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A CONJECTURE BY DE PIERRO IS TRUE FOR TRANSLATES OF REGULAR SUBSPACES

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Abstract. Suppose we are given finitely many nonempty closed convex sets in a real Hilbert space and their associated projections. For suitable arrangements of the sets, it is known that the sequence obtained by iterating the composition of the underrelaxed projections is weakly convergent. The question arises how these weak limits vary as the underrelaxation parameter tends to zero. In 2001, De Pierro conjectured that the weak limits approach the least squares solution nearest to the starting point of the sequence. In fact, a result by Censor, Eggermont, and Gordon implies De Pierro's conjecture for affine subspaces in Euclidean space.

This paper extends the result by Censor et al. from Euclidean to Hilbert space. We show that De Pierro's conjecture is true for translates of regular subspaces and the limits all exist with respect to the norm topology. Regularity always holds in Euclidean space. However, this condition is not automatic in infinite-dimensional Hilbert space. Two subspaces are constructed to illustrate the possible divergence of the iterates of the composition of the underrelaxed projections. Somewhat surprisingly, examples in the Euclidean plane demonstrate that the approach to the least squares solution can be nonlinear.

1. Introduction

Throughout this paper, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle \text{ and induced norm } \|\cdot\|, \tag{1.1}$$

and that

$$C_1,\dots,C_N \text{ are finitely many nonempty closed convex subsets of } X \tag{1.2}$$

with

$$\text{corresponding projectors } P_{C_1},\dots,P_{C_N}. \tag{1.3}$$

Recall (see, e.g., [15, Chapter 5]) that the *projector* (or *nearest point mapping*) associated with a nonempty closed convex set C in X is the mapping $P_C\colon X \to C\colon x \mapsto P_C x$, where $P_C x$ is the unique minimizer of the optimization problem

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defining the *distance* of x to C,

$$d(x, C) := \inf_{c \in C} \|x - c\|, \tag{1.4}$$

i.e., $P_C x$ is the *projection* of x onto C, characterized by

$$P_C x \in C \quad \text{and} \quad \sup_{c \in C} \langle c - P_C x, x - P_C x \rangle \leq 0. \tag{1.5}$$

For every $\lambda \in \left]0, 1\right]$, we set

$$Q_\lambda := \big((1-\lambda)\operatorname{Id} + \lambda P_{C_N}\big) \cdots \big((1-\lambda)\operatorname{Id} + \lambda P_{C_1}\big) \tag{1.6}$$

and we define the corresponding sets of *fixed points* by

$$F_\lambda := \operatorname{Fix} Q_\lambda := \{x \in X : x = Q_\lambda x\}. \tag{1.7}$$

We aim to understand the behaviour of the sequence $(Q_\lambda^n x)_{n \in \mathbb{N}}$ in terms of $\lambda \in \left]0, 1\right]$, for an arbitrary $x \in X$. Bruck and Reich's seminal work on strongly nonexpansive mappings (see [9] and also [20]), specialized to our present setting, aids us greatly in this task.

Definition 1.1 (strongly nonexpansive)**.** Let $T \colon X \to X$ be *nonexpansive*, i.e.,

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\| \leq \|x - y\|. \tag{1.8}$$

Then T is said to be *strongly nonexpansive*, if $(x_n - y_n) - (Tx_n - Ty_n) \to 0$ whenever $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences in X such that $(x_n - y_n)_{n \in \mathbb{N}}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$.

Fact 1.2 (basic properties of strongly nonexpansive mappings)**.** *The following statements are true for mappings from X to X.*

(i) *Every projector associated with a nonempty closed convex set is strongly nonexpansive.*

(ii) *The class of strongly nonexpansive mappings is closed under convex combinations. Moreover, if $T_1, \ldots, T_M$ are strongly nonexpansive, $\{\lambda_1, \ldots, \lambda_M\} \subset \left]0, 1\right]$, $\sum_{i=1}^M \lambda_i = 1$, and $\bigcap_{i=1}^M \operatorname{Fix} T_i \neq \varnothing$, then*

$$\operatorname{Fix} \textstyle\sum_{i=1}^M \lambda_i T_i = \bigcap_{i=1}^M \operatorname{Fix} T_i. \tag{1.9}$$

(iii) *The class of strongly nonexpansive mappings is closed under composition. Moreover, if $T_1, \ldots, T_M$ are strongly nonexpansive and $\bigcap_{i=1}^M \operatorname{Fix} T_i \neq \varnothing$, then*

$$\operatorname{Fix} T_M \cdots T_1 = \textstyle\bigcap_{i=1}^M \operatorname{Fix} T_i. \tag{1.10}$$

(iv) *If a strongly nonexpansive mapping possesses at least one fixed point, then every sequence generated by iterating the mapping converges weakly to some fixed point; otherwise, the sequence has no bounded subsequence.*

(v) *The iterates of an odd strongly nonexpansive mapping converge strongly to a fixed point.*

Proof. (i): It is well-known that every projector is firmly nonexpansive (see, e.g., [15, Chapter 5]). The conclusion now follows from [9, Proposition 2.1], which states that every firmly nonexpansive mapping is strongly nonexpansive. (ii): See [9, Proposition 1.3] and [20, Lemma 1.3 and Lemma 1.4]. (iii): [9, Proposition 1.1 and Lemma 2.1]. (iv): [9, Corollary 1.3 and Corollary 1.4]. (v): [9, Corollary 1.2]. ☐

Given a nonempty closed convex set $C \subset X$ and $\lambda \in \,]0,1]$, it will be convenient to set

$$R_{\lambda,C} := (1-\lambda)\operatorname{Id} + \lambda P_C. \tag{1.11}$$

Clearly, $\operatorname{Id} = P_X$ is strongly nonexpansive and so is $R_{\lambda,C}$ by Fact 1.2(i)&(ii).

Corollary 1.3. *Let $\lambda \in \,]0,1]$ and $x \in X$. Then:*

(i) *The mapping Q_λ is strongly nonexpansive.*
(ii) *If $\bigcap_{i=1}^N C_i \neq \varnothing$, then $F_\lambda = \bigcap_{i=1}^N C_i$.*
(iii) *If $F_\lambda \neq \varnothing$, then $(Q_\lambda^n x)_{n\in\mathbb{N}}$ converges weakly to some point in F_λ.*
(iv) *If $F_\lambda = \varnothing$, then $\lim_{n\to+\infty} \|Q_\lambda^n x\| = +\infty$.*

Proof. Our observation following (1.11) and Fact 1.2(ii) show that

$$\text{each } R_{\lambda,C_i} \text{ is strongly nonexpansive and } \operatorname{Fix} R_{\lambda,C_i} = C_i. \tag{1.12}$$

(i)&(ii): Combine (1.12) with Fact 1.2(iii). (iii)&(iv): This follows from (i) and Fact 1.2(iv). ☐

If $F_\lambda \neq \varnothing$, then an alternative proof of Corollary 1.3(iii) can be based upon convergence results on averaged mappings. See [12, Section 7] for further details.

The following notion of a least squares solution is required throughout the paper.

Definition 1.4 (least squares solutions)**.** A point $x \in X$ is a *least square solution (associated with the given sets $C_1, \ldots, C_N$)*, if and only if

$$\textstyle\sum_{i=1}^N \|x - P_{C_i}x\|^2 = \inf_{y\in X} \sum_{i=1}^N \|y - P_{C_i}y\|^2; \tag{1.13}$$

the set of all such points is denoted by $\mathcal{L}$.

Note that $\mathcal{L}$ coincides with $\bigcap_{i=1}^N C_i$ provided the intersection is nonempty. The set $\mathcal{L}$ serves as a tremendously useful "generalized intersection" and numerous algorithms approximate a point in it; see, e.g., [3], [11], [14], and the references therein. Moreover, $\mathcal{L}$ can be characterized as follows.

Fact 1.5. $\mathcal{L} = \operatorname{Fix} \sum_{i=1}^N \frac{1}{N} P_{C_i}$.

Proof. See, e.g., [3, Section 6]. ☐

We are now in a position to formulate De Pierro's conjecture.

Conjecture 1.6 (De Pierro)**.** *(See* [13, Section 3, Conjecture II]*.) Suppose that $F_\lambda \neq \varnothing$, for every $\lambda \in \,]0,1]$. In view of Corollary 1.3(iii), all weak limits*

$$(\forall x \in X)(\forall \lambda \in \,]0,1]) \qquad x_\lambda := \operatorname{weak}\lim_{n\to+\infty} Q_\lambda^n x \tag{1.14}$$

are well defined. Then the conjecture of De Pierro *states that $(x_\lambda)_{\lambda\in]0,1]}$ approaches $P_{\mathcal{L}}x$ as $\lambda \to 0^+$.*

The reader is referred to the nice article [13] for further information on this conjecture as well as its relevance to applications.

Remark 1.7. Several comments on Conjecture 1.6 are in order.

(i) De Pierro points out in [13, Section 1] that Conjecture 1.6 is true for affine subspaces in Euclidean space; in fact, this is a consequence of a result by Censor, Eggermont, and Gordon [10, Theorem 1]. Only the case when each affine subspaces is a hyperplane is covered directly by [10, Theorem 1]; to deal with general affine subspaces, one has to utilize the reformulation techniques outlined at the end of [10, Section 2] and on [16, page 41].

(ii) We implemented interactive JAVA code for the visualization of De Pierro's conjecture in the Euclidean plane, for various types of convex sets. Numerous experiments were performed, all of which strongly support De Pierro's conjecture.

(iii) Little is known about De Pierro's conjecture for general nonempty closed convex sets; this setting remains a challenging topic for future research.

(iv) Suppose the sequence $(x_n)_{n\in\mathbb{N}}$ is generated by

$$x_0 = x, \qquad (\forall n \in \mathbb{N}) \quad x_{n+1} := (1-\lambda_n)x_n + \lambda_n Q_{\lambda_n} x_n, \tag{1.15}$$

where

$$(\lambda_n)_{n\in\mathbb{N}} \text{ is a sequence satisfying } \textstyle\sum_{n\in\mathbb{N}} \lambda_n = +\infty \text{ and } 0 \leftarrow \lambda_n \in \,]0,1]. \tag{1.16}$$

Iteration (1.15) can be interpreted as a "diagonalization" of Conjecture 1.6. De Pierro also conjectures (see [13, Section 3, Conjecture I]) that $\mathcal{L} \neq \varnothing$ if and only if each curve $(x_\lambda)_{\lambda\in]0,1]}$ (where x_λ is given by (1.14)) approaches $P_{\mathcal{L}}x$ and each sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.15) approaches some point in $\mathcal{L}$. Note that in the general convex case, it is possible for the sequence $(x_n)_{n\in\mathbb{N}}$ to converge to a point in $\mathcal{L}\setminus\{P_{\mathcal{L}}x\}$; see De Pierro's example at the end of [13, Section 3]. In the case of translates of subspaces in Euclidean space, however, De Pierro proved that the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (1.15) converges to $P_{\mathcal{L}}x$; see [13, Theorem 4.3.1]. It would be interesting to know whether the techniques utilized in the present paper can be modified to make them applicable to analyze (1.15).

