

ITERATING BREGMAN RETRACTIONS*

HEINZ H. BAUSCHKE[†] AND PATRICK L. COMBETTES[‡]

Abstract. The notion of a Bregman retraction of a closed convex set in Euclidean space is introduced. Bregman retractions include backward Bregman projections and forward Bregman projections, as well as their convex combinations, and are thus quite flexible. The main result on iterating Bregman retractions unifies several convergence results on projection methods for solving convex feasibility problems. It is also used to construct new sequential and parallel algorithms.

Key words. backward Bregman projection, Bregman distance, Bregman function, Bregman projection, Bregman retraction, convex feasibility problem, forward Bregman projection, Legendre function, paracontraction, projection algorithm

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1. Standing assumptions, problem statement, and motivation. We assume throughout this paper that

$$(1.1) \quad X \text{ is a Euclidean space with scalar product } \langle \cdot, \cdot \rangle \text{ and induced norm } \|\cdot\|$$

and that

$$(1.2) \quad f: X \rightarrow]-\infty, +\infty] \text{ is a proper closed convex Legendre function such that } \text{dom } f^* \text{ is open,}$$

where f^* denotes the conjugate of f . Recall that a function is Legendre if it is both essentially smooth and essentially strictly convex (see, e.g., [31] for basic facts and notions from convex analysis). In addition, we assume that

$$(1.3) \quad \begin{aligned} &(C_i)_{i \in I} \text{ are finitely many closed convex sets in } X \\ &\text{such that } (\text{int dom } f) \cap \bigcap_{i \in I} C_i \neq \emptyset. \end{aligned}$$

Our aim is to study algorithms for solving the fundamental *convex feasibility problem* (see [4], [14], [17], [20], and [27] for further information and references)

$$(1.4) \quad \text{find } x \in \bigcap_{i \in I} C_i.$$

Assumption (1.2) guarantees that we capture a large class of functions (see Example 2.1 below) for which the corresponding *Bregman distance*

$$(1.5) \quad D_f: X \times X \rightarrow [0, +\infty] : (x, y) \mapsto \begin{cases} f(x) - f(y) - \langle x - y, \nabla f(y) \rangle & \text{if } y \in \text{int dom } f; \\ +\infty & \text{otherwise} \end{cases}$$

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[†]Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario N1G 2W1, Canada (hbausch@uoguelph.ca). The research of this author was supported by the Natural Sciences and Engineering Research Council of Canada.

[‡]Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie – Paris 6, 75005 Paris, France (plc@math.jussieu.fr).

enjoys useful properties (Proposition 2.2). This type of directed distance was first introduced by Bregman in [8]; see [17] for a historical account. Now fix a closed convex set C in X such that $C \cap \text{int dom } f \neq \emptyset$ and a point $y \in \text{int dom } f$. Then there is a unique point in $C \cap \text{int dom } f$, called the *backward Bregman projection* (or simply the *Bregman projection*) of y onto C and denoted by $\overleftarrow{P}_C y$, which satisfies (Fact 2.3)

$$(1.6) \quad (\forall c \in C) \quad D_f(\overleftarrow{P}_C y, y) \leq D_f(c, y).$$

Moreover, if f allows forward Bregman projections (Definition 2.4), then there is analogously a unique point in $C \cap \text{int dom } f$, called the *forward Bregman projection* of y onto C and denoted by $\overrightarrow{P}_C y$, which satisfies (Fact 2.6)

$$(1.7) \quad (\forall c \in C) \quad D_f(y, \overrightarrow{P}_C y) \leq D_f(y, c).$$

If $f = \frac{1}{2} \|\cdot\|^2$, then both $\overleftarrow{P}_C y$ and $\overrightarrow{P}_C y$ coincide with the *orthogonal projection* of y onto C ; however, the backward and forward Bregman projections differ generally, due to the asymmetry of D_f .

With backward and forward Bregman projections in place, we now describe three projection methods for solving (1.4). To this end, fix an index selector map $i: \mathbb{N} = \{0, 1, 2, \dots\} \rightarrow I$ that takes on each value in I infinitely often, and a starting point $y_0 \in \text{int dom } f$. The *method of backward Bregman projections* generates a sequence $(y_n)_{n \in \mathbb{N}}$ by

$$(1.8) \quad (\forall n \in \mathbb{N}) \quad y_{n+1} = \overleftarrow{P}_{C_{i(n+1)}} y_n.$$

Analogously, if f allows forward Bregman projections, then the update rule for the *method of forward Bregman projections* is

$$(1.9) \quad (\forall n \in \mathbb{N}) \quad y_{n+1} = \overrightarrow{P}_{C_{i(n+1)}} y_n.$$

Well-known *cyclic* versions arise if $I = \{1, \dots, N\}$ and $i(n) = n \bmod N$, where the range of the mod function is assumed to be $\{1, \dots, N\}$. The sequence $(y_n)_{n \in \mathbb{N}}$ generated by (1.8) (or by (1.9), if f allows forward Bregman projections) is known to solve (1.4) asymptotically: indeed, $(y_n)_{n \in \mathbb{N}}$ converges to some point in $\bigcap_{i \in I} C_i$; see [5] and [16] (or [7], respectively).

The third algorithm is due to Byrne and Censor [12], who adapted Csiszár and Tusnády's classical alternating minimization procedure [22] to a product space setting (see also section 5). Their algorithm assumes two constraints, $I = \{1, 2\}$, and a sequence $(y_n)_{n \in \mathbb{N}}$ is generated using *alternating backward-forward Bregman projections*:

$$(1.10) \quad (\forall n \in \mathbb{N}) \quad y_{n+1} = (\overleftarrow{P}_{C_2} \circ \overrightarrow{P}_{C_1}) y_n.$$

They show that, under appropriate conditions, $(y_n)_{n \in \mathbb{N}}$ converges to some point in $C_1 \cap C_2$; see [12, Theorem 1].

The striking resemblance in the update rules of the three preceding algorithms motivates this paper. Our objective is to provide a unified convergence analysis of these algorithms using the notion of a *Bregman retraction*, which encompasses both backward and forward Bregman projections. The main theorem not only recovers known convergence results but also provides a theoretical basis for the application of new sequential and parallel methods.

It is instructive to contrast our Bregman retraction-based framework with Censor and Reich's [16] framework, which is built on *paracontractions* (Definition 3.11).

While backward Bregman projections are both Bregman retractions and paracontractions, the two notions differ in general; actually, Examples 3.12 and 3.13 show that neither framework contains the other.

The key advantage of the Bregman retraction-based framework presented here is its applicability: the conditions on f are mild and easy to check. Moreover, simple constraint qualifications guarantee that Bregman retractions—in the form of backward Bregman projections (and forward Bregman projections, if f allows them)—always exist.

The paper is organized as follows. Background material on Bregman distances and associated projections is included in section 2. In section 3, Bregman retractions are introduced, analyzed, and illustrated by examples. The main result is proved in section 4, and applications are presented in section 5.

