrm{J})$ $\mathcal{S}_j \subseteq \{T_1,\ldots,T_m\}$ such that $\mathrm{Id} \in \mathcal{S}_j$ and $\cup_{t=1}^{p} \mathcal{S}_t = \{T_1,\ldots,T_m\}$. Then the following hold:*

(i) $\cap_{j=1}^{p} \operatorname{Fix} CC_{\mathcal{S}_j} = \cap_{i=1}^{m} \operatorname{Fix} T_i$.

(ii) *$CC_{\mathcal{S}_p} \cdots CC_{\mathcal{S}_1}$ is strictly quasinonexpansive and* $\operatorname{Fix} CC_{\mathcal{S}_p} \cdots CC_{\mathcal{S}_1} = \cap_{i=1}^{m} \operatorname{Fix} T_i$.

(iii) *Let $(\omega_j)_{j\in\mathrm{J}}$ be real numbers in $]0,1]$ such that $\sum_{j\in\mathrm{J}} \omega_j = 1$. Then $\sum_{j\in\mathrm{J}} \omega_j CC_{\mathcal{S}_j}$ is firmly quasinonexpansive. Moreover,* $\operatorname{Fix} \sum_{j\in\mathrm{J}} \omega_j CC_{\mathcal{S}_j} = \cap_{i=1}^{m} \operatorname{Fix} T_i$.

Proof. (i): Because $(\forall j \in \mathrm{J})$ $\mathrm{Id} \in \mathcal{S}_j \subseteq \{T_1,\ldots,T_m\}$, by Fact 4.6(ii), $(\forall j \in \mathrm{J})$ $\cap_{i=1}^{m} \operatorname{Fix} T_i \subseteq \cap_{T\in\mathcal{S}_j} \operatorname{Fix} T = \operatorname{Fix} CC_{\mathcal{S}_j}$. Hence,

$$\cap_{i=1}^{m} \operatorname{Fix} T_i \subseteq \cap_{j=1}^{p} \operatorname{Fix} CC_{\mathcal{S}_j}. \tag{4.2}$$

On the other hand, because $\cup_{t=1}^{p} \mathcal{S}_t = \{T_1,\ldots,T_m\}$, for every $i \in \mathrm{I}$ there exists $t_i \in \mathrm{J}$ such that $T_i \in \mathcal{S}_{t_i}$. By the assumption, $\mathrm{Id} \in \mathcal{S}_{t_i}$, and by Fact 4.6(ii) again, $\operatorname{Fix} CC_{\mathcal{S}_{t_i}} = \cap_{T\in\mathcal{S}_{t_i}} \operatorname{Fix} T \subseteq \operatorname{Fix} T_i$, which implies that $\cap_{j=1}^{p} \operatorname{Fix} CC_{\mathcal{S}_j} \subseteq \operatorname{Fix} CC_{\mathcal{S}_{t_i}} \subseteq \operatorname{Fix} T_i$. Moreover, because the above $i \in \mathrm{I}$ is chosen arbitrarily, we have

$$\cap_{j=1}^{p} \operatorname{Fix} CC_{\mathcal{S}_j} \subseteq \cap_{i=1}^{m} \operatorname{Fix} T_i. \tag{4.3}$$

Therefore, (4.2) and (4.3) yield (i).

(ii): Let $j \in \mathrm{J}$. By assumption, $\mathrm{Id} \in \mathcal{S}_j$, and by Proposition 4.11(ii), $CC_{\mathcal{S}_j}$ is firmly quasinonexpansive. By Remark 2.14, we know that $CC_{\mathcal{S}_j}$ is strictly quasinonexpansive. In addition, the global assumption $\cap_{i=1}^{m} \operatorname{Fix} T_i \neq \varnothing$ and the assumption, $\mathcal{S}_j \subseteq \{T_1,\ldots,T_m\}$, imply that $\cap_{T\in\mathcal{S}_j} \operatorname{Fix} T \neq \varnothing$. Hence, by [3, Corollary 4.50], $CC_{\mathcal{S}_p} \cdots CC_{\mathcal{S}_1}$ is strictly quasinonexpansive and $\operatorname{Fix} CC_{\mathcal{S}_p} \cdots CC_{\mathcal{S}_1} = \cap_{j=1}^{p} \operatorname{Fix} CC_{\mathcal{S}_j}$. Combine the identity with the (i) above to deduce $\operatorname{Fix} CC_{\mathcal{S}_p} \cdots CC_{\mathcal{S}_1} = \cap_{i=1}^{m} \operatorname{Fix} T_i$.

(iii): Let $j \in \mathrm{J}$. By assumption, $\mathrm{Id} \in \mathcal{S}_j$, and by Proposition 4.11(ii), $CC_{\mathcal{S}_j}$ is firmly quasinonexpansive. By [3, Corollary 4.48], $\sum_{j\in\mathrm{J}} \omega_j CC_{\mathcal{S}_j}$ is firmly quasinonexpansive.

In addition, by Remark 2.14, for every $j \in \mathrm{J}$, $CC_{\mathcal{S}_j}$ is firmly quasinonexpansive implies that $CC_{\mathcal{S}_j}$ is quasinonexpansive. By [3, Proposition 4.47] and the (i) above, we obtain that $\operatorname{Fix} \sum_{j\in\mathrm{J}} \omega_j CC_{\mathcal{S}_j} = \cap_{j=1}^{p} \operatorname{Fix} CC_{\mathcal{S}_j} = \cap_{i=1}^{m} \operatorname{Fix} T_i$. ■

Lemma 4.13 *Suppose that $T_1,\ldots,T_m$ are linear. Then* $(\forall x \in \mathcal{H})$ $(\forall i \in \{1,\ldots,m\})$ $T_i x - x \in (\cap_{j=1}^{m} \operatorname{Fix} T_j)^{\perp}$.

Proof. Let $x \in \mathcal{H}$ and let $i \in \{1,\ldots,m\}$. By Lemma 2.29, $T_i x - x \in \operatorname{ran}(T_i - \mathrm{Id}) \subseteq \overline{\operatorname{ran}}\,(\mathrm{Id} - T_i) = (\operatorname{Fix} T_i)^{\perp} \subseteq (\cap_{j=1}^{m} \operatorname{Fix} T_j)^{\perp}$. ■

Lemma 4.14 *Suppose that $T_1,\ldots,T_m$ are linear and that $T_1 = \mathrm{Id}$. Then the following statements hold:*

(i) $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}} x - x \in (\cap_{i=1}^{m} \operatorname{Fix} T_i)^{\perp}$.

(ii) $(\forall x \in (\cap_{i=1}^{m} \operatorname{Fix} T_i)^{\perp})$ $(\forall k \in \mathbb{N})$ $CC_{\mathcal{S}}^{k} x \in (\cap_{i=1}^{m} \operatorname{Fix} T_i)^{\perp}$.

Proof. (i): Let $x \in \mathcal{H}$. Since $\mathrm{Id} \in \mathcal{S}$, by Fact 4.9(i) and Fact 4.5, there exist $\alpha_1,\ldots,\alpha_{m-1}$ in $\mathbb{R}$ such that

$$CC_{\mathcal{S}} x - x = \sum_{i=1}^{m-1} \alpha_i (T_{i+1} x - x).$$

On the other hand, by Lemma 4.13, we have that $(\forall i \in \{1,\ldots,m-1\})$ $T_{i+1} x - x \in (\cap_{j=1}^{m} \operatorname{Fix} T_j)^{\perp}$. Since $(\cap_{j=1}^{m} \operatorname{Fix} T_j)^{\perp}$ is a linear subspace, (i) is true.

(ii): For every $x \in (\cap_{i=1}^{m} \operatorname{Fix} T_i)^{\perp}$, by (i) above, $CC_{\mathcal{S}} x = x + (CC_{\mathcal{S}} x - x) \in (\cap_{i=1}^{m} \operatorname{Fix} T_i)^{\perp}$. Therefore, the required result follows from (i) inductively. ■

Remark 4.15 (i) In view of Fact 4.10, we note that Lemma 4.13 and Lemma 4.14 reduce to [7, Propositions 5.4 and 5.5] respectively when the related isometries are reflectors.

(ii) Lemma 4.14(ii) implies that when we use the CIM, $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$, to find the best approximation $\mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x$, if we choose our initial point x in the linear subspace $(\cap_{i=1}^m \mathrm{Fix}\, T_i)^{\perp}$ and if $\mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x \neq 0$, then it is impossible for us to find the $\mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x$ in finitely many steps. This is consistent with [10, Section 4] which shows that to satisfy one step convergence of CRM for hyperplane intersection, there are certain requirements for the initial points.

The following result reduces to [7, Proposition 5.3] when the related isometries are reflectors.

Theorem 4.16 *Let $z \in \mathcal{H}$. Set $(\forall i \in \{1,\ldots,m\})$ $(\forall x \in \mathcal{H})$ $F_i x := T_i(x+z) - z$ and $\mathcal{S}_F = \{F_1,\ldots,F_m\}$. Let $\gamma \in [0,1[$. Then for every $x \in \mathcal{H}$, the following statements are equivalent:*

(i) $(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x\| \leq \gamma^k \|x - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x\|$.