The purpose of this paper is to extend the known positive results on De Pierro's conjecture on affine subspaces from Euclidean to general Hilbert space, and to illustrate the subtleties in infinite-dimensional settings by examples.

Note that, since there are two notions of convergence in Hilbert space (weak and strong), it is not immediately obvious how De Pierro's conjecture should even be formulated. In fact, De Pierro (see [13, first page]) asserts that probably most of the results in [13] remain valid in Hilbert space with *weak* convergence. As our main results show, the following two strikingly different scenarios occur in the affine setting:

- If $C_1,\dots,C_N$ are translates of *regular* subspaces, a notion we review in Section 5 below, then each set F_λ is nonempty and for every $x \in X$ the sequence $(Q_\lambda^n x)_{n\in\mathbb{N}}$ converges *strongly* to some point x_λ. Moreover, the resulting curve $(x_\lambda)_{\lambda\in]0,1]}$ converges *strongly* to $P_{\mathcal{L}}x$, the least squares solution nearest to x (see Theorem 6.4). In other words,

De Pierro's conjecture is true for translates of regular subspaces.

This result also provides additional support of another conjecture by De Pierro (see Remark 1.7(iv)), since in this setting $\mathcal{L}$ and each F_λ are nonempty.

- If $C_1, \ldots, C_N$ are translates of subspaces that are not regular, then it can happen that $\mathcal{L}$ and all sets F_λ are empty (see Example 4.1).

The second irregular case has not been observed previously. This is probably due to the fact that all arrangements of affine subspaces in Euclidean space are automatically regular (see Remark 5.3(iii)).

With an eye towards the general convex case, we have aimed to keep our proofs as geometrical as possible; for instance, in contrast to [10], the Moore-Penrose inverse is not utilized here.

The paper is organized as follows. In Section 2, we fix the notation for the rest of this paper and develop several auxiliary results. The first main result, which concerns the asymptotic behaviour of the sequence $(Q_\lambda^n x)_{n\in\mathbb{N}}$, is presented in Section 3. Section 4 contains a systematic construction of two closed affine subspaces in $\ell_2(\mathbb{N})$ that exhibit "bad" behaviour with respect to iterating Q_λ. Regularity is the focus of Section 5. We provide a novel parallel characterization and show that it guarantees linear convergence of $(Q_\lambda^n x)_{n\in\mathbb{N}}$. Section 6 establishes De Pierro's conjecture for translates of regular subspaces. Some illustrative examples in the Euclidean plane conclude the paper.

2. Standing assumptions and auxiliary results

From now on, we assume that the sets

$$C_1, \ldots, C_N \text{ are closed affine subspaces.} \tag{2.1}$$

For each $i \in \{1, \ldots, N\}$, we set

$$c_i := P_{C_i}0 \in C_i \cap L_i^\perp, \text{ so that } C_i = c_i + L_i, \text{ and } L_i \text{ is a closed linear subspace.} \tag{2.2}$$

Further, let

$$L := \bigcap_{i=1}^N L_i. \tag{2.3}$$

For the reader's convenience, we present and prove in this section various useful auxiliary results that are scattered throughout the literature; see, e.g., [3], [11], [14], [15], and [18]. We used implicitly the following result in the formulation of (2.2).

Proposition 2.1. *Suppose that S is a nonempty closed convex set in X and that $c \in X$. Let $C := c + S$ and fix $x \in X$. Then:*

(i) $P_C x = c + P_S(x - c)$.

(ii) *If S is a closed linear subspace, then $C \cap S^\perp = \{P_C 0\}$; consequently, $P_C x = P_C 0 + P_S x$.*

Proof. (i): Note that $c + P_S(x - c) \in C$ and that, for every $s \in S$,

$$\|x - (c + P_S(x - c))\| = \|(x - c) - P_S(x - c) \leq \|(x - c) - s\| = \|x - (c + s)\|; \tag{2.4}$$

therefore, $P_C x = c + P_S(x - c)$. (ii): Pick $y \in C \cap S^\perp$. Then $y \in C$ so that $C = y + S$. Now $\sup\langle C - y, 0 - y\rangle = \sup\langle (y + S) - y, -y\rangle = \sup\langle S, -y\rangle = 0$, since $y \in S^\perp$. Using (1.5), we obtain $y = P_C 0$. Conversely, $P_C 0 \in C$ and $\sup\langle C - P_C 0, 0 - P_C 0\rangle \leq 0$. Since $C = P_C 0 + S$ and S is a linear subspace, it follows that $P_C 0 \in S^\perp$. Altogether, $C \cap S^\perp = \{P_C 0\}$, as claimed. This identity and (i) imply $P_C x = P_C 0 + P_S(x - P_C 0) = P_C 0 + P_S x - P_S P_C 0 = P_C 0 + P_S x$. ☐

Corollary 2.2. *Let $\lambda \in \,]0, 1]$, $i \in \{1, \ldots, N\}$, and $x \in X$. Then:*

(i) $P_{C_i} x = c_i + P_{L_i} x$;
(ii) $R_{\lambda, C_i} x = \lambda c_i + R_{\lambda, L_i} x$.

Proof. (i): Combine Proposition 2.1(ii) with the standing assumption that $c_i = P_{C_i} 0$ (see (2.2)). (ii): Using (i), we have $R_{\lambda, C_i} x = (1 - \lambda)x + \lambda P_{C_i} x = (1 - \lambda)x + \lambda(c_i + P_{L_i} x) = R_{\lambda, L_i} x + \lambda c_i$. ☐

Our standing assumptions (2.1)–(2.3) make the following refinement of Fact 1.5 possible.

Theorem 2.3. *Let $x \in X$. Then:*

(i) $x \in \mathcal{L} \Leftrightarrow x \in \operatorname{Fix} \sum_{i=1}^N \frac{1}{N} P_{C_i} \Leftrightarrow (\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})x = \sum_{i=1}^N \frac{1}{N} c_i$.
(ii) *$\mathcal{L}$ is a closed affine subspace of X.*
(iii) *If $\mathcal{L} \neq \varnothing$, then the* parallel space *of $\mathcal{L}$ is L, i.e., $\mathcal{L} + L \subset \mathcal{L}$ and $\mathcal{L} - \mathcal{L} \subset L$.*

Proof. (i): The first equivalence follows at once from Fact 1.5. Using Corollary 2.2(i), we observe that $x = \sum_{i=1}^N \frac{1}{N} P_{C_i} x \Leftrightarrow x = \sum_{i=1}^N \frac{1}{N}(c_i + P_{L_i} x) \Leftrightarrow (\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})x = \sum_{i=1}^N \frac{1}{N} c_i$.

(ii): This is a direct consequence of (i).

(iii): Pick $x \in \mathcal{L}$ and $l \in L$. Using (i), we have $(\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})(x + l) = (\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})(x) + (\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})(l) = \sum_{i=1}^N \frac{1}{N} c_i + (l - \sum_{i=1}^N \frac{1}{N} l) = \sum_{i=1}^N \frac{1}{N} c_i$. Thus, again by (i), we conclude $x + l \in \mathcal{L}$. Hence, $\mathcal{L} + L \subset \mathcal{L}$. Now, take any $y \in \mathcal{L}$. Utilizing (i) once more, we have $(\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})(x) = \sum_{i=1}^N \frac{1}{N} c_i = (\operatorname{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i})(y)$. Subtraction of the last term from the first yields $x - y = \sum_{i=1}^N \frac{1}{N} P_{L_i}(x - y)$. Using Fact 1.2(i)&(ii), we obtain $x - y \in \operatorname{Fix} \sum_{i=1}^N \frac{1}{N} P_{L_i} = \bigcap_{i=1}^N \operatorname{Fix} P_{L_i} = \bigcap_{i=1}^N L_i = L$. Consequently, $\mathcal{L} - \mathcal{L} \subset L$ and the proof is complete. ☐

The remaining results of this section extend results from [5] and [18], where $\lambda = 1$.

Proposition 2.4. *Let $\lambda \in \,]0, 1]$ and $i \in \{1, \ldots, N\}$. Then:*

(i) $P_{L_i}(L^\perp) \subset L^\perp$;
(ii) $R_{\lambda, L_i}(L^\perp) = \big((1 - \lambda)I + \lambda P_{L_i}\big)(L^\perp) \subset L^\perp$;
(iii) $P_{L_i}(\operatorname{Id} - P_L) = P_{L_i \cap L^\perp}$.

Proof. (i): If $l \in L$ and $\widetilde{l} \in L^\perp$, then $0 = \langle l, \widetilde{l}\rangle = \langle P_{L_i} l, \widetilde{l}\rangle = \langle l, P_{L_i} \widetilde{l}\rangle$. Hence $P_{L_i}(L^\perp) \subset L^\perp$. (ii): This is a consequence of (i). (iii): (See also [18, Lemma 6].)

For brevity, set $T := P_{L_i}(\operatorname{Id} - P_L)$. If $x \in L_i \cap L^\perp$, then $Tx = x$ and hence $\operatorname{ran} T \supset L_i \cap L^\perp$. Conversely, $\operatorname{ran} T \subset \operatorname{ran} P_{L_i} = L_i$ and also, using (i), $\operatorname{ran} T \subset L^\perp$. Altogether,

$$\operatorname{ran} T = L_i \cap L^\perp. \tag{2.5}$$

This implies

$$T^2 = T. \tag{2.6}$$

Now (2.5) also yields $T = P_{L_i}(\operatorname{Id} - P_L) = (\operatorname{Id} - P_L)P_{L_i}(\operatorname{Id} - P_L)$, which in turn shows that

$$T = T^*. \tag{2.7}$$

In view of (2.5)–(2.7), $T = P_{L_i}(\operatorname{Id} - P_L)$ is the orthogonal projector onto $\operatorname{ran} T = L_i \cap L^\perp$. ☐

Corollary 2.5. *Let $\lambda \in]0, 1]$. Then:*

(i) $(\operatorname{Id} - P_L)R_{\lambda,L_i}(\operatorname{Id} - P_L) = R_{\lambda,L_i}(\operatorname{Id} - P_L) = (1 - \lambda)P_{L^\perp} + \lambda P_{L_i \cap L^\perp}$;

(ii) $R_{\lambda,L_N} \cdots R_{\lambda,L_1}(\operatorname{Id} - P_L) = \big(R_{\lambda,L_N}(\operatorname{Id} - P_L)\big) \cdots \big(R_{\lambda,L_1}(\operatorname{Id} - P_L)\big)$;

(iii) $R_{\lambda,L_N} \cdots R_{\lambda,L_1}(\operatorname{Id} - P_L) = \big((1 - \lambda)P_{L^\perp} + \lambda P_{L_N \cap L^\perp}\big) \cdots \big((1 - \lambda)P_{L^\perp} + \lambda P_{L_1 \cap L^\perp}\big)$.