2. Preliminary results. Below is a selection of functions satisfying our assumptions (see [5] for additional examples).

EXAMPLE 2.1 (see [5]). *Suppose that $X = \mathbb{R}^J$ and, for every $x \in X$, write $x = (\xi_j)_{j=1}^J$. Then the following functions satisfy (1.2) (here and elsewhere, we use the convention $0 \cdot \ln(0) = 0$):*

- (i) $f: x \mapsto \frac{1}{2}\|x\|^2 = \frac{1}{2} \sum_{j=1}^J |\xi_j|^2$, with $\text{dom } f = \mathbb{R}^J$ (energy);
- (ii) $f: x \mapsto \sum_{j=1}^J \xi_j \ln(\xi_j) - \xi_j$, with $\text{dom } f = [0, +\infty[^J$ (negative entropy);
- (iii) $f: x \mapsto \sum_{j=1}^J \xi_j \ln(\xi_j) + (1 - \xi_j) \ln(1 - \xi_j)$, with $\text{dom } f = [0, 1]^J$ (Fermi–Dirac entropy);
- (iv) $f: x \mapsto -\sum_{j=1}^J \ln(\xi_j)$, with $\text{dom } f =]0, +\infty[^J$ (Burg entropy);
- (v) $f: x \mapsto -\sum_{j=1}^J \sqrt{\xi_j}$, with $\text{dom } f = [0, +\infty[^J$.

The assumptions imposed on f in (1.2) guarantee the following very useful properties of D_f .

PROPOSITION 2.2. *Let D_f be defined as in (1.5). Then we have the following:*

- (i) D_f is continuous on $(\text{int dom } f)^2$.
- (ii) If $x \in \text{dom } f$ and $y \in \text{int dom } f$, then $D_f(x, y) \geq 0$, and $D_f(x, y) = 0 \Leftrightarrow x = y$.
- (iii) If $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are two sequences in $\text{int dom } f$ converging to $x \in \text{int dom } f$ and $y \in \text{int dom } f$, respectively, then $D_f(x_n, y_n) \rightarrow 0 \Leftrightarrow x = y$.
- (iv) If $x \in \text{int dom } f$ and $(y_n)_{n \in \mathbb{N}}$ is a sequence in $\text{int dom } f$ such that the sequence $(D_f(x, y_n))_{n \in \mathbb{N}}$ is bounded, then $(y_n)_{n \in \mathbb{N}}$ is bounded and all its cluster points belong to $\text{int dom } f$.
- (v) If $x \in \text{int dom } f$ and $(y_n)_{n \in \mathbb{N}}$ is a sequence in $\text{int dom } f$ such that $D_f(x, y_n) \rightarrow 0$, then $y_n \rightarrow x$.

Proof. (i) This follows from the definition of D_f and the continuity of f (respectively, ∇f) on $\text{int dom } f$; see [31, Theorem 10.1] (respectively, [31, Theorem 25.5]). (ii) [5, Theorem 3.7.(iv)]. (iii) This is a consequence of (i) and (ii). (iv) [5, Theorem 3.7.(vi) and Theorem 3.8.(ii)]. (v) (See also [7, Fact 2.18].) By (iv), $(y_n)_{n \in \mathbb{N}}$ is bounded and has all its cluster points in $\text{int dom } f$. Pick an arbitrary cluster point of $(y_n)_{n \in \mathbb{N}}$, say $y_{k_n} \rightarrow y \in \text{int dom } f$. Then $D_f(x, y_{k_n}) \rightarrow 0$ and thus $x = y$ by (iii). \square

We now turn to backward and forward Bregman projections.

FACT 2.3 (backward Bregman projection). *Suppose that C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Then, for every $y \in \text{int dom } f$, there exists a unique point $\overleftarrow{P}_C y \in C \cap \text{dom } f$ such that $D_f(\overleftarrow{P}_C y, y) \leq D_f(c, y)$ for all $c \in C$. The point $\overleftarrow{P}_C y$ is called the backward Bregman projection (or simply the Bregman*

projection) of y onto C , and it is characterized by

$$(2.1) \quad \overleftarrow{P}_C \in C \cap \text{int dom } f \quad \text{and} \quad (\forall c \in C) \quad \langle c - \overleftarrow{P}_C y, \nabla f(y) - \nabla f(\overleftarrow{P}_C y) \rangle \leq 0;$$

equivalently, it is characterized by

$$(2.2) \quad \overleftarrow{P}_C \in C \cap \text{int dom } f \quad \text{and} \quad (\forall c \in C) \quad D_f(c, y) \geq D_f(c, \overleftarrow{P}_C y) + D_f(\overleftarrow{P}_C y, y).$$

Finally, the operator \overleftarrow{P}_C is continuous on $\text{int dom } f$.

Proof. Under the present assumptions on f , the claims follow from [5, Theorem 3.14 and Proposition 3.16], except for the continuity of \overleftarrow{P}_C , which we derive now. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\text{int dom } f$ converging to $\bar{x} \in \text{int dom } f$. Set $(c_n)_{n \in \mathbb{N}} = (\overleftarrow{P}_C x_n)_{n \in \mathbb{N}}$ and $\bar{c} = \overleftarrow{P}_C \bar{x}$. We must show that $(c_n)_{n \in \mathbb{N}}$ converges to \bar{c} . Using Proposition 2.2(i) and (2.2), we have

$$(2.3) \quad D_f(\bar{c}, \bar{x}) \leftarrow D_f(\bar{c}, x_n) \geq D_f(\bar{c}, c_n) + D_f(c_n, x_n) \geq D_f(\bar{c}, c_n).$$

Hence $(D_f(\bar{c}, c_n))_{n \in \mathbb{N}}$ is bounded. By Proposition 2.2(iv), $(c_n)_{n \in \mathbb{N}}$ is bounded and all its cluster points belong to $C \cap \text{int dom } f$. Let \hat{c} be such a cluster point, say $c_{k_n} \rightarrow \hat{c} \in \text{int dom } f$. Using the definition of \bar{c} , Proposition 2.2(i), and (2.2), we deduce $D_f(\hat{c}, \bar{x}) \geq D_f(\bar{c}, \bar{x}) \leftarrow D_f(\bar{c}, x_{k_n}) \geq D_f(\bar{c}, c_{k_n}) + D_f(c_{k_n}, x_{k_n}) \rightarrow D_f(\bar{c}, \hat{c}) + D_f(\hat{c}, \bar{x}) \geq D_f(\hat{c}, \bar{x})$; thus $D_f(\bar{c}, \hat{c}) = 0$, and hence, by Proposition 2.2(ii), $\bar{c} = \hat{c}$. \square

DEFINITION 2.4. *The function f allows forward Bregman projections if it satisfies the following additional properties:*

- (i) $\nabla^2 f$ exists and is continuous on $\text{int dom } f$;
- (ii) D_f is convex on $(\text{int dom } f)^2$;
- (iii) for every $x \in \text{int dom } f$, $D_f(x, \cdot)$ is strictly convex on $\text{int dom } f$.