(ii) $(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}_F}^k (x-z) - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, F_i}(x-z)\| \leq \gamma^k \|(x-z) - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, F_i}(x-z)\|$.

Consequently, the following assertions hold:

(a) Given $x \in \mathcal{H}$, $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\cap_{i=1}^m T_i} x$ with linear rate γ if and only if $(CC_{\mathcal{S}_F}^k (x-z))_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\cap_{i=1}^m F_i}(x-z)$ with linear rate γ.

(b) $(\forall x \in \mathcal{H})$ $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\cap_{i=1}^m T_i} x$ with linear rate γ if and only if $(\forall y \in \mathcal{H})$ $(CC_{\mathcal{S}_F}^k y)_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\cap_{i=1}^m F_i} y$ with linear rate γ.

Proof. By Lemma 4.8(iii), $\cap_{i=1}^m T_i \neq \varnothing$ is equivalent to $\cap_{i=1}^m F_i \neq \varnothing$. Hence, Fact 4.9(i) and Lemma 3.8(iv) yield that for every $x \in \mathcal{H}$, both $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ and $(CC_{\mathcal{S}_F}^k x)_{k\in\mathbb{N}}$ are well-defined.

By Lemma 4.8(iv)&(ii), for every $x \in \mathcal{H}$,

$$x - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x = (x-z) - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, F_i}(x-z),$$

and

$$(\forall k \in \mathbb{N}) \quad CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, T_i} x = CC_{\mathcal{S}_F}^k (x-z) - \mathrm{P}_{\cap_{i=1}^m \mathrm{Fix}\, F_i}(x-z).$$

Therefore, we obtain that (i) $\Leftrightarrow$ (ii). Moreover, it is clear that both (a) and (b) follow easily from the equivalence of (i) and (ii). The proof is complete. ■

Remark 4.17 Theorem 4.16, Proposition 3.4 and Lemma 3.8(ii) allow us to assume that all of the associated isometries are linear when we study the linear convergence of CIMs.

5 Linear convergence of circumcentered isometry methods

The linear convergence results in this section hinge on the following two facts.

Fact 5.1 [7, Theorem 4.14] *Recall that $(\forall i \in \{1,\ldots,m\})$ $T_i : \mathcal{H} \to \mathcal{H}$ is an affine isometry with $\cap_{j=1}^m \mathrm{Fix}\, T_j \neq \varnothing$ and that $\mathcal{S} := \{T_1,\ldots,T_m\}$. Let W be a nonempty closed affine subspace of $\cap_{j=1}^m \mathrm{Fix}\, T_j$. Assume that there exist $F : \mathcal{H} \to \mathcal{H}$ and $\gamma \in [0,1[$ such that $(\forall x \in \mathcal{H})$ $F(x) \in \mathrm{aff}\,(\mathcal{S}(x))$ and $(\forall x \in \mathcal{H})$ $\|Fx - \mathrm{P}_W x\| \leq \gamma \|x - \mathrm{P}_W x\|$. Then*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_W x\| \leq \gamma^k \|x - \mathrm{P}_W x\|.$$

Fact 5.2 [7, Theorem 4.15] *Suppose that $\mathcal{H} = \mathbb{R}^n$. Recall that $(\forall i \in \{1,\ldots,m\})$ $T_i : \mathcal{H} \to \mathcal{H}$ is an affine isometry with $\cap_{j=1}^m \operatorname{Fix} T_j \neq \varnothing$ and that $\mathcal{S} := \{T_1,\ldots,T_m\}$. Let $T_{\mathcal{S}} \in \operatorname{aff} \mathcal{S}$ satisfy that $\operatorname{Fix} T_{\mathcal{S}} \subseteq \cap_{T\in\mathcal{S}} \operatorname{Fix} T$. Assume that $T_{\mathcal{S}}$ is linear and α-averaged with $\alpha \in \left]0,1\right[$. Then $\|T_{\mathcal{S}} \mathrm{P}_{(\cap_{T\in\mathcal{S}} \operatorname{Fix} T)^\perp}\| \in \left[0,1\right[$. Moreover,*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{T\in\mathcal{S}} \operatorname{Fix} T}\, x\| \leq \|T_{\mathcal{S}} \mathrm{P}_{(\cap_{T\in\mathcal{S}} \operatorname{Fix} T)^\perp}\|^k \|x - \mathrm{P}_{\cap_{T\in\mathcal{S}} \operatorname{Fix} T}\, x\|.$$

Note that because $T_{\mathcal{S}}$ is linear, $0 \in \operatorname{Fix} T_{\mathcal{S}} \subseteq \cap_{T\in\mathcal{S}} \operatorname{Fix} T$, which implies that $(\forall T \in \mathcal{S})$, T must be linear. In addition, actually, $T_{\mathcal{S}} \in \operatorname{aff} \mathcal{S}$ and $\operatorname{Fix} T_{\mathcal{S}} \subseteq \cap_{T\in\mathcal{S}} \operatorname{Fix} T$ imply that $\operatorname{Fix} T_{\mathcal{S}} = \cap_{T\in\mathcal{S}} \operatorname{Fix} T$.

Linear convergence of CIMs in finite-dimensional spaces

Lemma 5.3 *Let $t \in \mathbb{N} \smallsetminus \{0\}$ and let $\mathrm{I} := \{1,2,\ldots,t\}$. Let $F_1, F_2, \ldots, F_t$ be nonexpansive and linear on $\mathcal{H}$. Let $(\omega_i)_{i\in\mathrm{I}}$ be real numbers in $\left]0,1\right]$ such that $\sum_{i\in\mathrm{I}} \omega_i = 1$ and let $(\alpha_i)_{i\in\mathrm{I}}$ be real numbers in $\left]0,1\right[$. Denote*

$$A := \sum_{i\in\mathrm{I}} \omega_i A_i \quad \textit{where} \quad (\forall i \in \mathrm{I}) \quad A_i := (1-\alpha_i)\operatorname{Id} + \alpha_i F_i.$$

Then the following assertions hold:

(i) *Let $\alpha := \sum_{i\in\mathrm{I}} \omega_i \alpha_i$. Then A is α-averaged and linear.*

(ii) $\operatorname{Fix} A = \cap_{i\in\mathrm{I}} \operatorname{Fix} F_i$.

(iii) *Assume that $\widetilde{\mathcal{S}}$ is a finite set of operators such that $\{\operatorname{Id}, F_1, F_2, \ldots, F_t\} \subseteq \widetilde{\mathcal{S}}$. Then $A \in \operatorname{aff} \widetilde{\mathcal{S}}$.*

Proof. (i): Because $F_1, F_2, \ldots, F_t$ are linear, A is linear. Since $F_1, F_2, \ldots, F_t$ are nonexpansive, $(\forall i \in \mathrm{I})$ A_i is α_i-averaged. Hence, the required result follows from Fact 2.18.

(ii): The result follows from Fact 2.19.

(iii): By definition, $(\forall i \in \mathrm{I})$ $A_i \in \operatorname{aff}\{\operatorname{Id}, F_1, F_2, \ldots, F_t\}$. Hence, $A \in \operatorname{aff}\{\operatorname{Id}, F_1, F_2, \ldots, F_t\} \subseteq \operatorname{aff} \widetilde{\mathcal{S}}$. ■

The following result reduces to [7, Proposition 5.15] when the isometries are reflectors.

Theorem 5.4 *Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $F_1, F_2, \ldots, F_t$ be linear isometries on $\mathcal{H}$. Assume that $\widetilde{\mathcal{S}}$ is a finite subset of $\Omega(F_1, \ldots, F_t)$, where $\Omega(F_1, \ldots, F_t)$ consists of all finite compositions of operators from $\{F_1, \ldots, F_t\}$. Assume that $\{\operatorname{Id}, F_1, F_2, \ldots, F_t\} \subseteq \widetilde{\mathcal{S}}$. Let $(\omega_i)_{i\in\mathrm{I}}$ be real numbers in $\left]0,1\right]$ such that $\sum_{i\in\mathrm{I}} \omega_i = 1$ and let $(\alpha_i)_{i\in\mathrm{I}}$ be real numbers in $\left]0,1\right[$. Denote $A := \sum_{i=1}^t \omega_i A_i$ where $(\forall i \in \{1,\ldots,t\})$ $A_i := (1-\alpha_i)\operatorname{Id} + \alpha_i F_i$. Then the following statements hold:*

(i) $\operatorname{Fix} CC_{\widetilde{\mathcal{S}}} = \cap_{T\in\widetilde{\mathcal{S}}} \operatorname{Fix} T = \cap_{i=1}^t \operatorname{Fix} F_i = \operatorname{Fix} A$.

(ii) $\|A \mathrm{P}_{(\cap_{i=1}^t \operatorname{Fix} F_i)^\perp}\| < 1$. *Moreover,*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\widetilde{\mathcal{S}}}^k x - \mathrm{P}_{\cap_{i=1}^t \operatorname{Fix} F_i}\, x\| \leq \|A \mathrm{P}_{(\cap_{i=1}^t \operatorname{Fix} F_i)^\perp}\|^k \|x - \mathrm{P}_{\cap_{i=1}^t \operatorname{Fix} F_i}\, x\|.$$

Consequently, $(CC_{\widetilde{\mathcal{S}}}^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\cap_{i=1}^t \operatorname{Fix} F_i}\, x$ with a linear rate $\|A \mathrm{P}_{(\cap_{i=1}^t \operatorname{Fix} F_i)^\perp}\|$.