Proof. (i): The second identity follows from the definition (see (1.11)) and Proposition 2.4(iii): indeed, $R_{\lambda,L_i}(\operatorname{Id} - P_L) = \big((1 - \lambda)\operatorname{Id} + \lambda P_{L_i}\big)P_{L^\perp} = (1 - \lambda)P_{L^\perp} + \lambda P_{L_i}P_{L^\perp} = (1 - \lambda)P_{L^\perp} + \lambda P_{L_i \cap L^\perp}$. Hence $\operatorname{ran} R_{\lambda,L_i}(\operatorname{Id} - P_L) \subset L^\perp$ and the first identity is now clear. (ii)&(iii): Using (i) repeatedly, we see that

$$\begin{aligned} R_{\lambda,L_N} \cdots R_{\lambda,L_1}(\operatorname{Id} - P_L) &= R_{\lambda,L_N} \cdots R_{\lambda,L_3}\big(R_{\lambda,L_2}(\operatorname{Id} - P_L)\big)\big(R_{\lambda,L_1}(\operatorname{Id} - P_L)\big) \\ &\ \vdots \\ &= \big(R_{\lambda,L_N}(\operatorname{Id} - P_L)\big) \cdots \big(R_{\lambda,L_1}(\operatorname{Id} - P_L)\big) \\ &= \big((1 - \lambda)P_{L^\perp} + \lambda P_{L_N \cap L^\perp}\big) \cdots \big((1 - \lambda)P_{L^\perp} + \lambda P_{L_1 \cap L^\perp}\big). \end{aligned} \tag{2.8}$$ ☐

Proposition 2.6. *Let $\lambda \in]0, 1]$ and $i \in \{1, \ldots, N\}$. Then, for every $k \in \{1, 2, \ldots\}$,*

$$(R_{\lambda,L_N} \cdots R_{\lambda,L_1})^k = P_L \oplus (R_{\lambda,L_N} \cdots R_{\lambda,L_1}P_{L^\perp})^k. \tag{2.9}$$

Proof. Proposition 2.4(ii) implies that $R_{\lambda,L_i}(L^\perp) \subset L^\perp$. We now prove (2.9) by induction on k. For $k = 1$, we have $R_{\lambda,L_N} \cdots R_{\lambda,L_1} = R_{\lambda,L_N} \cdots R_{\lambda,L_1}(P_L \oplus P_{L^\perp}) = P_L \oplus (R_{\lambda,L_N} \cdots R_{\lambda,L_1}P_{L^\perp})$. If the identity holds true for $k \in \{1, 2, \ldots\}$, then the same is true for $k + 1$ because

$$\begin{aligned} (R_{\lambda,L_N} \cdots R_{\lambda,L_1})^{k+1} &= (R_{\lambda,L_N} \cdots R_{\lambda,L_1})\big(P_L \oplus (R_{\lambda,L_N} \cdots R_{\lambda,L_1}P_{L^\perp})^k\big) \\ &= P_L \oplus \big((R_{\lambda,L_N} \cdots R_{\lambda,L_1})(R_{\lambda,L_N} \cdots R_{\lambda,L_1}P_{L^\perp})^k\big) \\ &= P_L \oplus (R_{\lambda,L_N} \cdots R_{\lambda,L_1}P_{L^\perp})^{k+1}. \end{aligned} \tag{2.10}$$ ☐

3. Dichotomy

The results in this section extend their counterparts in [5, Section 5.7], where $\lambda = 1$. For every $i \in \{1, \ldots, N\}$, recall (see (1.11)) that $R_{\lambda,L_i} = (1-\lambda)\operatorname{Id} + \lambda P_{L_i}$. It will be convenient to abbreviate, for every $\lambda \in \,]0,1]$,

$$R_\lambda := R_{\lambda,L_N} \cdots R_{\lambda,L_1}, \tag{3.1}$$

and to define

$$\mathbf{T}_\lambda \colon L_1^\perp \times \cdots \times L_N^\perp \to L^\perp \colon \mathbf{y} = (y_1, \ldots, y_N) \mapsto \lambda \textstyle\sum_{i=1}^N R_{\lambda,L_N} \cdots R_{\lambda,L_{i+1}} y_i. \tag{3.2}$$

Let us verify that $\operatorname{ran} \mathbf{T}_\lambda \subset L^\perp$. If $\mathbf{y} = (y_1, \ldots, y_N) \in L_1^\perp \times \cdots \times L_N^\perp$, then each y_i belongs to $L^\perp$; thus, by Proposition 2.4(ii), $R_{L_N} \cdots R_{L_{i+1}} y_i \in L^\perp$ and hence $\mathbf{T}_\lambda \mathbf{y} \in L^\perp$.

We let (see (2.2))

$$\mathbf{c} := (c_1, \ldots, c_N) \in L_1^\perp \times \cdots \times L_N^\perp. \tag{3.3}$$

The behaviour of the iterates of R_λ is the topic of the next result.

Proposition 3.1. *Let $\lambda \in \,]0,1]$ and $x \in X$. Then:*

(i) *R_λ is linear and strongly nonexpansive, with* $\operatorname{Fix} R_\lambda = L$*;*

(ii) *$R_\lambda^n x \to P_L x$;*

(iii) *$(R_\lambda P_{L^\perp})^n x \to 0$.*

Proof. Recall that

$$R_\lambda = R_{\lambda,L_N} \cdots R_{\lambda,L_1} = \big((1-\lambda)\operatorname{Id} + \lambda P_{L_N}\big) \cdots \big((1-\lambda)\operatorname{Id} + \lambda P_{L_1}\big). \tag{3.4}$$

It is thus clear that R_λ is linear. Moreover, by Fact 1.2(i)–(iii), the operator R_λ is strongly nonexpansive with

$$\operatorname{Fix} R_\lambda = \textstyle\bigcap_{i=1}^N \operatorname{Fix} R_{\lambda,L_i} = \bigcap_{i=1}^N \big(\operatorname{Fix}(\operatorname{Id}) \cap \operatorname{Fix}(P_{L_i})\big) = \bigcap_{i=1}^N L_i = L. \tag{3.5}$$

Hence (i) is verified. (ii): The *strong* convergence of $(R_\lambda^n x)_{n\in\mathbb{N}}$ to some point in L is guaranteed by Fact 1.2(v) (see also [2, Corollary 2.4]); it remains to determine this limit. To this end, observe that, for every $i \in \{1, \ldots, N\}$,

$$R_{\lambda,L_i} = R_{\lambda,L_i}^* \quad \text{and} \quad (\forall y \in X) \quad \langle y, R_{\lambda,L_i} y\rangle = (1-\lambda)\|y\|^2 + \lambda \|P_{L_i} y\|^2 \geq 0. \tag{3.6}$$

In view of (3.4) and (3.6), R_λ is the composition of finitely many operators that are self-adjoint, positive (semidefinite), and nonexpansive. Since $\operatorname{Fix} R_\lambda = L$, [7, Theorem 2.5] now implies that

$$\lim_{n\to+\infty} R_\lambda^n x = P_L x, \tag{3.7}$$

which completes the proof of (ii). (An alternative proof can be based upon Fejér monotonicity and [6, Fact 2.2].) (iii): Analogous to the derivation of (i), we see that $R_\lambda P_{L^\perp}$ is linear and strongly nonexpansive, with $\operatorname{Fix} R_\lambda P_{L^\perp} = \{0\}$. The conclusion is now a consequence of Fact 1.2(v). ☐

The following result provides a useful decomposition of the iterates of Q_λ.

Theorem 3.2. *Let* $\lambda \in \,]0,1]$ *and* $n \in \{1, 2, \dots, \}$. *Then:*

$$Q_\lambda^n x = R_\lambda^n x + \textstyle\sum_{k=0}^{n-1} R_\lambda^k \mathbf{T}_\lambda \mathbf{c} = P_L x + (R_\lambda P_{L^\perp})^n x + \sum_{k=0}^{n-1} (R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c}. \tag{3.8}$$

Proof. We prove the first identity by induction on n. Recall that $R_{\lambda,C_i} = \lambda c_i + R_{\lambda,L_i}$, for every $i \in \{1, \dots, N\}$ (Corollary 2.2(ii)). This readily implies

$$Q_\lambda x = R_{\lambda,L_N} R_{\lambda,L_{N-1}} \cdots R_{\lambda,L_1} x + \mathbf{T}_\lambda \mathbf{c} = R_\lambda x + \mathbf{T}_\lambda \mathbf{c}. \tag{3.9}$$

Hence the first identity in (3.8) is true for $n = 1$. Now assume the first identity holds for $n \in \{1, 2, \dots, \}$. Then, using (3.9) (with x replaced by $Q_\lambda^n x$), we obtain

(3.10)
$$\begin{aligned} Q_\lambda^{n+1} x &= Q_\lambda (Q_\lambda^n x) = R_\lambda (Q_\lambda^n x) + \mathbf{T}_\lambda \mathbf{c} \\ &= R_\lambda \big(R_\lambda^n x + \textstyle\sum_{k=0}^{n-1} R_\lambda^k \mathbf{T}_\lambda \mathbf{c} \big) + \mathbf{T}_\lambda \mathbf{c} = R_\lambda^{n+1} x + \sum_{k=0}^{n-1} R_\lambda^{k+1} \mathbf{T}_\lambda \mathbf{c} + \mathbf{T}_\lambda \mathbf{c} \\ &= R_\lambda^{n+1} x + \textstyle\sum_{k=0}^{n} R_\lambda^k \mathbf{T}_\lambda \mathbf{c}. \end{aligned}$$

Thus the formula holds true for $n + 1$ and the first identity is verified. The second identity now follows from Proposition 2.6, the fact that $\mathbf{Tc} \in L^\perp$, and Proposition 2.4(ii). ☐

Corollary 3.3. *Let* $\lambda \in \,]0,1]$, $n \in \mathbb{N}$, *and* $x \in X$. *Then:*

(i) $F_\lambda = (\operatorname{Id} - R_\lambda)^{-1} (\mathbf{T}_\lambda \mathbf{c})$.
(ii) $F_\lambda = y + L$, *for every* $y \in F_\lambda$. *In other words, if* $F_\lambda \neq \varnothing$, *then its parallel space is* L.
(iii) $Q_\lambda^n x = R_\lambda^n (x - y) + y$, *for every* $y \in F_\lambda$.