REMARK 2.5. *The function f allows forward Bregman projections if and only if it satisfies the standing assumptions of [7], which allows us to apply the results of [7]. This equivalence follows from [7, Remark 2.1] and*

$$(2.4) \quad D_f \text{ is convex on } (\text{int dom } f)^2 \iff D_f \text{ is convex on } X^2.$$

We now verify (2.4). The implication “ \Leftarrow ” is clear. To establish “ \Rightarrow ”, let us fix $(y_1, y_2) \in (\text{int dom } f)^2$, $(x_1, x_2) \in (\text{dom } f)^2$, and $(\lambda_1, \lambda_2) \in]0, 1[{}^2$ such that $\lambda_1 + \lambda_2 = 1$. For $\varepsilon \in]0, 1[$ and $i \in \{1, 2\}$, set $x_{i,\varepsilon} = (1-\varepsilon)x_i + \varepsilon y_i \in \text{int dom } f$. Then $D_f(\lambda_1 x_{1,\varepsilon} + \lambda_2 x_{2,\varepsilon}, \lambda_1 y_1 + \lambda_2 y_2) \leq \lambda_1 D_f(x_{1,\varepsilon}, y_1) + \lambda_2 D_f(x_{2,\varepsilon}, y_2)$. Now take $y \in \text{int dom } f$. Since f is closed and convex, so is $D_f(\cdot, y)$. Hence, as $\varepsilon \downarrow 0^+$, the line segment continuity property of $D_f(\cdot, y)$ [31, Corollary 7.5.1] results in $D_f(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \leq \lambda_1 D_f(x_1, y_1) + \lambda_2 D_f(x_2, y_2)$. Thus D_f is convex on $\text{dom } f \times \text{int dom } f = \text{dom } D_f$ and, thereby, on X^2 .

FACT 2.6 (forward Bregman projection). *Suppose that f allows forward Bregman projections and C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Then, for every $y \in \text{int dom } f$, there exists a unique point $\overrightarrow{P}_C y \in C \cap \text{dom } f$ such that $D_f(y, \overrightarrow{P}_C y) \leq D_f(y, c)$ for all $c \in C$. The point $\overrightarrow{P}_C y$ is called the forward Bregman projection of y onto C and it is characterized by*

$$(2.5) \quad \overrightarrow{P}_C y \in C \cap \text{int dom } f \quad \text{and} \quad (\forall c \in C) \quad \langle c - \overrightarrow{P}_C y, \nabla^2 f(\overrightarrow{P}_C y)(y - \overrightarrow{P}_C y) \rangle \leq 0;$$

equivalently, it is characterized by

$$(2.6) \quad \overrightarrow{P}_C y \in C \cap \text{int dom } f \quad \text{and} \quad (\forall c \in C) \quad D_f(c, y) \geq D_f(c, \overrightarrow{P}_C y) + D_{D_f}((c, c), (y, \overrightarrow{P}_C y)).$$

Finally, the operator \overrightarrow{P}_C is continuous on $\text{int dom } f$.

Proof. This follows from [7, Lemma 2.9, Lemma 3.5, Lemma 3.6, and Corollary 3.7]. \square

The key requirement in Definition 2.4 is the convexity of D_f , which is studied separately in [6]. Not every Legendre function allows forward Bregman projections, but the most important ones from Example 2.1 do.

EXAMPLE 2.7 (functions allowing forward Bregman projections; see [7, Example 2.16]). *Let $X = \mathbb{R}^J$. Then the energy, the negative entropy, and the Fermi–Dirac entropy allow forward Bregman projections.*

The following example shows that backward and forward Bregman projections are different notions.

EXAMPLE 2.8 (entropic averaging in \mathbb{R}^2). *Let $f: \mathbb{R}^2 \rightarrow]-\infty, +\infty]: (\xi_1, \xi_2) \mapsto \sum_{i=1}^2 \xi_i \ln(\xi_i) - \xi_i$ be the negative entropy on \mathbb{R}^2 , and let $\Delta = \{(\xi_1, \xi_2) \in \mathbb{R}^2: \xi_1 = \xi_2\}$. Then $\text{dom } f = [0, +\infty[^2$ and clearly $\Delta \cap \text{int dom } f \neq \emptyset$. Using (2.1) and (2.5), it is straightforward to verify that, for every $(\xi_1, \xi_2) \in \text{int dom } f =]0, +\infty[^2$,*

$$(2.7) \quad \overleftarrow{P}_\Delta(\xi_1, \xi_2) = (\sqrt{\xi_1 \xi_2}, \sqrt{\xi_1 \xi_2}) \quad \text{and} \quad \overrightarrow{P}_\Delta(\xi_1, \xi_2) = \left(\frac{1}{2}(\xi_1 + \xi_2), \frac{1}{2}(\xi_1 + \xi_2)\right).$$

These formulae can also be deduced from Example 3.16 below.

We close this section with a characterization of convergence for Bregman monotone sequences. Note that when $f = \frac{1}{2}\|\cdot\|^2$, Bregman monotonicity reverts to the standard notion of Fejér monotonicity, which is discussed in detail in [4] and [21].

PROPOSITION 2.9 (Bregman monotonicity). *Suppose that C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Suppose further that $(y_n)_{n \in \mathbb{N}}$ is a sequence which is Bregman monotone with respect to $C \cap \text{int dom } f$; i.e., it lies in $\text{int dom } f$ and*

$$(2.8) \quad (\forall c \in C \cap \text{int dom } f)(\forall n \in \mathbb{N}) \quad D_f(c, y_{n+1}) \leq D_f(c, y_n).$$

Then $(y_n)_{n \in \mathbb{N}}$ converges to some point in $C \cap \text{int dom } f \Leftrightarrow$ all cluster points of $(y_n)_{n \in \mathbb{N}}$ are in C .

Proof. The implication “ \Rightarrow ” is clear. “ \Leftarrow ”: pick $c \in C \cap \text{int dom } f$. Then the sequence $(D_f(c, y_n))_{n \in \mathbb{N}}$ is decreasing and nonnegative, and hence bounded. By Proposition 2.2(iv), $(y_n)_{n \in \mathbb{N}}$ is bounded and all its cluster points lie in $\text{int dom } f$. Let $\{c, \hat{c}\} \subset C \cap \text{int dom } f$ be two cluster points of $(y_n)_{n \in \mathbb{N}}$, say $y_{k_n} \rightarrow c$ and $y_{l_n} \rightarrow \hat{c}$. By Proposition 2.2(iii), $D_f(c, y_{k_n}) \rightarrow 0$. Since $(y_n)_{n \in \mathbb{N}}$ is Bregman monotone, we have $D_f(c, y_n) \rightarrow 0$ and, in particular, $D_f(c, y_{l_n}) \rightarrow 0$. Using Proposition 2.2(v), we conclude that $c = \hat{c}$. \square

3. Bregman retractions.

3.1. Properties and examples.

DEFINITION 3.1 (Bregman retraction). *Suppose that C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$ and μ is a function from $\text{dom } \mu = (C \cap \text{int dom } f) \times \text{int dom } f$ to $[0, +\infty[$. Then $R: \text{dom } R = \text{int dom } f \rightarrow C \cap \text{int dom } f$ is a Bregman retraction of C with modulus μ if the following two properties hold for every $c \in C \cap \text{int dom } f$ and every $x \in \text{int dom } f$:*

- (i) $D_f(c, x) \geq D_f(c, Rx) + \mu(c, x)$.
- (ii) *If $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\text{int dom } f$ and y is a point in $\text{int dom } f$ such that $x_n \rightarrow x$, $Rx_n \rightarrow y$, and $\mu(c, x_n) \rightarrow 0$, then $x = y$.*

PROPOSITION 3.2 (basic properties of Bregman retractions). *Suppose that C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Suppose further that R is*

a Bregman retraction of C with modulus μ . Then the following holds true for every $c \in C \cap \text{int dom } f$ and every $x \in \text{int dom } f$:

(i) Suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\text{int dom } f$ and y is a point in $\text{int dom } f$ such that $x_n \rightarrow x$ and $Rx_n \rightarrow y$. Then $\mu(c, x_n) \rightarrow 0 \Leftrightarrow x = y$.