Proof. By assumptions and by Fact 4.9(i), $CC_{\widetilde{\mathcal{S}}}$ is proper.

(i): By assumption, $\widetilde{\mathcal{S}} \subseteq \Omega(F_1, \ldots, F_t)$, so every operator in $\widetilde{\mathcal{S}}$ is a finite composition of operators from $\{F_1, \ldots, F_t\}$. Hence, $\cap_{i=1}^t \operatorname{Fix} F_i \subseteq \cap_{T\in\widetilde{\mathcal{S}}} \operatorname{Fix} T$. Moreover, because $\{\operatorname{Id}, F_1, F_2, \ldots, F_t\} \subseteq \widetilde{\mathcal{S}}$, $\cap_{T\in\widetilde{\mathcal{S}}} \operatorname{Fix} T \subseteq \cap_{i=1}^t \operatorname{Fix} F_i$. Hence, $\cap_{i=1}^t \operatorname{Fix} F_i = \cap_{T\in\widetilde{\mathcal{S}}} \operatorname{Fix} T$. By Lemma 5.3(ii), $\operatorname{Fix} A = \cap_{i=1}^t \operatorname{Fix} F_i$. By Fact 4.6(ii), we have $\operatorname{Fix} CC_{\widetilde{\mathcal{S}}} = \cap_{T\in\widetilde{\mathcal{S}}} \operatorname{Fix} T$. Hence, $\operatorname{Fix} CC_{\widetilde{\mathcal{S}}} = \cap_{T\in\widetilde{\mathcal{S}}} \operatorname{Fix} T = \cap_{i=1}^t \operatorname{Fix} F_i = \operatorname{Fix} A$.

(ii): This follows from (i) above, Lemma 5.3(i)&(iii), Proposition 2.22(ii), and Fact 5.2. ■

Lemma 5.5 *Let $t \in \mathbb{N} \smallsetminus \{0\}$, let $\mathrm{I} := \{1, \ldots, t\}$, let $F_1, F_2, \ldots, F_t$ be nonexpansive and linear, let $(\omega_i)_{i\in\mathrm{I}}$ be real numbers in $]0,1]$ s.t. $\sum_{i\in\mathrm{I}} \omega_i = 1$ and let $(\alpha_i)_{i\in\mathrm{I}}$ and $(\lambda_i)_{i\in\mathrm{I}}$ be real numbers in $]0,1[$. Denote $A := \sum_{i\in\mathrm{I}} \omega_i A_i$ where $A_1 := (1-\alpha_1)\,\mathrm{Id} + \alpha_1 F_1$, $(\forall i \in \mathrm{I}\smallsetminus\{1\})$ $A_i := (1-\alpha_i)\,\mathrm{Id} + \alpha_i\left((1-\lambda_i)\,\mathrm{Id} + \lambda_i F_i\right) F_{i-1}\cdots F_1$. Then the following assertions hold:*

(i) *Let $\alpha := \sum_{i\in\mathrm{I}} \omega_i \alpha_i$. Then A is α-averaged and linear.*

(ii) $\mathrm{Fix}\, A = \cap_{i=1}^{t} \mathrm{Fix}\, F_i$.

(iii) *Assume that $\widetilde{\mathcal{S}}$ is a finite set of operators such that $\{\mathrm{Id}, F_1, F_2F_1, \ldots, F_t\cdots F_2F_1\} \subseteq \widetilde{\mathcal{S}}$. Then $A \in \mathrm{aff}\, \widetilde{\mathcal{S}}$.*

Proof. (i): Since $F_1, F_2, \ldots, F_t$ are linear, so is A. Since $F_1, F_2, \ldots, F_t$ are nonexpansive, thus $(\forall i \in \mathrm{I})$ A_i is α_i-averaged. Hence, the required result follows from Fact 2.18.

(ii): Since every averaged operator is strictly quasinonexpansive, we have $(\forall i \in \mathrm{I})$ $(\forall \lambda \in\,]0,1[\,)$ $(1-\lambda)\,\mathrm{Id} + \lambda F_i$ is strictly quasinonexpansive. Hence, the result follows from Facts 2.19 and 2.20, since $\cap_{i=1}^{t} \mathrm{Fix}\, F_i \neq \varnothing$.

(iii): Clearly, $A_1 := (1-\alpha_1)\,\mathrm{Id} + \alpha_1 F_1 \in \mathrm{aff}\,\{\mathrm{Id}, F_1, F_2F_1, \ldots, F_m\cdots F_2F_1\} \subseteq \mathrm{aff}\, \widetilde{\mathcal{S}}$. Moreover, for every $i \in \mathrm{I}\smallsetminus\{1\}$,

$$\begin{aligned}
A_i &= (1-\alpha_i)\,\mathrm{Id} + \alpha_i\left((1-\lambda_i)\,\mathrm{Id} + \lambda_i F_i\right) F_{i-1}\cdots F_1 \\
&= (1-\alpha_i)\,\mathrm{Id} + \alpha_i\left((1-\lambda_i)F_{i-1}\cdots F_1 + \lambda_i F_i F_{i-1}\cdots F_1\right) \\
&= (1-\alpha_i)\,\mathrm{Id} + \alpha_i(1-\lambda_i)F_{i-1}\cdots F_1 + \alpha_i\lambda_i F_i F_{i-1}\cdots F_1 \\
&\in \mathrm{aff}\,\{\mathrm{Id}, F_1, F_2F_1, \ldots, F_m\cdots F_2F_1\} \subseteq \mathrm{aff}\, \widetilde{\mathcal{S}}.
\end{aligned}$$

Hence, $A = \sum_{i\in\mathrm{I}} \omega_i A_i \in \mathrm{aff}\, \widetilde{\mathcal{S}}$. ∎

The following results is a generalization of [9, Theorem 3.3] and [7, Proposition 5.10].

Theorem 5.6 *Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $F_1, F_2, \ldots, F_t$ be linear isometries. Assume that $\widetilde{\mathcal{S}}$ is a finite subset of $\Omega(F_1, \ldots, F_t)$, where $\Omega(F_1, \ldots, F_t)$ consists of all finite compositions of operators from $\{F_1, \ldots, F_t\}$. Assume that $\{\mathrm{Id}, F_1, F_2F_1, \ldots, F_t\cdots F_2F_1\} \subseteq \widetilde{\mathcal{S}}$. Let $(\omega_i)_{i\in\mathrm{I}}$ be real numbers in $]0,1]$ such that $\sum_{i\in\mathrm{I}} \omega_i = 1$ and let $(\alpha_i)_{i\in\mathrm{I}}$ and $(\lambda_i)_{i\in\mathrm{I}}$ be real numbers in $]0,1[$. Set $A := \sum_{i\in\mathrm{I}} \omega_i A_i$ where $A_1 := (1-\alpha_1)\,\mathrm{Id} + \alpha_1 F_1$ and $(\forall i \in \mathrm{I}\smallsetminus\{1\})$ $A_i := (1-\alpha_i)\,\mathrm{Id} + \alpha_i\left((1-\lambda_i)\,\mathrm{Id} + \lambda_i F_i\right) F_{i-1}\cdots F_1$. Then the following assertions hold:*

(i) $\mathrm{Fix}\, CC_{\widetilde{\mathcal{S}}} = \cap_{T\in\widetilde{\mathcal{S}}} \mathrm{Fix}\, T = \cap_{i=1}^{t} \mathrm{Fix}\, F_i = \mathrm{Fix}\, A$.

(ii) $\|A\, \mathrm{P}_{(\cap_{i=1}^{t} \mathrm{Fix}\, F_i)^{\perp}}\| \in [0,1[$. *Moreover,*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\widetilde{\mathcal{S}}}^{k} x - \mathrm{P}_{\cap_{i=1}^{t} \mathrm{Fix}\, F_i}\, x\| \leq \|A\, \mathrm{P}_{(\cap_{i=1}^{t} \mathrm{Fix}\, F_i)^{\perp}}\|^{k} \|x - \mathrm{P}_{\cap_{i=1}^{t} \mathrm{Fix}\, F_i}\, x\|.$$

Consequently, $(CC_{\widetilde{\mathcal{S}}}^{k} x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\cap_{i=1}^{t} \mathrm{Fix}\, F_i}\, x$ with a linear rate $\|A\, \mathrm{P}_{(\cap_{i=1}^{t} \mathrm{Fix}\, F_i)^{\perp}}\|$.

Proof. By assumptions and by Fact 4.9(i), $CC_{\widetilde{\mathcal{S}}}$ is proper.

(i): Because $\widetilde{\mathcal{S}}$ is a finite subset of $\Omega(F_1, \ldots, F_t)$ such that $\{\mathrm{Id}, F_1, F_2F_1, \ldots, F_t\cdots F_2F_1\} \subseteq \widetilde{\mathcal{S}}$, by Fact 4.10, $\mathrm{Fix}\, CC_{\widetilde{\mathcal{S}}} = \cap_{T\in\widetilde{\mathcal{S}}} \mathrm{Fix}\, T = \cap_{i=1}^{t} \mathrm{Fix}\, F_i$. In addition, by Lemma 5.5(ii), $\mathrm{Fix}\, A = \cap_{i=1}^{t} \mathrm{Fix}\, F_i$. Hence, (i) is true.