Proof. Recall that $F_\lambda = \operatorname{Fix} Q_\lambda$ (see (1.7)). (i): Using Theorem 3.2, we obtain the equivalences $x \in F_\lambda \Leftrightarrow x - Q_\lambda x = 0 \Leftrightarrow x - (R_\lambda x + \mathbf{T}_\lambda \mathbf{c}) = 0 \Leftrightarrow (\operatorname{Id} - R_\lambda) x = \mathbf{T}_\lambda \mathbf{c} \Leftrightarrow x \in (\operatorname{Id} - R_\lambda)^{-1} (\mathbf{T}_\lambda \mathbf{c})$. We now turn to the remaining two items. Proposition 3.1(i) yields

$$\operatorname{Fix} R_\lambda = L. \tag{3.11}$$

Since (ii)&(iii) are trivially true if $F_\lambda = \varnothing$, let us assume that $F_\lambda \neq \varnothing$ and pick an arbitrary $y \in F_\lambda$. If $l \in L$, then (by (i) and (3.11)) $(y + l) - R_\lambda (y + l) = (y - R_\lambda y) + (l - R_\lambda l) = \mathbf{T}_\lambda \mathbf{c}$ and so $y + l \in F_\lambda$ by (i). We conclude that $y + L \subset F_\lambda$. Conversely, pick $x \in F_\lambda$ and set $h := x - y$. Then, by (i), $\mathbf{T}_\lambda \mathbf{c} = x - R_\lambda x = (y + h) - R_\lambda (y + h) = (y - R_\lambda y) + (h - R_\lambda h) = \mathbf{T}_\lambda \mathbf{c} + (h - R_\lambda h)$. Hence $h \in \operatorname{Fix} R_\lambda = L$ (see (3.11)), which implies $F_\lambda \subset y + L$. Altogether, we have established (ii). Since $y \in F_\lambda$, (i) implies that $y - R_\lambda y = \mathbf{T}_\lambda \mathbf{c}$. We prove the desired identity by induction on n. Clearly, the statement is true for $n = 0$ (and also for $n = 1$, using Theorem 3.2). Assume the identity holds for some $n \in \mathbb{N}$. Utilizing Theorem 3.2, we then obtain

$$\begin{aligned} Q_\lambda^{n+1} x &= Q_\lambda (Q_\lambda^n x) = R_\lambda Q_\lambda^n x + \mathbf{T}_\lambda \mathbf{c} \\ &= R_\lambda (R_\lambda^n (x - y) + y) + (y - R_\lambda y) = R_\lambda^{n+1} (x - y) + y. \end{aligned}$$ ☐

We are now ready for our first main result.

Theorem 3.4 (dichotomy)**.** *Let* $\lambda \in \,]0,1]$ *and* $x \in X$. *Then exactly one of the following two alternatives holds:*

(i) $F_\lambda = \varnothing$ *and* $\lim_{n\to+\infty} \|Q_\lambda^n x\| = +\infty$.

(ii) $F_\lambda \neq \varnothing$, $\lim_{n\to+\infty} Q_\lambda^n x = P_{F_\lambda} x = P_L x + P_{F_\lambda} 0$, *and* $\sum_{k=0}^{+\infty} (R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c} = P_{F_\lambda} 0 \in L^\perp$.

Proof. If $F_\lambda = \varnothing$, then (i) is precisely Corollary 1.3(iv). Henceforth, we assume that F_λ is nonempty. Now pick any $y \in F_\lambda$. By Proposition 3.1(ii),

$$\lim_{n\to+\infty} R_\lambda^n (x-y) = P_L(x-y). \tag{3.12}$$

Corollary 3.3(iii), (3.12), Proposition 2.1(i), Corollary 3.3(ii), and Proposition 2.1(ii) imply that

$$\lim_{n\to+\infty} Q_\lambda^n x = y + \lim_{n\to+\infty} R_\lambda^n(x-y) = y + P_L(x-y) = P_{y+L} x = P_{F_\lambda} x = P_{F_\lambda} 0 + P_L x. \tag{3.13}$$

Now we take the limit in (3.8) of Theorem 3.2 (recall that $\lim_{n\to+\infty} (R_\lambda P_{L^\perp})^n x = 0$ by Proposition 3.1(iii)) and deduce that

$$\lim_{n\to+\infty} Q_\lambda^n x = P_L x + \textstyle\sum_{k=0}^{+\infty} (R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c}. \tag{3.14}$$

Altogether, (3.13) and (3.14) show that

$$P_{F_\lambda} 0 = \textstyle\sum_{k=0}^{+\infty} (R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c}. \tag{3.15}$$

Finally, since $F_\lambda = y + L$ (see Corollary 3.3(ii)), Proposition 2.1(ii) yields $P_{F_\lambda} 0 \in L^\perp$. ☐

If $\lambda = 1$ and each affine subspace is linear, then Theorem 3.4(ii) reduces to the classical cyclic projections result by von Neumann [19] (for $N = 2$) and by Halperin [17] (for $N \geq 2$). See also [15, Chapter 9] for further information.

In the following section, we provide a striking example where all fixed point sets F_λ, as well as $\mathcal{L}$, are empty. This shows that alternative (i) of Theorem 3.4 does occur and it also provides additional support for another conjecture of De Pierro's (see Remark 1.7(iv)). We shall subsequently show that only alternative (ii) can be observed in the presence of regularity.

4. An irregular example

Example 4.1 (constructing two irregular affine subspaces)**.** Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$ and consider the Hilbert space of real square-summable sequences,

$$X = \ell_2(\mathbb{N}) = \{x = (\xi_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \colon \textstyle\sum_{n\in\mathbb{N}} |\xi_n|^2 < +\infty\}. \tag{4.1}$$

For each $k \in \mathbb{N}$, the k^{th} standard vector $u^{(k)}$ is defined by $u_n^{(k)} = 1$, if $n = k$; $u_n^{(k)} = 0$, otherwise. Then $\{u^{(k)} \colon k \in \mathbb{N}\}$ is the standard orthonormal basis of X. Consider a *decreasing* sequence of angles

$$(\gamma_n)_{n\in\{1,2,\ldots\}} \text{ in } \left]0, \tfrac{1}{2}\pi\right[, \text{ and let } \gamma_\infty := \inf_{n\in\{1,2,\ldots\}} \gamma_n = \lim_{n\in\{1,2,\ldots\}} \gamma_n \in \left[0, \tfrac{1}{2}\pi\right[. \tag{4.2}$$

The *irregular* case occurs when $\gamma_\infty = 0$. Now define two closed affine subspaces by

$$C_1 := \left(1, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\right) + \overline{\operatorname{span}}\left\{u^{(1)}, u^{(3)}, u^{(5)}, \ldots\right\} \tag{4.3}$$

and

$$C_2 := \overline{\operatorname{span}}\big\{\cos(\gamma_n)u^{(2n-1)} + \sin(\gamma_n)u^{(2n)} \colon n \in \{1,2,\dots\}\big\}. \tag{4.4}$$

Let $\lambda \in \,]0,1]$ and $x = (\xi_n)_{n\in\mathbb{N}} \in X$. Then:

(i) $P_{C_1}x = \big(1, \xi_1, \frac{1}{2}, \xi_3, \frac{1}{4}, \xi_5, \frac{1}{6}, \xi_7, \dots\big)$.

(ii) $P_{C_2}x = \sum_{n\in\{1,2,\dots\}} \big(\xi_{2n-1}\cos(\gamma_n) + \xi_{2n}\sin(\gamma_n)\big)\big(\cos(\gamma_n)u^{(2n-1)} + \sin(\gamma_n)u^{(2n)}\big)$.

(iii) $F_\lambda \neq \varnothing \Leftrightarrow \sum_{n\in\{1,2,\dots\}}(\cot(\gamma_n)/n)^2 < +\infty$; if this is the case, then F_λ is a singleton with the unique element

$$\tfrac{1-\lambda}{2-\lambda}u^{(0)} + \sum_{n\in\{1,2,\dots\}} \tfrac{1}{2n}\big(\cot(\gamma_n)u^{(2n-1)} + u^{(2n)}\big). \tag{4.5}$$

(iv) $\mathcal{L} \neq \varnothing \Leftrightarrow \sum_{n\in\{1,2,\dots\}}(\cot(\gamma_n)/n)^2 < +\infty$; if this is the case, then $\mathcal{L}$ is a singleton with the unique element

$$\tfrac{1}{2}u^{(0)} + \sum_{n\in\{1,2,\dots\}} \tfrac{1}{2n}\big(\cot(\gamma_n)u^{(2n-1)} + u^{(2n)}\big). \tag{4.6}$$

(v) The parallel spaces of C_1 and C_2 are

$$L_1 := \overline{\operatorname{span}}\big\{u^{(1)}, u^{(3)}, u^{(5)}, \dots\big\} \tag{4.7}$$

and

$$L_2 := C_2 = \overline{\operatorname{span}}\big\{\cos(\gamma_n)u^{(2n-1)} + \sin(\gamma_n)u^{(2n)} \colon n \in \{1,2,\dots\}\big\}, \tag{4.8}$$

respectively.

(vi) $L_1 + L_2$ is closed $\Leftrightarrow \gamma_\infty > 0$.

(vii) If $\gamma_\infty > 0$ or $(\gamma_n)_{n\in\{1,2,\dots\}}$ converges to 0 "slow enough," then F_λ and $\mathcal{L}$ are both nonempty. The latter case happens, for instance, when $\gamma_n := \operatorname{arccot}(\sqrt[4]{n})$, for every $n \in \{1,2,\dots\}$.

(viii) If $(\gamma_n)_{n\in\{1,2,\dots\}}$ converges to 0 "fast enough," then F_λ and $\mathcal{L}$ are both empty. This occurs when $\gamma_n := \operatorname{arccot}(\sqrt{n})$, for every $n \in \{1,2,\dots\}$.