(ii) $\mu(c, x) = 0 \Leftrightarrow x = Rx \Leftrightarrow x \in C$.

Proof. (i) The implication “ \Rightarrow ” is clear by Definition 3.1(ii). If $x = y$, then (using Proposition 2.2(i) and Definition 3.1(i)) $0 = D_f(c, x) - D_f(c, x) \leftarrow D_f(c, x_n) - D_f(c, Rx_n) \geq \mu(c, x_n) \geq 0$. Hence $\mu(c, x_n) \rightarrow 0$ and “ \Leftarrow ” is verified. (ii) The first equivalence is a special case of (i), while the implication $x = Rx \Rightarrow x \in C$ is clear. Now assume $x \in C$. Then Definition 3.1(i) with $c = x$ yields $0 = D_f(x, x) \geq D_f(x, Rx) + \mu(x, x) \geq D_f(x, Rx) \geq 0$. Hence $D_f(x, Rx) = 0$ and thus $x = Rx$ by Proposition 2.2(ii). \square

Every nonempty closed convex set in X possesses a Bregman retraction with respect to the energy.

EXAMPLE 3.3 (orthogonal projection). Suppose that $f = \frac{1}{2} \|\cdot\|^2$ and C is a nonempty closed convex set in X . Then its orthogonal projection P_C is a Bregman retraction with modulus $\mu: (c, x) \mapsto \frac{1}{2} \|x - P_C x\|^2$.

Proof. This will turn out to be a special case of Example 3.6 or Example 3.7. \square

However, the next example shows that there exist Bregman retractions that are not projections.

EXAMPLE 3.4. Let $f = \frac{1}{2} \|\cdot\|^2$ and $C = \{x \in X: \|x\| \leq 1\}$. Fix $\varepsilon \in]0, 1[$, define $\lambda: X \rightarrow [0, +\infty[: x \mapsto 1 + \min\{\varepsilon, \|x - P_C x\|\}$, and let $R: X \rightarrow C: x \mapsto (1 - \lambda(x))x + \lambda(x)P_C x$, where P_C is the orthogonal projection onto C . Then R is a Bregman retraction of C with modulus $\mu: (c, x) \mapsto \frac{1}{2}(2 - \lambda(x))\lambda(x)\|x - P_C x\|^2$.

Proof. Fix $x \in X$ and $c \in C$. It follows from standard properties of orthogonal projections (see, e.g., [4, Corollary 2.5]) that

$$(3.1) \quad \|x - c\|^2 - \|x - Rx\|^2 \geq (2 - \lambda(x))\lambda(x)\|x - P_C x\|^2,$$

which corresponds to Definition 3.1(i). Now assume $(x_n)_{n \in \mathbb{N}}$ converges to x . Since P_C , and hence λ , is continuous, we have $P_C x_n \rightarrow P_C x$ and $Rx_n \rightarrow Rx$. Assume further that $\mu(c, x_n) \rightarrow 0$. Then $x_n - P_C x_n \rightarrow 0$ and thus $Rx_n = x_n + \lambda(x_n)(P_C x_n - x_n) \rightarrow x$. Hence $Rx = x$ and therefore R is a Bregman retraction. \square

REMARK 3.5. In passing, note that if C is a closed convex set in X such that $(\text{int } C) \cap \text{int dom } f \neq \emptyset$ and $y \in \text{int dom } f \setminus C$, then both points $\overleftarrow{P}_C y$ and $\overrightarrow{P}_C y$ belong to $(\text{bdry } C) \cap \text{int dom } f$. Now let R and C be as in Example 3.4. Since R maps points outside C to the interior of C , there is no function f such that R is the backward or forward Bregman projection onto C with respect to D_f .

The following two examples contain Example 3.3 if we let $f = \frac{1}{2} \|\cdot\|^2$.

EXAMPLE 3.6 (backward Bregman projection). Suppose that C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Then the backward Bregman projection \overleftarrow{P}_C is a continuous Bregman retraction with modulus $\mu: (c, x) \mapsto D_f(\overleftarrow{P}_C x, x)$.

Proof. This follows from Fact 2.3 and Proposition 2.2(iii). \square

EXAMPLE 3.7 (forward Bregman projection). Suppose that f allows forward Bregman projections and C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Then the forward Bregman projection \overrightarrow{P}_C is a continuous Bregman retraction with

modulus

$$(3.2) \quad \begin{aligned} \mu: (c, x) &\mapsto D_{D_f}((c, c), (x, \vec{P}_C x)) \\ &= D_f(c, x) - D_f(c, \vec{P}_C x) + \langle c - \vec{P}_C x, \nabla^2 f(\vec{P}_C x)(x - \vec{P}_C x) \rangle. \end{aligned}$$

Proof. See [7, Lemma 2.9] for the nonnegativity of D_{D_f} and for the expression of D_{D_f} . Fact 2.6 states that \vec{P}_C is continuous and (2.6) verifies Definition 3.1(i). It remains to establish condition (ii) of Definition 3.1. So pick $c \in C \cap \text{int dom } f$ and $(x_n)_{n \in \mathbb{N}}$ in $\text{int dom } f$ such that $x_n \rightarrow x \in \text{int dom } f$, $\vec{P}_C x_n \rightarrow y \in \text{int dom } f$, and $\mu(c, x_n) \rightarrow 0$. By [7, Lemma 2.9], D_{D_f} is continuous on $(\text{int dom } f)^4$ and therefore $\mu(c, x_n) \rightarrow D_{D_f}((c, c), (x, y))$. Altogether, $D_{D_f}((c, c), (x, y)) = 0$ and [7, Lemma 2.10] implies $x = y$. \square

The following example is motivated by [30, section 4.7].

EXAMPLE 3.8. Let $X = \mathbb{R}^J$, f be the negative entropy, and let

$$(3.3) \quad C = \{x \in X : x \geq 0 \text{ and } \langle x, \mathbf{1} \rangle \leq 1\}, \quad \text{where } \mathbf{1} = (1, \dots, 1) \in X.$$

Let

$$(3.4) \quad R: \text{int dom } f \rightarrow C \cap \text{int dom } f: x \mapsto \begin{cases} x & \text{if } x \in C; \\ x/\langle x, \mathbf{1} \rangle & \text{otherwise.} \end{cases}$$

Then $R = \overleftarrow{P}_C = \vec{P}_C$. Consequently, R is a continuous Bregman retraction of C .