(ii): This follows from (i) above, Lemma 5.5(i)&(iii), Proposition 2.22(ii), and Fact 5.2. ∎

Corollary 5.7 *Suppose that $\mathcal{H} = \mathbb{R}^n$ and that $T_1, T_2, \ldots, T_m$ are linear isometries. Set $\mathcal{S}_1 := \{\mathrm{Id}, T_1, T_2, \ldots, T_m\}$ and $\mathcal{S}_2 := \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m\cdots T_2T_1\}$. Then*

(i) *$(CC_{\mathcal{S}_1}^{k} x)_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\cap_{i=1}^{m} \mathrm{Fix}\, T_i}\, x = \mathrm{P}_{\mathrm{Fix}\, CC_{\mathcal{S}_1}}\, x$.*

(ii) $(CC_{\mathcal{S}_2}^k x)_{k\in\mathbb{N}}$ *converges linearly to* $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\operatorname{Fix} CC_{\mathcal{S}_2}} x$.

Proof. (i): This is from Theorem 5.4 with $\widetilde{\mathcal{S}} = \{\mathrm{Id}, T_1, T_2, \ldots, T_m\}$ by applying $t = m+1$, and $F_1 = \mathrm{Id}, F_2 = T_1, \ldots, F_t = T_m$.

(ii): This comes from Theorem 5.6 with $\widetilde{\mathcal{S}} = \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m \cdots T_2T_1\}$ by applying $t = m+1$, and $F_1 = \mathrm{Id}, F_2 = T_1, F_3 = T_2, \ldots, F_t = T_m$. ■

Remark 5.8 (i) Corollary 5.7(i) states that for every nonempty set $\mathcal{S}$ of linear isometries in $\mathbb{R}^n$, if $\mathrm{Id} \in \mathcal{S}$, then for every $x \in \mathbb{R}^n$, $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ converges linearly to $\mathrm{P}_{\cap_{T\in\mathcal{S}} \operatorname{Fix} T} x = \mathrm{P}_{\operatorname{Fix} CC_{\mathcal{S}}} x$.

(ii) Corollary 5.7(i)&(ii) illustrate that given arbitrary linear isometries $T_1, T_2, \ldots, T_m$ in $\mathbb{R}^n$, we are able to construct multiple CIMs linearly converging to $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\operatorname{Fix} CC_{\mathcal{S}}} x$ for every $x \in \mathbb{R}^n$.

Example 5.9 Let U_1 and U_2 be closed linear subspaces of $\mathbb{R}^n$. Set $\mathcal{S}_1 := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ and $\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Let $x \in \mathcal{H}$. Then by Corollary 5.7, $(CC_{\mathcal{S}_1}^k x)_{k\in\mathbb{N}}$ and $(CC_{\mathcal{S}_2}^k x)_{k\in\mathbb{N}}$ both linearly converge to $\mathrm{P}_{U_1\cap U_2} x$.

Linear convergence of CIMs in Hilbert spaces with adjustment of the initial point

In view of [4, Page 3438], in order to better accelerate the symmetric MAP, the accelerated symmetric MAP first applies another operator to the initial point. (Similarly, to accelerate the DRM, the C–DRM first applies another operator to the initial point as well, see [8, Theorem 1].) The following results provide sufficient conditions for the linear convergence of CIMs with first applying an operator T to the initial point. We shall provide applications of the following results later.

Theorem 5.10 *Suppose that* $T_1, \ldots, T_m$ *are linear isometries from* $\mathcal{H}$ *to* $\mathcal{H}$ *and that* $\mathcal{S} = \{T_1, T_2, \ldots, T_m\}$ *with* $T_1 = \mathrm{Id}$. *Let* W *be a nonempty closed linear subspace of* $\cap_{i=1}^m \operatorname{Fix} T_i$. *Let* $F : \mathcal{H} \to \mathcal{H}$ *satisfy* $(\forall x \in \mathcal{H})$ $Fx \in \operatorname{aff}(\mathcal{S}(x))$. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be such that* $\mathrm{P}_W T = \mathrm{P}_W = T\mathrm{P}_W$. *Assume one of the following items holds:*

(i) *There exists* $\gamma \in [0,1[$ *such that* $(\forall x \in \mathcal{H})$ $\|Fx - \mathrm{P}_W x\| \leq \gamma \|x - \mathrm{P}_W x\|$.

(ii) *There exists* $\gamma \in [0,1[$ *such that* $(\forall x \in \mathcal{H})$ $\|Fx - \mathrm{P}_{\operatorname{Fix} F} x\| \leq \gamma \|x - \mathrm{P}_{\operatorname{Fix} F} x\|$ *and that* $(\forall k \in \mathbb{N})$ $\mathrm{P}_{\operatorname{Fix} F} CC_{\mathcal{S}}^k T = \mathrm{P}_W$.

Then

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k Tx - \mathrm{P}_W x\| \leq \gamma^k \|T\mathrm{P}_{W^\perp}\| \|x - \mathrm{P}_W x\|. \tag{5.1}$$

Proof. We prove (5.1) by induction on k.

Because

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad \|Tx - \mathrm{P}_W x\| &= \|Tx - T\mathrm{P}_W x\| \quad (\text{by } \mathrm{P}_W = T\mathrm{P}_W)\\ &= \|T(x - \mathrm{P}_W x)\| \quad (T \text{ is linear})\\ &= \|T\mathrm{P}_{W^\perp}\mathrm{P}_{W^\perp} x\| \quad (\text{by Fact 2.4(ii) and Fact 2.3(i)})\\ &\leq \|T\mathrm{P}_{W^\perp}\|\|x - \mathrm{P}_W x\|,\end{aligned}$$

(5.1) is true for $k = 0$.

Suppose that (5.1) is true for some $k \in \mathbb{N}$. Let $x \in \mathcal{H}$. First note that

$$\begin{aligned}\|CC_{\mathcal{S}}^{k+1} Tx - \mathrm{P}_W x\| &= \left\|CC_{\mathcal{S}}\left(CC_{\mathcal{S}}^k Tx\right) - \mathrm{P}_W(CC_{\mathcal{S}}^k Tx)\right\| \quad (\text{by Fact 4.9(v) and } \mathrm{P}_W T = \mathrm{P}_W)\\ &\leq \left\|F\left(CC_{\mathcal{S}}^k Tx\right) - \mathrm{P}_W(CC_{\mathcal{S}}^k Tx)\right\|. \quad \left(\text{apply Fact 4.9(iii) with } z = \mathrm{P}_W(CC_{\mathcal{S}}^k Tx)\right)\end{aligned}$$

Assume first that assumption (i) holds. Then

$$\begin{aligned}\|F\left(CC_{\mathcal{S}}^{k}Tx\right)-\mathrm{P}_W(CC_{\mathcal{S}}^{k}Tx)\| &\leq \gamma\left\|CC_{\mathcal{S}}^{k}Tx-\mathrm{P}_W(CC_{\mathcal{S}}^{k}Tx)\right\| \quad \text{(by assumption (i))}\\ &\leq \gamma\left\|CC_{\mathcal{S}}^{k}Tx-\mathrm{P}_W x\right\| \quad \text{(by Fact 4.9(v) and } \mathrm{P}_W T=\mathrm{P}_W)\\ &\leq \gamma^{k+1}\|T\,\mathrm{P}_{W^\perp}\|\|x-\mathrm{P}_W x\|. \quad \text{(by inductive hypothesis)}\end{aligned}$$

Now assume that assumption (ii) holds. By assumptions and by Fact 4.9(v), we have that

$$(\forall t\in\mathbb{N})\quad \mathrm{P}_{\operatorname{Fix}F}\,CC_{\mathcal{S}}^{t}T=\mathrm{P}_W=\mathrm{P}_W\,T=\mathrm{P}_W\,CC_{\mathcal{S}}^{t}T. \tag{5.2}$$

Hence,

$$\begin{aligned}\|F\left(CC_{\mathcal{S}}^{k}Tx\right)-\mathrm{P}_W(CC_{\mathcal{S}}^{k}Tx)\| &= \|F(CC_{\mathcal{S}}^{k}Tx)-\mathrm{P}_{\operatorname{Fix}F}(CC_{\mathcal{S}}^{k}Tx)\| \quad \text{(by (5.2))}\\ &\leq \gamma\left\|CC_{\mathcal{S}}^{k}Tx-\mathrm{P}_{\operatorname{Fix}F}(CC_{\mathcal{S}}^{k}Tx)\right\| \quad \text{(by assumption (ii))}\\ &= \gamma\left\|CC_{\mathcal{S}}^{k}Tx-\mathrm{P}_W x\right\| \quad \text{(by (5.2))}\\ &\leq \gamma^{k+1}\|T\,\mathrm{P}_{W^\perp}\|\|x-\mathrm{P}_W x\|. \quad \text{(by inductive hypothesis)}\end{aligned}$$

Altogether, the proof is complete. ■

Remark 5.11 (i) One application of Theorem 5.10(i) is shown in Theorem 6.10 below.