Proof. (i): Using Proposition 2.1(i), we see that

$$\begin{aligned} P_{C_1}x &= \big(1, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\big) \\ &\quad + P_{\overline{\operatorname{span}}\{u^{(1)},u^{(3)},u^{(5)},\dots\}}\big(\xi_0 - 1, \xi_1 - 1, \xi_2 - \tfrac{1}{2}, \xi_3 - \tfrac{1}{3}, \xi_4 - \tfrac{1}{4}, \dots\big) \\ &= \big(1, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\big) + \big(0, \xi_1 - 1, 0, \xi_3 - \tfrac{1}{3}, 0, \dots\big) \\ &= \big(1, \xi_1, \tfrac{1}{2}, \xi_3, \tfrac{1}{4}, \dots\big). \end{aligned} \tag{4.9}$$

(ii): The given orthonormal basis for the linear subspace C_2, $\big\{\cos(\gamma_n)u^{(2n-1)} + \sin(\gamma_n)u^{(2n)} \colon n \in \{1,2,\dots\}\big\}$ is orthonormal, so the formula is clear by Fourier expansion.

It will be instructive and convenient to make a geometric observation before we turn to the remaining items. Fix $n \in \{1,2,\dots\}$, identify the two-dimensional subspace $X_n := \operatorname{span}\{u^{(2n-1)}, u^{(2n)}\}$ of X with the Euclidean plane $\mathbb{R}^2$, and consider the restrictions to X_n of C_1 and C_2: $C_1 \cap X_n$ and $C_2 \cap X_n$. Then the first and second restrictions correspond to the lines

$$\big(\tfrac{1}{2n-1}, \tfrac{1}{2n}\big) + \mathbb{R}u^{(2n-1)}, \tag{4.10}$$

and

$$\mathbb{R}\big(\cos(\gamma_n)u^{(2n-1)} + \sin(\gamma_n)u^{(2n)}\big), \tag{4.11}$$

respectively. Since $\gamma_n \in \left]0, \frac{1}{2}\pi\right[$, the two lines are not parallel; in fact, the unique point in the intersection of the two lines is

$$\tfrac{1}{2n}\big(\cot(\gamma_n)u^{(2n-1)} + u^{(2n)}\big). \tag{4.12}$$

Suppose that $z = (\zeta_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

(iii): We assume that z is an *algebraic fixed point* of Q_λ, i.e.,

$$z = R_{\lambda,C_2}R_{\lambda,C_1}z, \tag{4.13}$$

but postpone the discussion whether z is actually a member of X momentarily. In view of (i) and (ii), the projections are separable in the sense that (4.13) decouples into an equation for ζ_0 and into a family of systems of two linear equations in two unknowns ζ_{2n-1}, ζ_{2n} (corresponding to the subspace X_n), for every $n \in \{1, 2, \ldots\}$. Using (i) and (ii), the equation determining ζ_0 is

$$\zeta_0 = (1-\lambda)^2\zeta_0 + (1-\lambda)\lambda; \quad \text{hence} \quad \zeta_0 = \tfrac{1-\lambda}{2-\lambda}. \tag{4.14}$$

It is possible (and tedious) to determine ζ_{2n-1}, ζ_{2n} in the same algebraic fashion (by utilizing (i) and (ii)); however, the above geometric observation allows for the following, much shorter, argument. Because of its separable nature, solving equation (4.13) restricted to X_n amounts to finding fixed points (in X_n) of the composition of the corresponding relaxed projections onto the lines given by (4.10) and (4.11). Since these two lines meet, Fact 1.2 implies that any fixed point must correspond to a point in the intersection. In view of (4.12), we deduce

$$(\forall n \in \{1, 2, \ldots, \}) \qquad \big(\zeta_{2n-1}, \zeta_{2n}\big) = \tfrac{1}{2n}\big(\cot(\gamma_n), 1\big). \tag{4.15}$$

Now let the coordinates of z be given by (4.14) and (4.15). Since $\big(\frac{1}{n+1}\big)_{n\in\mathbb{N}} \in X$, we have the equivalence

$$z \in X \Leftrightarrow \sum_{n\in\{1,2,\ldots\}} \left(\tfrac{\cot(\gamma_n)}{n}\right)^2 < +\infty. \tag{4.16}$$

This completes the proof of (iii).

The verification of (iv) is analogous, the only difference lying in the equation determining ζ_0, which stems from the characterization $z \in \mathcal{L}$ if and only if $2z = P_{C_1}z + P_{C_2}z$ (see Fact 1.5).

Item (v) is clear.

We now turn to (vi). Since $L = L_1 \cap L_2 = \{0\}$, we have $L^\perp = X$ and Fact 5.2 yields the equivalence

$$L_1 + L_2 \text{ is closed if and only if } \|P_{L_2}P_{L_1}\| < 1. \tag{4.17}$$

Let $z = (\zeta_n)_{n\in\mathbb{N}} \in X$ such that $\|z\| = 1$. It is easy to check that

$$\|P_{L_2}P_{L_1}z\|^2 = \sum_{n\in\{1,2,\ldots\}} \cos^2(\gamma_n)\zeta_{2n-1}^2. \tag{4.18}$$

If $\gamma_\infty > 0$, then (4.18) implies that $\|P_{L_2}P_{L_1}\| \leq \cos(\gamma_\infty) < 1$ and thus $L_1 + L_2$ is closed. Henceforth, we assume that $\gamma_\infty = 0$. If $z = u^{(2n-1)}$, then (4.18) shows that $\|P_{L_2}P_{L_1}u^{(2n-1)}\| = \cos(\gamma_n)$. Supremizing over $n \in \{1, 2, \ldots\}$ implies $\|P_{L_2}P_{L_1}\| \geq 1$ and therefore $L_1 + L_2$ is not closed.

(vii)&(viii): In view of (iii)&(iv), this depends precisely on whether or not the series

$$\sum_{n\in\{1,2,\ldots\}} \frac{\cot^2(\gamma_n)}{n^2} \tag{4.19}$$

converges. Clearly, this series converges if $\gamma_\infty > 0$. It remains to verify the proposed concrete assignments for the sequences of angles. If $\gamma_n = \operatorname{arccot}(\sqrt[4]{n})$, for every $n \in \{1,2,\ldots\}$, then (4.19) turns into the convergent series $\sum_{n\in\{1,2,\ldots\}} n^{-1.5}$. Similarly, if $\gamma_n = \operatorname{arccot}(\sqrt{n})$, for every $n \in \{1,2,\ldots\}$, then (4.19) becomes the harmonic series. ☐

Remark 4.2. Some comments on Example 4.1 are in order.

(i) If X is a Euclidean space, then F_λ and $\mathcal{L}$ are always nonempty. In essence, this holds because subspaces are automatically closed and the alternative seen in item (viii) can never occur.
(ii) If $\gamma_\infty = 0$ (the irregular case), then L_1+L_2 is not closed (see (vi)); however, no further conclusion can be drawn on whether F_λ and $\mathcal{L}$ are both nonempty (see (vii)&(viii)).
(iii) If $\gamma_\infty > 0$ (the regular case), then L_1+L_2 is closed (see (vi)) and De Pierro's conjecture is true in this setting (combine Remark 5.3(i) with Theorem 6.4).

5. Regularity

We now recall the notion of regular subspaces, which plays an important role in the study of projection methods (see [4, Section 5]).

Definition 5.1 (regular subspaces)**.** We say that the subspaces $L_1,\ldots,L_N$ are *regular*, if

$$d(x_n, L_1\cap\cdots\cap L_N) \to 0, \tag{5.1}$$

whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\max_{i\in\{1,\ldots N\}} d(x_n, L_i) \to 0$.

Consider the product Hilbert space $\mathbf{X} := X^N$, with inner product $\langle \mathbf{x},\mathbf{y}\rangle := \sum_{i=1}^N \frac{1}{N}\langle x_i, y_i\rangle$ and induced norm $\|\mathbf{x}\|^2 := \sum_{i=1}^N \frac{1}{N}\|x_i\|^2$ for $\mathbf{x} = (x_1,\ldots,x_N)$ and $\mathbf{y} = (y_1,\ldots,y_N)$ in $\mathbf{X}$. Now define the product set $\mathbf{L}$ and the *diagonal* $\mathbf{\Delta}$ by

$$\mathbf{L} := L_1\times L_2\times\cdots\times L_N \quad\text{and}\quad \mathbf{\Delta} := \{(x,x,\cdots,x)\in\mathbf{X} \colon x\in X\}, \tag{5.2}$$

respectively. Then regularity of $L_1,\ldots,L_N$ is characterized as follows.

Fact 5.2. *The following are equivalent.*

(i) $L_1,\ldots,L_N$ *are regular.*
(ii) $L_1\cap L^\perp,\ldots,L_N\cap L^\perp$ *are regular.*
(iii) $L_1^\perp+\cdots+L_N^\perp$ *is closed.*
(iv) $\|P_{L_N}\cdots P_{L_1}P_{L^\perp}\| = \|P_{L_N\cap L^\perp}\cdots P_{L_1\cap L^\perp}\| < 1.$
(v) $\mathbf{L},\mathbf{\Delta}$ *are regular in* $\mathbf{X}$.

Proof. The equivalences (i)⇔(iii)⇔(v) follow from [4, Lemma 5.18 and Theorem 5.19]. Next, [5, Proposition 3.7.3] shows that (i)⇔(ii). For the equivalence of (iii) and (iv), combine Corollary 2.5(iii) (for $\lambda = 1$) with [5, Theorem 3.7.4]. ☐

Remark 5.3. Some comments regarding Fact 5.2 are in order.

(i) If $N = 2$, then (by [4, Proposition 5.16] or by [15, Theorem 9.35]) regularity of L_1, L_2 is also equivalent to

$$L_1 + L_2 \text{ is closed.} \tag{5.3}$$

(If $N \geq 3$, then the closedness of $L_1 + \cdots + L_N$ is in general *independent* of the closedness of $L_1^\perp + \cdots + L_N^\perp$; see [4, Remark 5.20].)

(ii) Let us define the *angle* in $[0, \frac{1}{2}\pi]$ of the (ordered) subspaces $L_1, \ldots, L_N$ by its cosine as follows:

$$\cos\gamma(L_1, \cdots, L_N) := \|P_{L_N} \cdots P_{L_1} P_{L^\perp}\| = \|P_{L_N \cap L^\perp} \cdots P_{L_1 \cap L^\perp}\| \in [0, 1]. \tag{5.4}$$

Note that $L_1, \ldots, L_N$ are regular if and only if $\gamma(L_1, \ldots, L_N) > 0$. Moreover, the angle coincides with the classical *Friedrichs angle* when $N = 2$. We refer the reader to [5, Section 3.7] and [15, Chapter 9] for further information.

(iii) Each of the following conditions implies (see [5, Proposition 3.7.7]) that the subspaces $L_1, \ldots, L_N$ are regular:
- at least one $L_i \cap L^\perp$ is finite-dimensional;
- all L_i, except possibly one, are finite-codimensional;
- X is finite-dimensional;
- each L_i is a hyperplane.