Proof. Fix $c \in C$ and $x \in \text{int dom } f \setminus C$. Then, abusing notation slightly,

$$\begin{aligned} \langle c - Rx, \nabla f(x) - \nabla f(Rx) \rangle &= \langle c - x/\langle x, \mathbf{1} \rangle, \ln(x) - \ln(x/\langle x, \mathbf{1} \rangle) \rangle \\ &= \langle c - x/\langle x, \mathbf{1} \rangle, \ln(\langle x, \mathbf{1} \rangle) \mathbf{1} \rangle \\ &= \ln(\langle x, \mathbf{1} \rangle)(\langle c, \mathbf{1} \rangle - 1) \\ &\leq 0. \end{aligned}$$

By (2.1), we see that $Rx = \overleftarrow{P}_C x$. Similarly,

$$\begin{aligned} \langle c - Rx, \nabla^2 f(Rx)(x - Rx) \rangle &= \langle c - x/\langle x, \mathbf{1} \rangle, (\langle x, \mathbf{1} \rangle - 1) \cdot \mathbf{1} \rangle \\ &= (\langle x, \mathbf{1} \rangle - 1)(\langle c, \mathbf{1} \rangle - 1) \\ &\leq 0. \end{aligned}$$

Thus, using (2.5), $Rx = \vec{P}_C x$. \square

REMARK 3.9. In [30, Section 4.7], it is observed that the orthogonal projection of an arbitrary point in \mathbb{R}^J onto C is hard to compute explicitly, and hence the use of the following extension \tilde{R} of R is suggested. Denoting the nonnegative part of a vector $x \in \mathbb{R}^J$ by x^+ (i.e., x^+ is the orthogonal projection of x onto the nonnegative orthant), the extension \tilde{R} is defined by

$$(3.5) \quad \tilde{R}: X \rightarrow C: x \mapsto \begin{cases} x^+/\langle x^+, \mathbf{1} \rangle & \text{if } \langle x^+, \mathbf{1} \rangle > 1; \\ x^+ & \text{otherwise.} \end{cases}$$

It is important to note that for certain points $x \in \text{dom } f \setminus C$ and $c \in C$, the inequality $\|\tilde{R}x - c\| = \|Rx - c\| \leq \|x - c\|$ does not hold. Indeed, take $X = \mathbb{R}^2$, let $c = (1, 0)$, consider the ray emanating from 0 that makes an angle of $\pi/6$ with $[0, +\infty[\cdot c$, and let x be the orthogonal projection of c onto this ray. Then $\|\tilde{R}x - c\| = \|Rx - c\| > \|x - c\|$ (and this example can be lifted to \mathbb{R}^J , where $J \geq 3$). Therefore, Example 3.8 shows that an operator which is not a Bregman retraction with respect to the energy may turn out to be a Bregman retraction with respect to some other function.

3.2. Comparison with Censor and Reich’s paracontractions. Let us first recall the concept of a *Bregman function*, as defined in [15] or [17] (see also [5], [9], [25], and [32] for more concise definitions).

DEFINITION 3.10. *Let S be a nonempty open convex subset of \mathbb{R}^J , let $g: \bar{S} \rightarrow \mathbb{R}$ be a continuous and strictly convex function, and let D_g be the corresponding Bregman distance. Then g is a Bregman function with zone S if the following conditions hold:*

- (i) g is continuously differentiable on S ;
- (ii) for every $x \in \bar{S}$ the sets $(\{y \in S: D_g(x, y) \leq \eta\})_{\eta \in \mathbb{R}}$ are bounded;
- (iii) for every $y \in S$ the sets $(\{x \in \bar{S}: D_g(x, y) \leq \eta\})_{\eta \in \mathbb{R}}$ are bounded;
- (iv) if $(y_n)_{n \in \mathbb{N}}$ lies in S and $y_n \rightarrow y$, then $D_g(y, y_n) \rightarrow 0$;
- (v) if $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence in \bar{S} , $(y_n)_{n \in \mathbb{N}}$ lies in S , $y_n \rightarrow y$, and $D_g(x_n, y_n) \rightarrow 0$, then $x_n \rightarrow y$.

The following notion is due to Censor and Reich.

DEFINITION 3.11 (see [16, Definition 3.2]). *Suppose that g is a Bregman function with zone $S \subset \mathbb{R}^J$, and let $T: S \rightarrow \mathbb{R}^J$ be an operator with domain S . A point $\bar{x} \in \mathbb{R}^J$ is called an asymptotic fixed point of T if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in S such that $x_n \rightarrow \bar{x}$ and $Tx_n \rightarrow \bar{x}$. The set of asymptotic fixed points is denoted by $\widehat{F}(T)$. The operator T is a paracontraction if $\widehat{F}(T) \neq \emptyset$ and the following two conditions hold:*

- (i) $(\forall c \in \widehat{F}(T))(\forall x \in S) \quad D_g(c, Tx) \leq D_g(c, x)$.
- (ii) If $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence in S and $c \in \widehat{F}(T)$ satisfies $D_g(c, x_n) - D_g(c, Tx_n) \rightarrow 0$, then $D_g(Tx_n, x_n) \rightarrow 0$.

EXAMPLE 3.12 (Bregman retraction $\not\approx$ paracontraction). *Let f and Δ be as in Example 2.8, and set $T = \overrightarrow{P}_\Delta$. Then T is a continuous Bregman retraction but not a paracontraction.*

Proof. The first claim follows from Examples 2.7 and 3.7. We now show that T is not a paracontraction. First, f is a Bregman function with zone $S = \text{int dom } f =]0, +\infty[^2$. In addition,

$$(3.6) \quad D_f(x, y) = \xi_1 \ln(\xi_1/\eta_1) - \xi_1 + \eta_1 + \xi_2 \ln(\xi_2/\eta_2) - \xi_2 + \eta_2$$

for $x = (\xi_1, \xi_2) \in \text{dom } f =]0, +\infty[^2$ and $y = (\eta_1, \eta_2) \in \text{int dom } f =]0, +\infty[^2$. The set of asymptotic fixed points of T is seen to be

$$(3.7) \quad \widehat{F}(T) = \Delta \cap \text{dom } f = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 = \xi_2 \geq 0\} \neq \emptyset.$$

Fix $c = (0, 0) \in \widehat{F}(T)$ and pick an arbitrary $x = (\xi_1, \xi_2) \in \text{int dom } f \setminus \Delta$. By Example 2.8, $Tx = \overrightarrow{P}_\Delta x = \frac{1}{2}(\xi_1 + \xi_2, \xi_1 + \xi_2)$. Hence,

$$(3.8) \quad \begin{aligned} D_f(c, x) - D_f(c, Tx) \\ = D_f(0, x) - D_f(0, Tx) = (\xi_1 + \xi_2) - \left(\frac{1}{2}(\xi_1 + \xi_2) + \frac{1}{2}(\xi_1 + \xi_2)\right) = 0. \end{aligned}$$

However, since $x \notin \Delta$, we have $Tx = \overrightarrow{P}_\Delta x \neq x$ and so, by Proposition 2.2(ii), $D_f(Tx, x) > 0$. Therefore Definition 3.11(ii) fails, and it follows that T is not a paracontraction. \square

EXAMPLE 3.13 (paracontraction $\not\approx$ Bregman retraction). *Let $X = \mathbb{R}$ and $f = \frac{1}{2}|\cdot|^2$. Then f is a Bregman function with zone $S = X$ and $T: X \rightarrow X: x \mapsto \frac{1}{2}x$ is a paracontraction with $\widehat{F}(T) = \{0\}$. Now suppose that T is a Bregman retraction. Then, by Proposition 3.2(ii), the underlying set must be $C = \{0\}$. However, by Definition 3.1,*

this is absurd since the range of T is not a subset of C . Therefore T is not a Bregman retraction.