(ii) Let L_1 and L_2 be closed linear subspace in $\mathcal{H}$. Assume that $\widetilde{\mathcal{S}}$ is a finite subset of $\Omega(\mathrm{R}_{L_1},\mathrm{R}_{L_2})$, where $\Omega(\mathrm{R}_{L_1},\mathrm{R}_{L_2})$ consists of all finite compositions of operators from $\{\mathrm{R}_{L_1},\mathrm{R}_{L_2}\}$. Assume that $L_1\cap L_2\subseteq\cap_{T\in\widetilde{\mathcal{S}}}\operatorname{Fix}T$. Let K be a closed linear subspace of $\mathcal{H}$ such that $L_1\cap L_2\subseteq K\subseteq L_1+L_2$. Denote by T_{L_2,L_1} the Douglas-Rachford operator associated with L_1 and L_2. Assume $T^d_{L_2,L_1}\in\operatorname{aff}\widetilde{\mathcal{S}}$ for some $d\in\mathbb{N}\smallsetminus\{0\}$. By [7, Corollary 5.17], we know that $\mathrm{P}_{L_1\cap L_2}=\mathrm{P}_{\operatorname{Fix}T_{L_2,L_1}}\mathrm{P}_K=\mathrm{P}_{L_1\cap L_2}\mathrm{P}_K=\mathrm{P}_{L_1\cap L_2}CC^k_{\widetilde{\mathcal{S}}}\mathrm{P}_K=\mathrm{P}_{\operatorname{Fix}T_{L_2,L_1}}CC^k_{\widetilde{\mathcal{S}}}\mathrm{P}_K$. In fact, [7, Proposition 5.18] is a special case of Theorem 5.10(ii) when $W=L_1\cap L_2$, $F=T^d_{L_2,L_1}$ and $T=\mathrm{P}_K$. Because [7, Proposition 5.18] is a generalization of [8, Theorem 1], [8, Theorem 1] is also a special instance of Theorem 5.10(ii).

6 Linear convergence of CRMs in Hilbert spaces

Since reflectors associated with affine subspaces are isometries, we deduce from Fact 4.9(i) that all of the circumcenter mappings induced by finite sets of reflectors are proper. In particular, we call the circumcenter method induced by a finite set of reflectors the *circumcentered reflection method* (CRM).

In this section, we shall use the linear convergence of *method of alternating projections* (MAP) to deduce sufficient conditions for the linear convergence of CRMs for finding the best approximation onto the intersection of finitely many affine subspaces.

Proposition 3.4, Lemma 3.8(ii) and Theorem 4.16 imply that in order to study the linear convergence of CRMs, we are free to assume that all of the related reflectors are associated with linear subspaces.

Recall that $m\in\mathbb{N}\smallsetminus\{0\}$. In this section, we assume that

$$U_1,\ldots,U_m \text{ are closed linear subspaces in the real Hilbert space } \mathcal{H}.$$

Clearly, $\{0\}\subseteq\cap_{i=1}^m U_i\neq\varnothing$. Set

$$\Omega:=\Omega(\mathrm{R}_{U_1},\ldots,\mathrm{R}_{U_m})=\left\{\mathrm{R}_{U_{i_r}}\cdots\mathrm{R}_{U_{i_2}}\mathrm{R}_{U_{i_1}}\;\middle|\; r\in\mathbb{N},\text{ and } i_1,\ldots,i_r\in\{1,\ldots,m\}\right\},$$

and

$$\Psi := \left\{ \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \;\middle|\; r, i_1, i_2, \ldots, i_r \in \{0, 1, \ldots, m\} \text{ and } 0 < i_1 < \cdots < i_r \right\}.$$

Recall that we use the empty product convention, so for $r = 0$, $\mathrm{R}_{U_{i_0}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} = \mathrm{Id}$.
We also assume that

$$\Psi \subseteq \mathcal{S} \subseteq \Omega \quad \text{and} \quad \mathcal{S} \text{ consists finitely many elements.} \tag{6.1}$$

Recall that $x \in \mathcal{H}$,

$$\mathcal{S}(x) := \{Tx \mid T \in \mathcal{S}\}.$$

In this section, we will deduce some linear convergence results on CRMs induced by $\mathcal{S}$ satisfying (6.1). We shall show that some CRMs do not have worse convergence rate than the sharp convergence rate of MAP for finding best approximation on $\cap_{i=1}^m U_i$. Moreover, we shall prove that some CRMs attain the known convergence rate of the accelerated symmetric MAP shown in [4].

Remark 6.1 We claim that there are exactly 2^m possible combinations for the indices of the reflectors making up the elements of the set Ψ.
In fact, for every $r \in \mathrm{I} := \{0, 1, \ldots, m\}$, the r-combination of the set I is a subset of r distinct items of I.[1] In addition, the number of r-combinations of I equals to the binomial coefficient $\binom{m}{r}$. Moreover, by the Binomial Theorem,

$$2^m = (1+1)^m = \sum_{r=0}^{m} \binom{m}{r}.$$

Therefore, the claim is true.
Actually with consideration of duplication, there are at most 2^m pairwise distinct elements in Ψ. (For instance, if $U_1 = U_2$, then $\mathrm{R}_{U_2} \mathrm{R}_{U_1} = \mathrm{Id}$.)

For example, when $m = 1$, $\Psi = \{\mathrm{Id}, \mathrm{R}_{U_1}\}$. When $m = 2$,

$$\Psi = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_2} \mathrm{R}_{U_1}\}.$$

When $m = 3$,

$$\Psi = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_3}, \mathrm{R}_{U_2} \mathrm{R}_{U_1}, \mathrm{R}_{U_3} \mathrm{R}_{U_1}, \mathrm{R}_{U_3} \mathrm{R}_{U_2}, \mathrm{R}_{U_3} \mathrm{R}_{U_2} \mathrm{R}_{U_1}\}.$$

Examples of linear convergent CRMs

First, let's see two examples where $m = 2$ to get some intuition about our upcoming main result Theorem 6.6. Actually, these examples are also corollaries of Theorem 6.6 below.

Example 6.2 Assume that $m = 2$, that $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_2} \mathrm{R}_{U_1}\}$, and that $U_1 + U_2$ is closed. Set $\gamma := \|\mathrm{P}_{U_2} \mathrm{P}_{U_1} \mathrm{P}_{(U_1 \cap U_2)^\perp}\|$. Then $\gamma \in [0, 1[$ and

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{U_1 \cap U_2} x\| \leq \gamma^k \|x - \mathrm{P}_{U_1 \cap U_2} x\|.$$

Consequently, $(CC_{\mathcal{S}}^k x)_{k \in \mathbb{N}}$ converges to $\mathrm{P}_{U_1 \cap U_2} x$ with a linear rate γ.

[1] Recall that we use the empty product convention that $\prod_{j=1}^{0} \mathrm{R}_{U_{i_j}} = \mathrm{Id}$, so the 0-combination of the set I is the Id.

Proof. By assumption and Facts 2.8 and 2.9, we know $\gamma \in [0,1[$. Set $T_{\mathcal{S}} := \mathrm{P}_{U_2}\mathrm{P}_{U_1}$. Using Fact 2.30, we get that $\operatorname{Fix} T_{\mathcal{S}} = \operatorname{Fix} \mathrm{P}_{U_2}\mathrm{P}_{U_1} = U_1 \cap U_2$. Hence, apply Fact 2.31(i) with T replaced by $T_{\mathcal{S}}$ to obtain that

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|T_{\mathcal{S}}^k x - \mathrm{P}_{U_1\cap U_2} x\| \leq \gamma^k \|x - \mathrm{P}_{U_1\cap U_2} x\|. \tag{6.2}$$

Moreover, because

$$T_{\mathcal{S}} = \mathrm{P}_{U_2}\mathrm{P}_{U_1} = \frac{1}{2}(\mathrm{R}_{U_2} + \mathrm{Id})\frac{1}{2}(\mathrm{R}_{U_1} + \mathrm{Id}) = \frac{1}{2^2}(\mathrm{R}_{U_2}\mathrm{R}_{U_1} + \mathrm{R}_{U_2} + \mathrm{R}_{U_1} + \mathrm{Id}) \in \operatorname{aff}(\mathcal{S})$$

and $U_1 \cap U_2$ is a closed linear subspace of $\cap_{T\in\mathcal{S}} \operatorname{Fix} T$, the desired results are from Fact 5.1 and (6.2). ■

Example 6.3 Assume that $m = 2$, that $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_1}\mathrm{R}_{U_2}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$, and that $U_1 + U_2$ is closed. Set $\gamma := \|\mathrm{P}_{U_2}\mathrm{P}_{U_1}\mathrm{P}_{(U_1\cap U_2)^\perp}\|$. Then $\gamma \in [0,1[$ and

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{U_1\cap U_2} x\| \leq \gamma^{2k} \|x - \mathrm{P}_{U_1\cap U_2} x\|.$$

Consequently, $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{U_1\cap U_2} x$ with a linear rate γ^2.