The next result can be viewed as the parallel counterpart of Fact 5.2(iv).

Theorem 5.4. *The subspaces $L_1, \ldots, L_N$ are regular if and only if*

$$\|P_{L_1 \cap L^\perp} + \cdots + P_{L_N \cap L^\perp}\| = \|P_{L_1} P_{L^\perp} + \cdots + P_{L_N} P_{L^\perp}\| < N. \tag{5.5}$$

Proof. Suppose first that $L_1, \ldots, L_N$ are regular. In view of Fact 5.2, the subspaces $\mathbf{L}, \mathbf{\Delta}$ are regular in $\mathbf{X}$ and hence

$$\|P_{\mathbf{\Delta}} P_{\mathbf{L}} P_{(\mathbf{\Delta} \cap \mathbf{L})^\perp}\| < 1. \tag{5.6}$$

Pick an arbitrary $x \in X$ and set $\mathbf{x} = (x, \ldots, x) \in \mathbf{X}$. It is straightforward to verify that

$$P_{\mathbf{\Delta}} P_{\mathbf{L}} P_{(\mathbf{\Delta} \cap \mathbf{L})^\perp} \mathbf{x} = \mathbf{y}, \text{ where } \mathbf{y} = (y, \ldots, y) \text{ and } y = \tfrac{1}{N}(P_{L_1} + \cdots + P_{L_N}) P_{L^\perp} x. \tag{5.7}$$

Using (5.6), we deduce that

$$\|y\|^2 = \|\mathbf{y}\|^2 \leq \|P_{\mathbf{\Delta}} P_{\mathbf{L}} P_{(\mathbf{\Delta} \cap \mathbf{L})^\perp}\|^2 \|\mathbf{x}\|^2 = \|P_{\mathbf{\Delta}} P_{\mathbf{L}} P_{(\mathbf{\Delta} \cap \mathbf{L})^\perp}\|^2 \|x\|^2. \tag{5.8}$$

Altogether, (5.6)—(5.8) imply

$$\tfrac{1}{N}\|(P_{L_1} + \cdots P_{L_N}) P_{L^\perp}\| < 1. \tag{5.9}$$

Therefore, (5.5) now follows from (5.9) and Proposition 2.4(iii).

Now suppose that $L_1, \ldots, L_N$ are not regular. Fact 5.2 shows that $L_1 \cap L^\perp, \ldots, L_N \cap L^\perp$ are not regular either. Since $\bigcap_{i=1}^N L_i \cap L^\perp = \{0\}$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that

$$\|x_n\| = d(x_n, \{0\}) \equiv 1, \tag{5.10}$$

while each $d(x_n, L_i \cap L^\perp) \to 0$. Hence

$$x_n - P_{L_1\cap L^\perp} x_n \to 0, \\ \vdots \\ x_n - P_{L_N\cap L^\perp} x_n \to 0. \tag{5.11}$$

Adding yields

$$Nx_n - (P_{L_1\cap L^\perp} + \cdots + P_{L_N\cap L^\perp})x_n \to 0. \tag{5.12}$$

The nonexpansivity of each $P_{L_i\cap L^\perp}$, (5.10), the triangle inequality, and (5.12) imply

$$\begin{aligned} 0 &\le N - \|(P_{L_1\cap L^\perp} + \cdots + P_{L_N\cap L^\perp})x_n\| \\ &= \|Nx_n\| - \|(P_{L_1\cap L^\perp} + \cdots + P_{L_N\cap L^\perp})x_n\| \\ &\le \|Nx_n - (P_{L_1\cap L^\perp} + \cdots + P_{L_N\cap L^\perp})x_n\| \to 0. \end{aligned} \tag{5.13}$$

Therefore, $\|(P_{L_1\cap L^\perp} + \cdots + P_{L_N\cap L^\perp})x_n\| \to N$, which (recall again the nonexpansivity of each $P_{L_i\cap L^\perp}$ and (5.10)) yields $\|P_{L_1\cap L^\perp} + \cdots + P_{L_N\cap L^\perp}\| = N$. ☐

Proposition 5.5. *Suppose that $L_1, \ldots, L_N$ are regular and let $\lambda \in\,]0,1]$. Recall that the angle $\gamma(L_1, \ldots, L_N)$ is defined in Remark 5.3(ii). Then:*

(i) $\|R_\lambda(\mathrm{Id} - P_L)\| \le \lambda^N \cos\big(\gamma(L_1, \ldots, L_N)\big) + 1 - \lambda^N < 1$;

(ii) $\dfrac{1}{1 - \|R_\lambda P_{L^\perp}\|} \le \dfrac{1}{\lambda^N\big(1 - \cos\gamma(L_1, \ldots, L_N)\big)}$.

Proof. Expanding Corollary 2.5(iii) shows that

$$\begin{aligned} R_\lambda(\mathrm{Id} - P_L) &= R_{\lambda,L_N} \cdots R_{\lambda,L_1}(\mathrm{Id} - P_L) \\ &= \big((1-\lambda)P_{L^\perp} + \lambda P_{L_N\cap L^\perp}\big) \cdots \big((1-\lambda)P_{L^\perp} + \lambda P_{L_1\cap L^\perp}\big) \\ &= \lambda^N P_{L_N\cap L^\perp} \cdots P_{L_1\cap L^\perp} + (1-\lambda^N)S, \end{aligned} \tag{5.14}$$

for some nonexpansive linear operator $S\colon X \to X$. Thus, using Fact 5.2 and (5.4), we deduce that

$$\begin{aligned} \|R_\lambda(\mathrm{Id} - P_L)\| &= \|\lambda^N P_{L_N\cap L^\perp} \cdots P_{L_1\cap L^\perp} + (1-\lambda^N)S\| \\ &\le \lambda^N \|P_{L_N\cap L^\perp} \cdots P_{L_1\cap L^\perp}\| + (1-\lambda^N) \\ &= \lambda^N \cos\big(\gamma(L_1, \ldots, L_N)\big) + (1-\lambda^N) \\ &< \lambda^N + (1-\lambda^N) \\ &= 1. \end{aligned} \tag{5.15}$$

Hence (i) is verified, and (ii) follows readily. ☐

We are now ready for our second main result which states that in the presence of regularity, all fixed point sets F_λ are nonempty and the convergence guaranteed by Theorem 3.4(ii) is linear. This coincides with [5, Theorem 5.7.8] when $\lambda = 1$.

Theorem 5.6 (regularity implies linear convergence)**.** *Suppose $L_1, \ldots, L_N$ are regular, let $\lambda \in\,]0,1]$ and $x \in X$. Then $F_\lambda \ne \varnothing$, $\|R_\lambda P_{L^\perp}\| < 1$, and for every $n \in \{1, 2, \ldots\}$,*

$$\|Q_\lambda^{n+1}x - P_{F_\lambda}x\| \le \|R_\lambda P_{L^\perp}\| \|Q_\lambda^n x - P_{F_\lambda}x\|. \tag{5.16}$$

In other words, $(Q_\lambda^n x)_{n\in\mathbb{N}}$ converges linearly to $P_{F_\lambda}x$ with a rate no worse than $\|R_\lambda P_{L^\perp}\| < 1$.

Proof. On the one hand, Theorem 3.2 implies

$$(\forall n \in \{1,2,\ldots\}) \quad Q_\lambda^n x = P_L x + (R_\lambda P_{L^\perp})^n x + \textstyle\sum_{k=0}^{n-1}(R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c}. \tag{5.17}$$

On the other hand, Proposition 5.5(i) yields $\|R_\lambda P_{L^\perp}\| < 1$. Therefore, we take the limit as $n \to +\infty$ in (5.17) (see also Proposition 3.1(iii)) and conclude that $Q_\lambda^n x \to P_L x + (\operatorname{Id} - R_\lambda P_{L^\perp})^{-1}(\mathbf{T}_\lambda \mathbf{c})$ (see, e.g., [8, Theorem 12.1]). In view of Theorem 3.4, we further deduce $F_\lambda \neq \varnothing$ and

$$Q_\lambda^n x \to P_L x + (\operatorname{Id} - R_\lambda P_{L^\perp})^{-1}(\mathbf{T}_\lambda \mathbf{c}) = P_{F_\lambda} x = P_L x + P_{F_\lambda} 0. \tag{5.18}$$

Now fix $n \in \{1,2,\ldots\}$. By (5.17) and (5.18),

$$\begin{aligned} Q_\lambda^{n+1} x - P_{F_\lambda} x &= (R_\lambda P_{L^\perp})^{n+1} x + \textstyle\sum_{k=0}^{n}(R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c} - (\operatorname{Id} - R_\lambda P_{L^\perp})^{-1}(\mathbf{T}_\lambda \mathbf{c}) \\ &= (R_\lambda P_{L^\perp})^{n+1} x - \textstyle\sum_{k=n+1}^{+\infty}(R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c} \\ &= (R_\lambda P_{L^\perp})\big((R_\lambda P_{L^\perp})^n x - \textstyle\sum_{k=n}^{+\infty}(R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c}\big) \\ &= (R_\lambda P_{L^\perp})(Q_\lambda^n x - P_{F_\lambda} x). \end{aligned} \tag{5.19}$$

Therefore, $\|Q_\lambda^{n+1} x - P_{F_\lambda} x\| = \|(R_\lambda P_{L^\perp})(Q_\lambda^n x - P_{F_\lambda} x)\| \leq \|R_\lambda P_{L^\perp}\| \|Q_\lambda^n x - P_{F_\lambda} x\|$. ☐

6. Main result

The following result will be utilized later.