3.3. New Bregman retractions via averages and products.

PROPOSITION 3.14 (averaged Bregman retractions). *Suppose that f allows forward Bregman projections and C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Suppose further that R_1 and R_2 are two continuous Bregman retractions of C with moduli μ_1 and μ_2 . Fix $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = 1$, and set $R = \lambda_1 R_1 + \lambda_2 R_2$. Then R is a Bregman retraction of C with modulus $\mu = \lambda_1 \mu_1 + \lambda_2 \mu_2$.*

Proof. It is clear that the range of R is contained in $C \cap \text{int dom } f$ and that $\text{dom } R = \text{int dom } f$. Fix $c \in C \cap \text{int dom } f$ and $x \in \text{int dom } f$. Since both R_1 and R_2 are Bregman retractions of C and since $D_f(c, \cdot)$ is convex on $\text{int dom } f$, we have

$$\begin{aligned} D_f(c, x) &= \lambda_1 D_f(c, x) + \lambda_2 D_f(c, x) \\ &\geq \lambda_1 (D_f(c, R_1 x) + \mu_1(c, x)) + \lambda_2 (D_f(c, R_2 x) + \mu_2(c, x)) \\ &= (\lambda_1 D_f(c, R_1 x) + \lambda_2 D_f(c, R_2 x)) + \mu(c, x) \\ &\geq D_f(c, Rx) + \mu(c, x). \end{aligned}$$

Hence condition (i) of Definition 3.1 holds. Next, assume $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\text{int dom } f$ converging to x such that $Rx_n \rightarrow y \in \text{int dom } f$ and $\mu(c, x_n) \rightarrow 0$. Then $\mu_1(c, x_n) \rightarrow 0$ and $\mu_2(c, x_n) \rightarrow 0$. On the other hand, since R_1 and R_2 are continuous on $\text{int dom } f$, $(R_1 x_n, R_2 x_n) \rightarrow (R_1 x, R_2 x)$, and hence $Rx_n \rightarrow Rx$. Thus $y = Rx$. Using condition (ii) of Definition 3.1 on each R_i , we also have $x = R_1 x = R_2 x$ and thus $x = Rx$. Altogether, $x = y$ and condition (ii) of Definition 3.1 is verified as well. \square

EXAMPLE 3.15 (averaged backward-forward Bregman projections). *Suppose that f allows forward Bregman projections and C is a closed convex set in X such that $C \cap \text{int dom } f \neq \emptyset$. Denote the Bregman retraction and its modulus from Example 3.6 (respectively, Example 3.7) by R_1 and μ_1 (respectively, R_2 and μ_2). Fix $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = 1$, and set $R = \lambda_1 R_1 + \lambda_2 R_2$. Then R is a Bregman retraction of C with modulus $\mu = \lambda_1 \mu_1 + \lambda_2 \mu_2$.*

We conclude this section with a product space construction first introduced by Pierra in [28] (see also [29]). The extension to a Bregman distance setting is due to Censor and Elfving [13]. The product space technique will be extremely useful for analyzing the parallel projection methods presented in section 5.

EXAMPLE 3.16 (product space setup). *For convenience, let $I = \{1, \dots, N\}$ in (1.3). Denote the standard Euclidean product space X^N by \mathbf{X} , and write $\mathbf{x} = (x_i)_{i \in I}$ for $\mathbf{x} \in \mathbf{X}$. Let*

$$(3.9) \quad \Delta = \{(x, \dots, x) \in \mathbf{X} : x \in X\} \quad \text{and} \quad \mathbf{C} = C_1 \times \dots \times C_N.$$

Fix $(\lambda_i)_{i \in I}$ in $]0, 1]$ such that $\sum_{i \in I} \lambda_i = 1$, and set

$$(3.10) \quad \mathbf{f}: \mathbf{X} \rightarrow]-\infty, +\infty]: \mathbf{x} \mapsto \sum_{i \in I} \lambda_i f(x_i).$$

Then \mathbf{f} is Legendre, $\text{dom } \mathbf{f}^*$ is open, and $\Delta \cap \mathbf{C} \cap \text{int dom } \mathbf{f} \neq \emptyset$. In addition, if $\mathbf{x} \in \text{dom } \mathbf{f}$ and $\mathbf{y} \in \text{int dom } \mathbf{f}$, then $D_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) = \sum_{i \in I} \lambda_i D_f(x_i, y_i)$. Moreover, we have the following:

- (i) The operators \overleftarrow{P}_Δ and \overleftarrow{P}_C are continuous Bregman retractions of Δ and C , respectively, and

$$(3.11) \quad \begin{aligned} \overleftarrow{P}_\Delta \mathbf{y} &= (z, \dots, z), \quad \text{where } z = \nabla f^* \left(\sum_{i \in I} \lambda_i \nabla f(y_i) \right), \\ \overleftarrow{P}_C \mathbf{y} &= (\overleftarrow{P}_{C_i} y_i)_{i \in I}. \end{aligned}$$

- (ii) Suppose that f allows forward Bregman projections. Then so does \mathbf{f} . The operators $\overrightarrow{P}_\Delta$ and \overrightarrow{P}_C are continuous Bregman retractions of Δ and C , respectively, and

$$(3.12) \quad \begin{aligned} \overrightarrow{P}_\Delta \mathbf{y} &= (z, \dots, z), \quad \text{where } z = \sum_{i \in I} \lambda_i y_i, \\ \overrightarrow{P}_C \mathbf{y} &= (\overrightarrow{P}_{C_i} y_i)_{i \in I}. \end{aligned}$$

Proof. The fact that the operators \overleftarrow{P}_Δ , \overleftarrow{P}_C (and $\overrightarrow{P}_\Delta$, \overrightarrow{P}_C , provided they exist) are continuous Bregman retractions follows from Example 3.6 (and Example 3.7, respectively). (i) See [5, Corollary 7.2] or [13, Lemmata 4.1 and 4.2]. (ii) Using Definition 2.4, it is straightforward to check that \mathbf{f} allows forward Bregman projections. Next, let $z = \sum_{i \in I} \lambda_i y_i$ and $\mathbf{z} = (z, \dots, z) \in \mathbf{X}$. Then $\mathbf{z} \in \Delta$. Observe that $\nabla^2 \mathbf{f}(\mathbf{z}) \mathbf{y} = (\lambda_i \nabla^2 f(z) y_i)_{i \in I}$ and $\nabla^2 \mathbf{f}(\mathbf{z}) \mathbf{z} = (\lambda_i \nabla^2 f(z) z)_{i \in I}$. Hence $\nabla^2 \mathbf{f}(\mathbf{z})(\mathbf{y} - \mathbf{z}) \in \Delta^\perp = \{\mathbf{x} \in \mathbf{X} : \sum_{i \in I} x_i = 0\}$, because $\sum_{i \in I} \lambda_i \nabla^2 f(z) y_i = \nabla^2 f(z) (\sum_{i \in I} \lambda_i y_i) = \nabla^2 f(z) z = \sum_{i \in I} \lambda_i \nabla^2 f(z) z$. Thus, it follows from (2.5) that $\mathbf{z} = \overrightarrow{P}_\Delta \mathbf{y}$. In view of the separability of $D_{\mathbf{f}}$ and C , the formula for \overrightarrow{P}_C is clear. \square

REMARK 3.17. The case when \mathbf{f} is replaced by $\sum_{i \in I} \lambda_i g_i(x_i)$, where $(g_i)_{i \in I}$ is a family of possibly different Bregman functions, was considered in [12] and [13]. This setup is too general to permit closed forms for \overleftarrow{P}_Δ or $\overrightarrow{P}_\Delta$. Furthermore, since Bregman functions are not necessarily Legendre, the existence of Bregman projections is not guaranteed and must therefore be imposed.