Proof. Denote $T_{\mathcal{S}} := \mathrm{P}_{U_1}\mathrm{P}_{U_2}\mathrm{P}_{U_1}$. Then $(\mathrm{P}_{U_2}\mathrm{P}_{U_1})^*\mathrm{P}_{U_2}\mathrm{P}_{U_1} = \mathrm{P}_{U_1}\mathrm{P}_{U_2}\mathrm{P}_{U_1} = T_{\mathcal{S}}$. Similarly with the proof of Example 6.2, we have that $\gamma \in [0,1[$ and $\operatorname{Fix} T_{\mathcal{S}} = U_1 \cap U_2$. Apply Fact 2.31(iv) with T replaced by $\mathrm{P}_{U_2}\mathrm{P}_{U_1}$ to obtain

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|T_{\mathcal{S}}^k x - \mathrm{P}_{U_1\cap U_2} x\| \leq \gamma^{2k} \|x - \mathrm{P}_{U_1\cap U_2} x\|. \tag{6.3}$$

Because

$$\begin{aligned}
&\frac{1}{2^3}(\mathrm{R}_{U_1} + \mathrm{Id})(\mathrm{R}_{U_2} + \mathrm{Id})(\mathrm{R}_{U_1} + \mathrm{Id})\\
=\ &\frac{1}{2^3}(\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1} + \mathrm{R}_{U_1}\mathrm{R}_{U_2} + \mathrm{R}_{U_1}\mathrm{R}_{U_1} + \mathrm{R}_{U_1} + \mathrm{R}_{U_2}\mathrm{R}_{U_1} + \mathrm{R}_{U_2} + \mathrm{R}_{U_1} + \mathrm{Id})\\
=\ &\frac{1}{2^3}(\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1} + \mathrm{R}_{U_1}\mathrm{R}_{U_2} + \mathrm{R}_{U_2}\mathrm{R}_{U_1} + 2\mathrm{R}_{U_1} + \mathrm{R}_{U_2} + 2\,\mathrm{Id})\\
\in\ &\operatorname{aff}\{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_1}\mathrm{R}_{U_2}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}\} = \operatorname{aff} \mathcal{S},
\end{aligned}$$

we obtain

$$T_{\mathcal{S}} = \mathrm{P}_{U_1}\mathrm{P}_{U_2}\mathrm{P}_{U_1} = \frac{1}{2^3}(\mathrm{R}_{U_1} + \mathrm{Id})(\mathrm{R}_{U_2} + \mathrm{Id})(\mathrm{R}_{U_1} + \mathrm{Id}) \in \operatorname{aff} \mathcal{S}.$$

Because $U_1 \cap U_2$ is a closed linear subspace of $\cap_{T\in\mathcal{S}} \operatorname{Fix} T$, the results come from Fact 5.1 and (6.3). ■

Remark 6.4 (i) From [12, Theorem 9.31] and Fact 2.8, we know that the sharp convergence rate of MAP associated with the two linear subspaces U_1 and U_2 is $\|\mathrm{P}_{U_2}\mathrm{P}_{U_1}\mathrm{P}_{(U_1\cap U_2)^\perp}\|^2$. Example 6.3 tells us that the linear convergence rate of some CRMs is no worse than $\|\mathrm{P}_{U_2}\mathrm{P}_{U_1}\mathrm{P}_{(U_1\cap U_2)^\perp}\|^2$.

(ii) Set $\mathcal{S}_1 := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$ and $\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_1}\mathrm{R}_{U_2}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. In [7, Section 6], our *numerical* experiments in $\mathbb{R}^{1000}$ showed that the CRMs induced by $\mathcal{S}_1$ and $\mathcal{S}_2$ given above perform better than the DRM, MAP and the C-DRM introduced in [8]. In [7], we didn't provide any *analytical* expanation for the outstanding performances of the CRMs induced by $\mathcal{S}_1$ and $\mathcal{S}_2$. Now Examples 6.2 and 6.3 present the theoretical support for the impressive performance.

CRMs associated with finitely many linear subspaces

In order to prove our more general results, we need the following lemma, which is also interesting itself.

Lemma 6.5 *Recall that $m \in \mathbb{N} \smallsetminus \{0\}$, $U_1, \ldots, U_m$ are closed linear subspaces in $\mathcal{H}$ and $\Psi \subseteq \mathcal{S} \subseteq \Omega$. Then the following statements hold:*

(i)

$$\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1} = 2^m \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} - \Big(\mathrm{Id} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_{k-1},i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big). \tag{6.4}$$

(ii)

$$\mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} = \frac{1}{2^m} \Big(\sum_{k=0}^{m} \sum_{\substack{i_1,\ldots,i_{k-1},i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big). \tag{6.5}$$

(iii) $\mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} \in \operatorname{conv} \Psi \subseteq \operatorname{aff} \mathcal{S}$.

Proof. When $k = m$, the only possibility for $\mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}}$ with $i_1, \ldots, i_k \in \{1, 2, \ldots, m\}$ and $i_1 < \cdots < i_k$ is $\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1}$. Hence, clearly (i) $\Leftrightarrow$ (ii). We thus only prove (i) and (iii).

(i): We prove this by induction on m. If $m = 1$, then by definition, $\mathrm{R}_{U_1} = 2\,\mathrm{P}_{U_1} - \mathrm{Id}$, which means that (6.4) is true for $m = 1$. Now assume (6.4) is true for some $m \geq 1$, i.e.,

$$\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1} = 2^m \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} - \Big(\mathrm{Id} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big) \tag{6.6}$$

Then

$$\begin{aligned}
& \mathrm{R}_{U_{m+1}} \mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1} \\
\overset{(6.6)}{=} & \mathrm{R}_{U_{m+1}} \Bigg(2^m \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} - \Big(\mathrm{Id} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big) \Bigg) \\
= & 2^m \mathrm{R}_{U_{m+1}} \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} - \Big(\mathrm{R}_{U_{m+1}} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{m+1}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big) \\
= & 2^m (2\,\mathrm{P}_{U_{m+1}} - \mathrm{Id}) \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} - \Big(\mathrm{R}_{U_{m+1}} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{m+1}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big) \\
= & 2^{m+1} \mathrm{P}_{U_{m+1}} \mathrm{P}_{U_m} \cdots \mathrm{P}_{U_1} - 2^m \mathrm{P}_{U_m} \mathrm{P}_{U_{m-1}} \cdots \mathrm{P}_{U_1} - \Big(\mathrm{R}_{U_{m+1}} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{m+1}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big) \\
\overset{(6.6)}{=} & 2^{m+1} \mathrm{P}_{U_{m+1}} \mathrm{P}_{U_m} \cdots \mathrm{P}_{U_1} - \mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1} - \Big(\mathrm{Id} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big) \\
& - \Big(R_{U_{m+1}} + \sum_{k=1}^{m-1} \sum_{\substack{i_1,\ldots,i_k \in \{1,2,\ldots,m\} \\ i_1 < \cdots < i_{k-1} < i_k}} \mathrm{R}_{U_{m+1}} \mathrm{R}_{U_{i_k}} \mathrm{R}_{U_{i_{k-1}}} \cdots \mathrm{R}_{U_{i_1}} \Big)
\end{aligned}$$

$$= 2^{m+1}\,\mathrm{P}_{U_{m+1}}\mathrm{P}_{U_m}\cdots\mathrm{P}_{U_1} - \Big(\mathrm{Id} + \sum_{k=1}^{m}\sum_{\substack{i_1,\ldots,i_k\in\{1,2,\ldots,m+1\}\\ i_1<\cdots<i_{k-1}<i_k}} \mathrm{R}_{U_{i_k}}\mathrm{R}_{U_{i_{k-1}}}\cdots\mathrm{R}_{U_{i_1}}\Big),$$

which is (6.4) with m being replaced by $m+1$. Therefore, (i) is true.

(iii): By Remark 6.1, we know there are exactly 2^m items in the big bracket on the right–hand side of (6.5), since these items in the big bracket are exactly all of the items in the set Ψ. Hence,

$$\mathrm{P}_{U_m}\mathrm{P}_{U_{m-1}}\cdots\mathrm{P}_{U_1} = \frac{1}{2^m}\Big(\sum_{k=0}^{m}\sum_{\substack{i_1,\ldots,i_{k-1},i_k\in\{1,2,\ldots,m\}\\ i_1<\cdots<i_{k-1}<i_k}} \mathrm{R}_{U_{i_k}}\mathrm{R}_{U_{i_{k-1}}}\cdots\mathrm{R}_{U_{i_1}}\Big) \in \operatorname{conv}\Psi \subseteq \operatorname{aff}\mathcal{S}.$$

Therefore, the proof is complete. ■

Now we are ready to use results on the linear convergence of MAPs or symmetric MAPs to prove the linear convergence of CRMs.