Proposition 6.1. *Let $\{\lambda, \mu\} \subset\,]0,1]$. Then $\|R_\lambda P_{L^\perp} - R_\mu P_{L^\perp}\| \leq \|R_\lambda - R_\mu\| \leq N|\lambda - \mu|$.*

Proof. The first inequality is clear. We establish the second inequality by induction on N, the number of subspaces. Fix $x \in X$. Then

$$\begin{aligned} \|R_{\lambda,L_1} x - R_{\mu,L_1} x\| &= \big\|\big((1-\lambda)\operatorname{Id} + \lambda P_{L_1}\big)x - \big((1-\mu)\operatorname{Id} + \mu P_{L_1}\big)x\big\| \\ &= |\mu - \lambda| \|x - P_{L_1} x\| \\ &\leq |\lambda - \mu| \|x\|, \end{aligned} \tag{6.1}$$

and the conclusion holds for $N = 1$. So let us assume that the desired inequality is verified for $N-1$. Abbreviate $S_\lambda := \big((1-\lambda)\operatorname{Id} + \lambda P_{L_{N-1}}\big) \cdots \big((1-\lambda)\operatorname{Id} + \lambda P_{L_1}\big)$ and similarly for S_μ. Then

$$\begin{aligned} R_\lambda x - R_\mu x &= R_{\lambda,L_N} S_\lambda x - R_{\mu,L_N} S_\mu x \\ &= \big(R_{\lambda,L_N} S_\lambda x - R_{\mu,L_N} S_\lambda x\big) + \big(R_{\mu,L_N} S_\lambda x - R_{\mu,L_N} S_\mu x\big) \\ &= (\mu - \lambda)(\operatorname{Id} - P_{L_N}) S_\lambda x + (1-\mu)(S_\lambda x - S_\mu x) + \mu P_{L_N}(S_\lambda x - S_\mu x). \end{aligned} \tag{6.2}$$

Now take the norm, apply the triangle inequality, recall that $(\mathrm{Id} - P_{L_N})S_\lambda$ and P_{L_N} are both nonexpansive, and use the induction hypothesis to conclude that

(6.3)
$$\begin{aligned}\|R_\lambda x - R_\mu x\| &\leq |\lambda-\mu|\|x\| + (1-\mu)(N-1)|\lambda-\mu|\|x\| + \mu(N-1)|\lambda-\mu|\|x\| \\ &= N|\lambda-\mu|\|x\|.\end{aligned}$$

Since $x \in X$ was chosen arbitrarily, it follows that $\|R_\lambda - R_\mu\| \leq N|\lambda-\mu|$. □

Suppose $L_1, \ldots, L_N$ are regular and set, for every $\lambda \in \left]0,1\right]$,

$$f_\lambda := P_{F_\lambda}0 = (\mathrm{Id} - R_\lambda P_{L^\perp})^{-1}(\mathbf{T}_\lambda \mathbf{c}) = \textstyle\sum_{k=0}^{+\infty}(R_\lambda P_{L^\perp})^k \mathbf{T}_\lambda \mathbf{c} \in L^\perp. \tag{6.4}$$

Note that these identities and inclusions are justified by Theorem 3.4(ii) and Theorem 5.6. The next result provides a useful estimate of the distance between two points on the curve $(f_\lambda)_{\lambda\in]0,1]}$.

Proposition 6.2. *Suppose that* $L_1, \ldots, L_N$ *are regular and let* $\{\lambda,\mu\} \subset \left]0,1\right]$. *Then:*

$$\|f_\lambda - f_\mu\| \leq \frac{|\lambda-\mu|N\big(\|f_\mu\| + \sum_{i=1}^N \|c_i\|\big)}{1 - \|R_\lambda P_{L^\perp}\|} \leq \frac{|\lambda-\mu|N\big(\|f_\mu\| + \sum_{i=1}^N \|c_i\|\big)}{\lambda^N\big(1 - \cos\gamma(L_1,\ldots,L_N)\big)}. \tag{6.5}$$

Proof. First, observe that the quotients are well-defined by Proposition 5.5(i). By definition, we have $f_\lambda = \mathbf{T}_\lambda\mathbf{c} + R_\lambda P_{L^\perp} f_\lambda$ and $f_\mu = \mathbf{T}_\mu\mathbf{c} + R_\mu P_{L^\perp} f_\mu$. Hence $f_\lambda - f_\mu = (\mathbf{T}_\lambda\mathbf{c} - \mathbf{T}_\mu\mathbf{c}) + (R_\lambda P_{L^\perp} f_\lambda - R_\mu P_{L^\perp} f_\mu)$ and thus

$$\|f_\lambda - f_\mu\| \leq \|\mathbf{T}_\lambda\mathbf{c} - \mathbf{T}_\mu\mathbf{c}\| + \|R_\lambda P_{L^\perp} f_\lambda - R_\mu P_{L^\perp} f_\mu\|. \tag{6.6}$$

On the one hand (recall (3.2)),

$$\begin{aligned}\mathbf{T}_\lambda\mathbf{c} - \mathbf{T}_\mu\mathbf{c} &= (\lambda-\mu)\textstyle\sum_{i=1}^N R_{\lambda,L_N}\cdots R_{\lambda,L_{i+1}} c_i \\ &\quad + \mu\textstyle\sum_{i=1}^N \big((R_{\lambda,L_N}\cdots R_{\lambda,L_{i+1}}) - (R_{\mu,L_N}\cdots R_{\mu,L_{i+1}})\big)c_i.\end{aligned} \tag{6.7}$$

Therefore, using the triangle inequality, the fact that each R_{λ,L_k} is nonexpansive, and Proposition 6.1, we obtain

$$\begin{aligned}\|\mathbf{T}_\lambda\mathbf{c} - \mathbf{T}_\mu\mathbf{c}\| &\leq |\lambda-\mu|\textstyle\sum_{i=1}^N \|c_i\| + \mu\sum_{i=1}^N (N-i)|\lambda-\mu|\|c_i\| \\ &\leq |\lambda-\mu|N\textstyle\sum_{i=1}^N \|c_i\|.\end{aligned} \tag{6.8}$$

On the other hand, $R_\lambda P_{L^\perp} f_\lambda - R_\mu P_{L^\perp} f_\mu = (R_\lambda P_{L^\perp} f_\lambda - R_\lambda P_{L^\perp} f_\mu) + (R_\lambda P_{L^\perp} f_\mu - R_\mu P_{L^\perp} f_\mu)$. Thus, using Proposition 6.1,

$$\|R_\lambda P_{L^\perp} f_\lambda - R_\mu P_{L^\perp} f_\mu\| \leq \|R_\lambda P_{L^\perp}\|\|f_\lambda - f_\mu\| + |\lambda-\mu|N\|f_\mu\|. \tag{6.9}$$

Combining (6.6), (6.8), and (6.9) yields altogether

$$\|f_\lambda - f_\mu\| \leq \|R_\lambda P_{L^\perp}\|\|f_\lambda - f_\mu\| + |\lambda-\mu|N\big(\|f_\mu\| + \textstyle\sum_{i=1}^N \|c_i\|\big). \tag{6.10}$$

This implies the first inequality of (6.5); the second then follows from Proposition 5.5(ii). □

We now follow the points along the curve $(f_\lambda)_{\lambda\in]0,1]}$ as λ approaches 0 and 1.

Theorem 6.3. *Suppose that* $L_1, \ldots, L_N$ *are regular. Then:*

(i) $f_0 := \lim_{\lambda\to0^+} f_\lambda = \big(\mathrm{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i\cap L^\perp}\big)^{-1}\big(\sum_{i=1}^N \frac{1}{N} c_i\big)$;

(ii) $f_1 = \lim_{\lambda\to 1^-} f_\lambda = (\mathrm{Id} - P_{L_N}\cdots P_{L_1}P_{L^\perp})^{-1}(\mathbf{T}_1\mathbf{c})$.

Moreover, the corresponding map $[0,1]\to L^\perp\colon \lambda\mapsto f_\lambda$ *is continuous, and* $(f_\lambda)_{\lambda\in[0,1]}$ *is a connected compact curve in* $L^\perp$.

Proof. Recall (see (3.3)) that $\mathbf{c} = (c_1,\dots,c_N)$, where each $c_i \in L_i^\perp \subset L^\perp$, and (see (6.4)) that each f_λ belongs to $L^\perp$. Define, for each $i\in\{1,\dots,N\}$,

$$K_i := L_i \cap L^\perp \tag{6.11}$$

and fix $\lambda \in \left]0,1\right]$. With the help of Proposition 2.4(ii) and Corollary 2.5, we obtain the following identities.

$$\begin{aligned}(\forall y\in L^\perp)\quad R_\lambda y &= R_\lambda P_{L^\perp} y\\ &= R_{\lambda,K_N}\cdots R_{\lambda,K_1} y\\ &= \big((1-\lambda)\,\mathrm{Id} + \lambda P_{K_N}\big)\cdots\big((1-\lambda)\,\mathrm{Id}+\lambda P_{K_1}\big)y\\ &= \big(\mathrm{Id}+\lambda(P_{K_N}-\mathrm{Id})\big)\cdots\big(\mathrm{Id}+\lambda(P_{K_1}-\mathrm{Id})\big)y\\ &= \big(\mathrm{Id}+\lambda\textstyle\sum_{i=1}^N (P_{K_i}-\mathrm{Id}) + A_\lambda\big)y,\end{aligned} \tag{6.12}$$

where $A_\lambda\colon L^\perp\to L^\perp$ is a bounded linear operator and

$$\|A_\lambda\| \leq \lambda^2\tbinom{N}{2} + \lambda^3\tbinom{N}{3} + \cdots + \lambda^N\tbinom{N}{N} = (1+\lambda)^N - (1+N\lambda). \tag{6.13}$$

Thus

$$(\forall y\in L^\perp)\quad \tfrac{1}{\lambda}(y - R_\lambda y) = -\tfrac{1}{\lambda}A_\lambda y + \textstyle\sum_{i=1}^N(\mathrm{Id}-P_{K_i})y. \tag{6.14}$$

Since $\frac{1}{\lambda}\|A_\lambda\| \leq \frac{1}{\lambda}\big((1+\lambda)^N - 1\big) - N \to 0$ as $\lambda\to 0^+$, we see that

$$(\forall y\in L^\perp)\quad \lim_{\lambda\to 0^+}\tfrac{1}{\lambda}(\mathrm{Id}-R_\lambda)y = \big(\textstyle\sum_{i=1}^N(\mathrm{Id}-P_{K_i})\big)y. \tag{6.15}$$

On the other hand, the regularity of $L_1,\dots,L_N$ and Theorem 5.4 imply $\|\sum_{i=1}^N P_{K_i}\| < N$, hence $\|\frac{1}{N}\sum_{i=1}^N P_{K_i}\| < 1$. Thus $\mathrm{Id} - \frac{1}{N}\sum_{i=1}^N P_{K_i}$ has a bounded inverse (by [8, Theorem 12.1]), and the same is true for $\sum_{i=1}^N(\mathrm{Id}-P_{K_i})$. In view of (6.15) and the continuity of inversion (see, e.g., [8, Corollary 12.3]), this yields the following operator limit identity

$$\lim_{\lambda\to 0^+}(\mathrm{Id}-R_\lambda)^{-1}(\lambda\,\mathrm{Id}) = \big(\textstyle\sum_{i=1}^N(\mathrm{Id}-P_{K_i})\big)^{-1}\quad\text{on } L^\perp. \tag{6.16}$$

Furthermore, we note that

$$\lim_{\lambda\to 0^+}\tfrac{1}{\lambda}\mathbf{T}_\lambda\mathbf{c} = \lim_{\lambda\to 0^+}\textstyle\sum_{i=1}^N R_{\lambda,L_N}\cdots R_{\lambda,L_{i+1}}c_i = \sum_{i=1}^N c_i, \tag{6.17}$$

and (since $\{f_\lambda, \mathbf{T}_\lambda\mathbf{c}\}\subset L^\perp$ and $R_\lambda = R_\lambda P_{L^\perp}$ on $L^\perp$) that

$$f_\lambda = (\mathrm{Id}-R_\lambda)^{-1}(\mathbf{T}_\lambda\mathbf{c}), \tag{6.18}$$

where $(\mathrm{Id}-R_\lambda)^{-1}$ is viewed as an operator on $L^\perp$. Combining (6.16), (6.17), and (6.18) now results in

$$\begin{aligned}\lim_{\lambda\to 0^+} f_\lambda &= \lim_{\lambda\to 0^+}(\mathrm{Id}-R_\lambda)^{-1}\big(\lambda(\tfrac{1}{\lambda}\mathbf{T}_\lambda\mathbf{c})\big)\\ &= \big(\textstyle\sum_{i=1}^N(\mathrm{Id}-P_{K_i})\big)^{-1}\big(\sum_{i=1}^N c_i\big)\end{aligned} \tag{6.19}$$

and (i) follows.