4. Main result. Going back to (1.4), we henceforth set

$$(4.1) \quad C = \bigcap_{i \in I} C_i$$

and assume that (the existence of the Bregman retractions is guaranteed by (1.3) and Example 3.6)

$$(4.2) \quad (\forall i \in I) \quad R_i \text{ is a Bregman retraction of } C_i \text{ with modulus } \mu_i.$$

We now formulate our main result.

THEOREM 4.1 (method of Bregman retractions). *Given an arbitrary starting point $y_0 \in \text{int dom } f$, generate a sequence by*

$$(4.3) \quad (\forall n \in \mathbb{N}) \quad y_{n+1} = R_{i(n+1)} y_n,$$

where $i: \mathbb{N} \rightarrow I$ takes on each value in I infinitely often. Then the sequence $(y_n)_{n \in \mathbb{N}}$ converges to a point in $C \cap \text{int dom } f$.

Proof. We proceed in several steps.

Step 1. We have

$$(\forall n \in \mathbb{N})(\forall c \in C_{i(n+1)} \cap \text{int dom } f) \quad D_f(c, y_n) \geq D_f(c, y_{n+1}) + \mu_{i(n+1)}(c, y_n).$$

Indeed, $y_{n+1} = R_{i(n+1)}y_n$ and $R_{i(n+1)}$ is a Bregman retraction of $C_{i(n+1)}$.

Step 2. $(y_n)_{n \in \mathbb{N}}$ is Bregman monotone with respect to $C \cap \text{int dom } f$.

This is clear from Step 1.

Step 3. $(y_n)_{n \in \mathbb{N}}$ is bounded and all its cluster points belong to $\text{int dom } f$.

Fix $c \in C \cap \text{int dom } f$. In view of Step 2, the sequence $(D_f(c, y_n))_{n \in \mathbb{N}}$ is decreasing, and hence bounded. Now apply Proposition 2.2(iv).

Next, let us consider an arbitrary cluster point of $(y_n)_{n \in \mathbb{N}}$, say $y_{k_n} \rightarrow y$.

Step 4. $y \in \text{int dom } f$ and $D_f(y, y_{k_n}) \rightarrow 0$.

This follows from Step 3 and Proposition 2.2(i).

Because I is finite, after passing to a further subsequence and relabelling if necessary, we assume that $i(k_n) \equiv i_{\text{in}}$. Since $C_{i_{\text{in}}}$ is closed, we have $y \in C_{i_{\text{in}}}$.

We now define $I_{\text{in}} = \{i \in I : y \in C_i\}$ and $I_{\text{out}} = \{i \in I : y \notin C_i\}$.

Step 5. $I_{\text{out}} = \emptyset$.

Suppose to the contrary that $I_{\text{out}} \neq \emptyset$. After passing to a further subsequence and relabelling if necessary, we assume that $\{i(k_n), i(k_n + 1), \dots, i(k_{n+1} - 1)\} = I$ —this is possible by our assumptions on the index selector i . For each $n \in \mathbb{N}$, let

$$(4.4) \quad m_n = \min \{k_n \leq k \leq k_{n+1} - 1 : i(k) \in I_{\text{out}}\} - 1.$$

The current assumptions imply that each m_n is a well-defined integer in $[k_n, k_{n+1} - 2]$ satisfying $y \in \bigcap_{k_n \leq k \leq m_n} C_{i(k)}$. Repeated use of Step 1 thus yields

$$(4.5) \quad (\forall n \in \mathbb{N}) \quad D_f(y, y_{m_n}) \leq D_f(y, y_{k_n}).$$

Using Step 4 and Proposition 2.2(v), we deduce $y_{m_n} \rightarrow y$. After passing to a further subsequence and relabelling if necessary, we assume that $i(m_n + 1) \equiv i_{\text{out}}$ and that $y_{m_n+1} = R_{i_{\text{out}}}y_{m_n} \rightarrow z \in C_{i_{\text{out}}} \cap \text{int dom } f$ (using Step 3 again). Now fix $c \in C \cap \text{int dom } f$. Step 1 implies that $(\mu_{i(n+1)}(c, y_n))_{n \in \mathbb{N}}$ is summable; in particular,

$$(4.6) \quad \mu_{i_{\text{out}}}(c, y_{m_n}) = \mu_{i(m_n+1)}(c, y_{m_n}) \rightarrow 0.$$

Since $R_{i_{\text{out}}}$ is a Bregman retraction, we obtain $y = z \in C_{i_{\text{out}}}$. But this in turn implies $i_{\text{out}} \in I_{\text{in}}$, which is the desired contradiction.

Last step. We have shown that $(y_n)_{n \in \mathbb{N}}$ is Bregman monotone with respect to $C \cap \text{int dom } f$ (Step 2) and that all its cluster points lie in $C \cap \text{int dom } f$ (Step 4 and Step 5). Therefore, by Proposition 2.9, the entire sequence $(y_n)_{n \in \mathbb{N}}$ converges to some point in $C \cap \text{int dom } f$. \square

REMARK 4.2. *The proof of Theorem 4.1 is guided by the proof of [7, Theorem 4.1] and similar convergence results on iterating operators under such general control; see [5], [16], and [26]. The present proof clearly shows when properties of the Bregman distance are used, as opposed to those of the modulus. This distinction is blurred in other proofs, because the implicit surrogates for the modulus depend on D_f : see the roles of $D_f(\overleftarrow{P}_{C_{r(n+1)}}y_n, y_n)$, $D_{D_f}((c, c), (y_n, \overrightarrow{P}_{C_{r(n+1)}}y_n))$, $D_f(T_s z(t), z(t))$, and $D_h^k(x^{k+1}, x^k)$ in the proofs of [5, Theorem 8.1], [7, Theorem 4.1], and [16, Theorem 3.1], [26, Theorem 4.1], respectively.*

REMARK 4.3 (Bregman retractions must correspond to the same Bregman distance). *It is natural to ask whether it is possible to use iterates of Bregman retractions*

coming from possibly different underlying Bregman distances to solve convex feasibility problems. Unfortunately, this approach is not successful in general. To see this, let $X = \mathbb{R}^2$, and set $R_C = \overleftarrow{P}_C = \overrightarrow{P}_C$, where f is the negative entropy and C is as in Example 3.8 (with $J = 2$). Further, let L be the straight line through the points $(0, \frac{53}{56})$ and $(\frac{1}{4}, \frac{7}{8})$, and let R_L be the orthogonal projection, i.e., the backward or forward Bregman projection with respect to the energy. Then, although $L \cap \text{int } C \neq \emptyset$, iterating the map $T = R_L \circ R_C$ may not lead to a point in $C \cap L$: indeed, $(\frac{1}{4}, \frac{7}{8})$ is a fixed point of T outside C .