Theorem 6.6 *Recall that $U_1,\ldots,U_m$ are closed linear subspaces of $\mathcal{H}$ and that $\Psi \subseteq \mathcal{S} \subseteq \Omega$. Set $\gamma := \|\mathrm{P}_{U_m}\mathrm{P}_{U_{m-1}}\cdots\mathrm{P}_{U_1}\mathrm{P}_{(\cap_{i=1}^m U_i)^\perp}\|$. Assume that $m \geq 2$ and that $U_1^\perp + \cdots + U_m^\perp$ is closed. Then $\gamma \in [0,1[$ and*

$$(\forall x\in\mathcal{H})(\forall k\in\mathbb{N})\quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^m U_i}x\| \leq \gamma^k\|x - \mathrm{P}_{\cap_{i=1}^m U_i}x\|.$$

Consequently, $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\cap_{i=1}^m U_i}x$ with a linear rate γ.

Proof. Set $T_{\mathcal{S}} := \mathrm{P}_{U_m}\mathrm{P}_{U_{m-1}}\cdots\mathrm{P}_{U_1}$. Fact 2.30 yields $\operatorname{Fix} T_{\mathcal{S}} = \cap_{i=1}^m U_i$. The assumptions and Corollary 2.12 imply

$$\gamma = \|T_{\mathcal{S}}\mathrm{P}_{(\operatorname{Fix}T_{\mathcal{S}})^\perp}\| \in [0,1[\,. \tag{6.7}$$

Applying Fact 2.31(i) with T replaced by $T_{\mathcal{S}}$, we obtain

$$(\forall x\in\mathcal{H})(\forall k\in\mathbb{N})\quad \|T_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^m U_i}x\| \leq \gamma^k\|x - \mathrm{P}_{\cap_{i=1}^m U_i}x\|. \tag{6.8}$$

By Lemma 6.5(iii), $T_{\mathcal{S}} \in \operatorname{aff}(\mathcal{S})$. By the construction of Ω and by $\Psi \subseteq \mathcal{S} \subseteq \Omega$, we obtain that $(\forall T\in\mathcal{S})$, $\cap_{i=1}^m U_i \subseteq \operatorname{Fix}T$, which implies that $\cap_{i=1}^m U_i$ is a closed linear subspace of $\cap_{T\in\mathcal{S}}\operatorname{Fix}T$. Hence, Fact 5.1, (6.7) and (6.8) yield the required results. ■

Corollary 6.7 *Assume that $m = 2n-1$ for some $n \in \mathbb{N}\smallsetminus\{0\}$, that $U_1,\ldots,U_n$ are closed linear subspaces of $\mathcal{H}$ with $U_1^\perp+\cdots+U_n^\perp$ being closed, and $(\forall i\in\{1,\ldots,n-1\})$ $U_{n+i} := U_{n-i}$. Recall that $\Psi\subseteq\mathcal{S}\subseteq\Omega$. Denote $\gamma := \|\mathrm{P}_{U_n}\mathrm{P}_{U_{n-1}}\cdots\mathrm{P}_{U_1}\mathrm{P}_{(\cap_{i=1}^n U_i)^\perp}\|$. Then $\gamma\in[0,1[$ and*

$$(\forall x\in\mathcal{H})(\forall k\in\mathbb{N})\quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^n U_i}x\| \leq \gamma^{2k}\|x - \mathrm{P}_{\cap_{i=1}^n U_i}x\|,$$

that is $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ converges to $\mathrm{P}_{\cap_{i=1}^n U_i}x$ with a linear rate γ^2.

Proof. First note that, $\mathrm{P}_{U_m}\mathrm{P}_{U_{m-1}}\cdots\mathrm{P}_{U_1} = \mathrm{P}_{U_1}\cdots\mathrm{P}_{U_{n-1}}\mathrm{P}_{U_n}\mathrm{P}_{U_{n-1}}\cdots\mathrm{P}_{U_1}$, and that $U_1^\perp+\cdots+U_{n-1}^\perp+U_n^\perp+U_{n+1}^\perp+\cdots+U_m^\perp = U_1^\perp+\cdots+U_n^\perp$. Set $\rho := \|\mathrm{P}_{U_m}\mathrm{P}_{U_{m-1}}\cdots\mathrm{P}_{U_1}\mathrm{P}_{(\cap_{i=1}^m U_i)^\perp}\|$. Since $U_1^\perp+\cdots+U_n^\perp$ is closed, Theorem 6.6 implies

$$(\forall x\in\mathcal{H})(\forall k\in\mathbb{N})\quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^m U_i}x\| \leq \rho^k\|x - \mathrm{P}_{\cap_{i=1}^m U_i}x\|. \tag{6.9}$$

Also set $T := \mathrm{P}_{U_n}\mathrm{P}_{U_{n-1}}\cdots\mathrm{P}_{U_1}$. Then Fact 2.30, Fact 2.31(ii) and Corollary 2.12 yield

$$\gamma = \|T\mathrm{P}_{(\cap_{i=1}^n U_i)^\perp}\| = \|T^*T\mathrm{P}_{(\cap_{i=1}^n U_i)^\perp}\|^{\frac{1}{2}} = \rho^{\frac{1}{2}} \in [0,1[\,, \tag{6.10}$$

because $\cap_{i=1}^n U_i = \cap_{i=1}^m U_i$. Hence, (6.9) and (6.10) yield

$$(\forall x\in\mathcal{H})(\forall k\in\mathbb{N})\quad \|CC_{\mathcal{S}}^k x - \mathrm{P}_{\cap_{i=1}^n U_i}x\| \leq \gamma^{2k}\|x - \mathrm{P}_{\cap_{i=1}^n U_i}x\|,$$

as claimed. ■

Applications of the accelerated symmetric MAP

In this section, set $T := \mathrm{P}_{U_m} \cdots \mathrm{P}_{U_1}$ and let A_T be the accelerated mapping of T defined in Definition 2.34. In the following two results we take advantage of the linear convergence of iteration sequence from A_T as a bridge to show the linear convergence of certain classes of CRMs.

Theorem 6.8 *Assume that $m = 2n-1$ for some $n \in \mathbb{N} \smallsetminus \{0\}$, $U_1, \ldots, U_n$ are closed linear subspaces of $\mathcal{H}$ with $U_1^{\perp} + \cdots + U_n^{\perp}$ being closed, and $(\forall i \in \{1, \ldots, n-1\})$ $U_{n+i} := U_{n-i}$. Recall that $\Psi \subseteq \mathcal{S} \subseteq \Omega$. Let c_1 and c_2 be defined as in (2.12) and (2.13). Set $T := \mathrm{P}_{U_m} \cdots \mathrm{P}_{U_1} = \mathrm{P}_{U_1} \cdots \mathrm{P}_{U_{n-1}} \mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1}$, $c(T) := \|T \mathrm{P}_{(\operatorname{Fix} T)^{\perp}}\|$, $\gamma := \|\mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1} \mathrm{P}_{(\cap_{i=1}^m U_i)^{\perp}}\|$, and $\eta := \frac{c_2 - c_1}{2 - c_1 - c_2}$. Then the following statements hold:*

(i) $0 \leq \eta \leq \frac{c(T)}{2 - c(T)} \leq c(T) = \gamma^2 < 1$.

(ii) $(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k(x) - \mathrm{P}_{\cap_{i=1}^n U_i} x\| \leq \eta^k \|x - \mathrm{P}_{\cap_{i=1}^n U_i} x\|$.

Proof. By Facts 2.3 and 2.4, we know that T is a linear, nonexpansive and self-adjoint. Because $T = \mathrm{P}_{U_1} \cdots \mathrm{P}_{U_{n-1}} \mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1} = (\mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1})^* \mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1}$, by [3, Example 20.16(ii)], T is monotone.

(i): By Fact 2.30, $\operatorname{Fix} T = \cap_{i=1}^n U_i \neq \varnothing$. By Fact 2.31(ii) and Corollary 2.12, we know that $c(T) = \gamma^2 < 1$. Hence, the inequalities follow from Lemma 2.36.

(ii): Let A_T be the accelerated mapping defined in Definition 2.34 of T. For every $x \in \mathcal{H}$, since $x \in \operatorname{aff} \mathcal{S}(x)$, and since by Lemma 6.5, $Tx \in \operatorname{aff} \mathcal{S}(x)$, thus $A_T x \in \operatorname{aff} \{x, Tx\} \subseteq \operatorname{aff} \mathcal{S}(x)$. Since $\operatorname{Fix} T = \cap_{i=1}^m U_i$, using Fact 2.35, we obtain

$$(\forall x \in \mathcal{H}) \quad \|A_T x - \mathrm{P}_{\cap_{i=1}^m U_i} x\| \leq \eta \|x - \mathrm{P}_{\cap_{i=1}^m U_i} x\|.$$

As we proved in Theorem 6.6, the assumption $\Psi \subseteq \mathcal{S} \subseteq \Omega$ implies that $\cap_{i=1}^m U_i$ is a closed linear subspace of $\cap_{G \in \mathcal{S}} \operatorname{Fix} G$. Hence, apply Fact 5.1 with $F = A_T$ and $W = \cap_{i=1}^m U_i$ to obtain (ii). ■

Example 6.9 [4, page 3438] Let T be the product of two orthogonal projections onto two 1–dimensional (nonorthogonal) subspaces in the Euclidean plane. Then the accelerated algorithm, $(A_T^k(Tx))_{k \in \mathbb{N}}$, converges in two steps, that is, $A_T(Tx) = \mathrm{P}_{\operatorname{Fix} T} x$ for any starting point. However, for any choice of x which is not in the range of T, none of the terms of the sequence $(A_T^k(x))_{k \in \mathbb{N}}$ is equal to $\mathrm{P}_{\operatorname{Fix} T} x$, which means that $(A_T^k(x))_{k \in \mathbb{N}}$ does not converge to $\mathrm{P}_{\operatorname{Fix} T} x$ in a finite number of steps.