5. Applications. Continuing to work under assumptions (4.1) and (4.2), we now discuss various sequential and parallel algorithms derived from Theorem 4.1.

5.1. Sequential algorithms.

APPLICATION 5.1 (sequential Bregman projections). For each $i \in I$, let $R_i = \overleftarrow{P}_{C_i}$. Then Theorem 4.1 coincides with [5, Theorem 8.1.(ii)]; see also [16, Theorem 3.2]. For cyclic Bregman projections, see Bregman’s classical paper [8].

APPLICATION 5.2 (new method of mixed backward-forward Bregman projections). Suppose that f allows forward Bregman projections. For each $i \in I$, let either $R_i = \overleftarrow{P}_{C_i}$ or $R_i = \overrightarrow{P}_{C_i}$. Then Theorem 4.1 yields a convergence result on iterating a mixture of backward and forward Bregman projections. Note: If desired, it is possible to use both \overleftarrow{P}_{C_i} and \overrightarrow{P}_{C_i} for a given set C_i infinitely often, by counting this set twice.

The following three algorithms are special instances of Application 5.2.

APPLICATION 5.3 (sequential forward Bregman projections). Suppose that f allows forward Bregman projections and let $R_i = \overrightarrow{P}_{C_i}$ for every $i \in I$. Then Theorem 4.1 reduces to [7, Theorem 4.1].

APPLICATION 5.4 (sequential orthogonal projections). Suppose that $f = \frac{1}{2} \|\cdot\|^2$, and let each R_i be the orthogonal projection P_{C_i} . Then Theorem 4.1 turns into a convergence result on (chaotic or random) iterations of orthogonal projections; see also [2], [19], and references therein.

APPLICATION 5.5 (alternating backward-forward Bregman projections). Suppose that f allows forward Bregman projections, and let $I = \{1, 2\}$, $R_1 = \overrightarrow{P}_{C_1}$, and $R_2 = \overleftarrow{P}_{C_2}$. Then the method of Bregman retractions (4.3) corresponds to an alternating backward-forward Bregman projection method, which can be viewed as Csiszár and Tusnády’s alternating minimization procedure [22] applied to D_f (this covers the Expectation-Maximization method for a specific Poisson model; see [22] and [24]).

5.2. Parallel algorithms. Various parallel algorithms arise by specializing Application 5.2 to the product space setting of Example 3.16. Using Example 3.16 and its notation, we deduce that the sequence $(\mathbf{T}^n \mathbf{x}_0)_{n \in \mathbb{N}}$, where $\mathbf{x}_0 \in \mathbf{\Delta}$ and $\mathbf{T} = \overleftarrow{P}_{\mathbf{\Delta}} \circ \overleftarrow{P}_{\mathbf{C}}$, converges to some point in $\mathbf{\Delta} \cap \mathbf{C} \cap \text{int dom } \mathbf{f}$. The same holds true when $\mathbf{T} \in \{\overleftarrow{P}_{\mathbf{\Delta}} \circ \overleftarrow{P}_{\mathbf{C}}, \overleftarrow{P}_{\mathbf{\Delta}} \circ \overrightarrow{P}_{\mathbf{C}}, \overrightarrow{P}_{\mathbf{\Delta}} \circ \overrightarrow{P}_{\mathbf{C}}\}$, provided that f allows forward Bregman projections.

Translating back to the original space X , we obtain the following four parallel algorithms.

APPLICATION 5.6 (parallel projections à la Censor and Elfving). Given $x_0 \in \text{int dom } f$, the sequence generated by

$$(5.1) \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = \nabla f^* \left(\sum_{i \in I} \lambda_i \nabla f(\overleftarrow{P}_{C_i} x_n) \right)$$

converges to a point in $C \cap \text{int dom } f$. This method, which amounts to iterating $\overleftarrow{P}_{\mathbf{\Delta}} \circ \overleftarrow{P}_{\mathbf{C}}$ in \mathbf{X} , was first suggested implicitly in [13]; see also [5] and Remark 3.17.

APPLICATION 5.7 (parallel projections à la Byrne and Censor I). *Suppose that f allows forward Bregman projections. Given $x_0 \in \text{int dom } f$, the sequence generated by*

$$(5.2) \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = \sum_{i \in I} \lambda_i \overleftarrow{P}_{C_i} x_n$$

converges to a point in $C \cap \text{int dom } f$. This method, which amounts to iterating $\overrightarrow{P}_\Delta \circ \overleftarrow{P}_C$ in \mathbf{X} , can be found implicitly in [11, section 4.1] (see also Remark 3.17).

APPLICATION 5.8 (parallel projections à la Byrne and Censor II). *Suppose that f allows forward Bregman projections. Given $x_0 \in \text{int dom } f$, the sequence generated by*

$$(5.3) \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = \nabla f^* \left(\sum_{i \in I} \lambda_i \nabla f(\overrightarrow{P}_{C_i} x_n) \right)$$

converges to a point in $C \cap \text{int dom } f$. This method, which amounts to iterating $\overrightarrow{P}_\Delta \circ \overrightarrow{P}_C$ in \mathbf{X} , can be found implicitly in [11, section 4.2] (see also Remark 3.17).

APPLICATION 5.9 (new parallel method). *Suppose that f allows forward Bregman projections. Given $x_0 \in \text{int dom } f$, the sequence generated by*

$$(5.4) \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = \sum_{i \in I} \lambda_i \overrightarrow{P}_{C_i} x_n$$

converges to a point in $C \cap \text{int dom } f$. This corresponds to iterating $\overrightarrow{P}_\Delta \circ \overrightarrow{P}_C$ in \mathbf{X} .

The negative entropy and the energy lead to concrete examples.

APPLICATION 5.10 (averaged entropic projections à la Butnariu, Censor, and Reich). *Let f be the negative entropy. Given $x_0 \in \text{int dom } f$, the sequence generated by*

$$(5.5) \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = \sum_{i \in I} \lambda_i \overleftarrow{P}_{C_i} x_n$$

converges to a point in $C \cap \text{int dom } f$. Convergence is guaranteed by [10, Theorem 3.3], which holds true in more general settings, or by Application 5.7.

We conclude with a classical method which can be obtained from Application 5.6, 5.7, 5.8, or 5.9 by setting $f = \frac{1}{2} \|\cdot\|^2$.

APPLICATION 5.11 (parallel orthogonal projections à la Auslender). *For each $i \in I$, let P_{C_i} be the orthogonal projection onto C_i . Given $x_0 \in X$, the sequence generated by*

$$(5.6) \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = \sum_{i \in I} \lambda_i P_{C_i} x_n$$

converges to some point in C [1] (see also [3], [18], and [23] for the case when $C = \emptyset$).

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