Inspired by Example 6.9, Fact 2.35 and Theorem 5.10(i), we show the following result, where we consider the special initial point $x_0 = \mathrm{P}_{U_m} \cdots \mathrm{P}_{U_1} x$.

Theorem 6.10 *Assume $m = 2n-1$ for some $n \in \mathbb{N} \smallsetminus \{0\}$, $U_1, \ldots, U_n$ are closed linear subspaces of $\mathcal{H}$ with $U_1^{\perp} + \cdots + U_n^{\perp}$ being closed, $(\forall i \in \{1, \ldots, n-1\})$ $U_{n+i} := U_{n-i}$. Recall that $\Psi \subseteq \mathcal{S} \subseteq \Omega$. Let c_1 and c_2 be defined as in (2.12) and (2.13). Set $T := \mathrm{P}_{U_m} \cdots \mathrm{P}_{U_1} = \mathrm{P}_{U_1} \cdots \mathrm{P}_{U_{n-1}} \mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1}$, $c(T) := \|T \mathrm{P}_{(\operatorname{Fix} T)^{\perp}}\|$, $\gamma := \|\mathrm{P}_{U_n} \mathrm{P}_{U_{n-1}} \cdots \mathrm{P}_{U_1} \mathrm{P}_{(\cap_{i=1}^m U_i)^{\perp}}\|$, and $\eta := \frac{c_2 - c_1}{2 - c_1 - c_2}$. Then $(\forall k \in \mathbb{N})$ $\eta^k c(T) \leq \gamma^{2(k+1)}$, and*

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC_{\mathcal{S}}^k(Tx) - \mathrm{P}_{\cap_{i=1}^n U_i} x\| \leq \eta^k c(T) \|x - \mathrm{P}_{\cap_{i=1}^n U_i} x\|.$$

Proof. By Theorem 6.8(i), we see that $0 \leq \eta \leq \frac{c(T)}{2 - c(T)} \leq c(T) = \gamma^2 < 1$. Hence, $(\forall k \in \mathbb{N})$ $\eta^k c(T) \leq \frac{c(T)^{k+1}}{(2 - c(T))^k} \leq c(T)^{k+1} = \gamma^{2(k+1)}$.

By the assumption, $\Psi \subseteq \mathcal{S} \subseteq \Omega$, $\operatorname{Fix} T = \cap_{i=1}^m U_i$ is a closed linear subspace of $\cap_{G \in \mathcal{S}} \operatorname{Fix} G$. Because by Lemma 6.5, $T \in \operatorname{aff} \mathcal{S}$, and $\mathrm{Id} \in \mathcal{S}$, it follows from Definition 2.34 that

$$(\forall x \in \mathcal{H}) \quad A_T x \in \operatorname{aff} \{x, Tx\} \subseteq \operatorname{aff} \mathcal{S}(x).$$

Using Fact 2.6, we get

$$T\,\mathrm{P}_{\cap_{i=1}^{n}U_i} = \mathrm{P}_{\cap_{i=1}^{n}U_i}\,T = \mathrm{P}_{\cap_{i=1}^{n}U_i}.$$

As we proved in Theorem 6.8, T is a linear, nonexpansive and self-adjoint operator on $\mathcal{H}$. Hence, by Fact 2.35, we obtain that

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \left\|A_T^k x - \mathrm{P}_{\cap_{i=1}^{n}U_i}\,x\right\| \leq \eta^k \|x - \mathrm{P}_{\cap_{i=1}^{n}U_i}\,x\|.$$

Therefore, the required result is obtained by applying Theorem 5.10(i) with $W = \cap_{i=1}^{n}U_i$, $F = A_T$ and $T = \mathrm{P}_{U_1}\cdots\mathrm{P}_{U_{n-1}}\mathrm{P}_{U_n}\mathrm{P}_{U_{n-1}}\cdots\mathrm{P}_{U_1}$. ■

Remark 6.11 Recall that in the whole section, $U_1,\ldots,U_m$ are closed linear subspaces of $\mathcal{H}$ and that the finite set $\mathcal{S}$ satisfies that $\Psi \subseteq \mathcal{S} \subseteq \Omega$. Set $T := \mathrm{P}_{U_m}\cdots\mathrm{P}_{U_1}$. By [4, Theorem 3.7], we know

$$(\forall x \in \mathcal{H}) \quad A_T(x) = \mathrm{P}_{\mathrm{aff}\,\{x,Tx\}}(\mathrm{P}_{\cap_{i=1}^{m}U_i}\,x).$$

By Lemma 6.5, we obtain that $(\forall x \in \mathcal{H})$, $\mathrm{aff}\,\{x,Tx\} \subseteq \mathrm{aff}\,(\mathcal{S}(x))$.

Moreover, $\Psi \subseteq \mathcal{S} \subseteq \Omega$ implies that $\cap_{i=1}^{m}U_i$ is a closed linear subspace of $\cap_{T\in\mathcal{S}}\mathrm{Fix}\,T$, so using Fact 4.9(ii), we get that

$$(\forall x \in \mathcal{H}) \quad CC_{\mathcal{S}}x = \mathrm{P}_{\mathrm{aff}\,(\mathcal{S}(x))}(\mathrm{P}_{\cap_{i=1}^{m}U_i}\,x).$$

Hence, in some sense the $CC_{\mathcal{S}}$ can be viewed as more aggressive than the A_T to converge to the point $\mathrm{P}_{\cap_{i=1}^{n}U_i}\,x$. Therefore, it is not surprised that the CRMs attain the linear convergence rate of the accelerated symmetric MAP in Theorems 6.8 and 6.10.

7 Conclusion and future work

In order to study the linear convergence of CIMs for finding the best approximation onto the intersection of fixed point sets of finitely many isometries, we first collected and proved some properties of isometries. Then, we showed the linear convergence of CIMs in finite-dimensional Hilbert spaces. Moreover, motivated by the accelerated symmetric MAP and the C-DRM, we presented two results on the linear convergence of CIMs in Hilbert spaces with first applying another operator to the initial point. In addition, we deduced sufficient conditions for the linear convergence of CRMs by using the linear convergence of (symmetric) MAP and accelerated symmetric MAP. In particular, we proved that the convergence rate of some CRMs is no worse than the sharp convergence rate of MAP and that some CRMs attain the known linear convergence rate of the accelerated symmetric MAP.

Let us comment on the relation between this paper and the related literature next.

We didn't consider properties of surjective or self-adjoint isometries before. In our previous paper [7], we proved the circumcenter mapping induced by finite set of isometries is proper, which deduces that the CIM is well-defined and is fundamental for our study on the linear convergence of CIMs in this paper. The linear convergence of CIMs in finite-dimensional Hilbert space are generalizations of the linear convergence of CRMs shown in [7, Propositions 5.10 and 5.15] and [9, Theorem 3.3]. The linear convergence of CIMs in Hilbert spaces shown in Theorem 5.10(ii) is a generalization of [8, Theorem 1] and [7, Proposition 5.18] from reflectors to isometries, while Theorem 5.10(i) is inspired by [4, page 3438]. Note that we proved that given a linear isometry T, T is a reflector associated with an affine subspace if and only if T is self-adjoint and that generally a linear isometry is not self-adjoint, our generalizations are indeed more flexible. The proof of linear convergence of CRMs in Hilbert spaces by using the linear convergence of (symmetric) MAP and accelerated symmetric MAP is new. In fact, compared with MAP and DRM, some instances of those CRMs showed outstanding performance numerically but not analytically in [7, Section 6]. Now Theorems 6.6 and 6.8 provide theoretical support for the numerical experiments presented in [7, Section 6].

Let $x \in \mathcal{H}$. Let $\mathcal{S}$ be a set of finitely many isometries. In Theorems 5.4 and 5.6, we constructed operators $T_{\mathcal{S}}$ (the operators named as A in Theorems 5.4 and 5.6) by using the elements of $\mathcal{S}$ and proved the linear convergence of the sequence $(T_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ for finding $\mathrm{P}_{\cap_{T\in\mathcal{S}} \operatorname{Fix} T}\, x$ when $\mathcal{H} = \mathbb{R}^n$. Then we took advantage of the linear convergence of the sequence $(T_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ to prove the linear convergence of the CIM induced by the $\mathcal{S}$ in $\mathbb{R}^n$. An interesting question is: *can we similarly construct a $T_{\mathcal{S}}$ such that the linear convergence of $(T_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ implies the linear convergence of the CIM induced by the $\mathcal{S}$ in infinite-dimensional Hilbert spaces?* In fact, in Section 6, we constructed some special sets $\mathcal{S}$ of reflectors such that the linear convergence of (symmetric) MAP or accelerated symmetric MAP implies the linear convergence of CRMs induced by those $\mathcal{S}$. If we can answer the question above, we might be able to obtain better results than those in Section 6.

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