(ii): clearly $\lim_{\lambda\to 1^-} R_\lambda P_{L^\perp} = P_{L_N}\cdots P_{L_1}P_{L^\perp}$, and this limit has norm less than 1 by Proposition 5.5(i). It follows that $f_\lambda = (\mathrm{Id} - R_\lambda P_{L^\perp})^{-1}\mathbf{T}_\lambda \mathbf{c} \to (\mathrm{Id} - P_{L_N}\cdots P_{L_1}P_{L^\perp})^{-1}\mathbf{T}_1\mathbf{c}$.

Now fix an arbitrary $\mu \in\,]0,1]$. Using Proposition 6.2, we see that

$$(\forall\lambda \in [\tfrac{1}{2}\mu, 1]) \quad \|f_\lambda - f_\mu\| \leq \frac{|\lambda-\mu| N 2^N \big(\|f_\mu\| + \sum_{i=1}^N \|c_i\|\big)}{\mu^N\big(1-\cos\gamma(L_1,\dots,L_N)\big)}. \tag{6.20}$$

Therefore, the map $\lambda \mapsto f_\lambda$ is continuous at μ, and hence on $]0,1]$. In view of (i), let us extend this map continuously on $[0,1]$. The final sentence of the theorem is now clear, since the curve $(f_\lambda)_{\lambda\in[0,1]}$ is the continuous image of the interval $[0,1]$. ☐

We are now in a position to state and prove our main result.

Theorem 6.4 (De Pierro's conjecture is true for translates of regular subspaces)**.** *Suppose that $L_1,\dots,L_N$ are regular and let $x \in X$. Then the following strong limits exist.*

$$(\forall\lambda \in\,]0,1]) \quad x_\lambda := \lim_{n\to+\infty} Q_\lambda^n x = P_{F_\lambda} x = P_L x + P_{F_\lambda} 0 = P_L x + f_\lambda. \tag{6.21}$$

Moreover, as $\lambda \to 0^+$, $(x_\lambda)_{\lambda\in]0,1]}$ converges strongly to the least squares solution nearest to x:

$$P_{\mathcal{L}} x = \lim_{\lambda\to 0^+} x_\lambda. \tag{6.22}$$

Proof. The statements concerning (6.21) are implied by Theorem 5.6, Theorem 3.4(ii), and (6.4). As in Theorem 6.3(i), we let

$$f_0 := \lim_{\lambda\to 0^+} f_\lambda \in L^\perp \tag{6.23}$$

so that $(\mathrm{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i\cap L^\perp}) f_0 = \sum_{i=1}^N \frac{1}{N} c_i$. Using Proposition 2.4(iii) and the fact that $f_0 \in L^\perp$, we obtain

$$\begin{aligned} \textstyle\sum_{i=1}^N \frac{1}{N} c_i &= (\mathrm{Id} - \textstyle\sum_{i=1}^N \frac{1}{N} P_{L_i\cap L^\perp}) f_0 \\ &= (\mathrm{Id} - \textstyle\sum_{i=1}^N \frac{1}{N} P_{L_i} P_{L^\perp}) f_0 = (\mathrm{Id} - \sum_{i=1}^N \frac{1}{N} P_{L_i}) f_0. \end{aligned} \tag{6.24}$$

Theorem 2.3(i) yields $f_0 \in \mathcal{L}$; thus altogether $f_0 \in \mathcal{L}\cap L^\perp$. Hence $f_0 = P_{\mathcal{L}} 0$ and

$$P_{\mathcal{L}} x = f_0 + P_L x \tag{6.25}$$

by Theorem 2.3(iii) and Proposition 2.1(ii). Using (6.21), (6.23), and (6.25), we see that

$$\lim_{\lambda\to 0^+} x_\lambda = P_L x + \lim_{\lambda\to 0^+} f_\lambda = P_L x + f_0 = P_{\mathcal{L}} x. \tag{6.26}$$ ☐

We conclude with some concrete examples in the Euclidean plane that illustrate the possible *nonlinearity* of the curve $(f_\lambda)_{\lambda\in]0,1]}$. See also [1] for some explicit computations of f_1 for hyperplanes in Euclidean space.

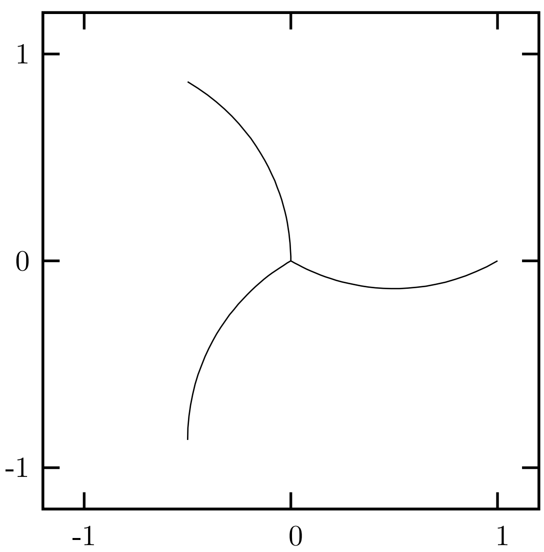

FIGURE 1. Three nonlinear curves $(f_\lambda)_{\lambda\in]0,1]}$ for u_1, u_2, u_3.

Example 6.5 (singletons)**.** Suppose that the affine subspaces are all singletons, i.e., $L_i = \{0\}$, for each $i \in \{1, \dots, N\}$. Let $\lambda \in]0, 1]$. Then F_λ is a singleton and its unique element is

$$f_\lambda = \lambda \frac{(1-\lambda)^{N-1}c_1 + (1-\lambda)^{N-2}c_2 + \cdots + (1-\lambda)c_{N-1} + c_N}{1-(1-\lambda)^N}. \tag{6.27}$$

Consequently, for every $x \in X$, $\lim_{n\to+\infty} Q_\lambda^n x = f_\lambda$ and $\lim_{\lambda\to 0^+} f_\lambda = \sum_{i=1}^N \frac{1}{N} c_i$.

Proof. Each C_i is a singleton, hence $L_i = \{0\}$ and thus $L = \{0\}$. By Remark 5.3(iii), the subspaces $L_1, \dots, L_N$ are regular, and Theorem 5.6 now implies that $F_\lambda \neq \varnothing$. In view of Corollary 3.3(ii), the set F_λ is a singleton and its only element is f_λ (see (6.4)). The formula for f_λ presented in (6.27) is a consequence of expanding and solving

$$\begin{aligned} f_\lambda &= \big((1-\lambda)\operatorname{Id} + \lambda P_{C_N}\big) \cdots \big((1-\lambda)\operatorname{Id} + \lambda P_{C_1}\big) f_\lambda \\ &= \big((1-\lambda)\operatorname{Id} + \lambda c_N\big) \cdots \big((1-\lambda)\operatorname{Id} + \lambda c_1\big) f_\lambda \end{aligned} \tag{6.28}$$

for f_λ. Furthermore, Theorem 2.3(i) results in $\mathcal{L} = \{\sum_{i=1}^N \frac{1}{N} c_i\}$. The remaining statements now follow from Theorem 6.4. ☐

Remark 6.6. We now illustrate the nonlinearity in the formula for f_λ given by (6.27). In the setting of Example 6.5, let $X = \mathbb{R}^2$, identified with $\mathbb{C}$, and $N = 3$. Let u_1, u_2, u_3 be all three cube roots of unity. Then $\mathcal{L} = \{0\}$ and Figure 1 shows three curves $(f_\lambda)_{\lambda\in]0,1]}$, obtained by assigning u_1, u_2, u_3 to c_1, c_2, c_3 in three different orderings. Note that these curves meet at the unique least squares solution 0, as predicted by Example 6.5.

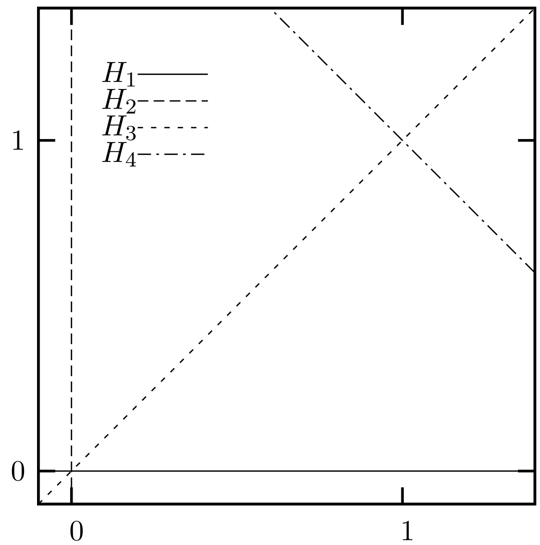

FIGURE 2. The hyperplanes H_1, H_2, H_3, and H_4.

In our last example, we show that the nature of the fixed point curves is highly dependent on the order of the sets.

Example 6.7. Let $X = \mathbb{R}^2$ and $N = 4$. Further, set $H_1 := \{(\xi_1, \xi_2) \in X : \xi_2 = 0\}$, $H_2 := \{(\xi_1, \xi_2) \in X : \xi_1 = 0\}$, $H_3 := \{(\xi_1, \xi_2) \in X : \xi_1 = \xi_2\}$, and $H_4 := \{(\xi_1, \xi_2) \in X : \xi_1 + \xi_2 = 2\}$; see Figure 2. Now take $\lambda \in]0, 1]$ and let

$$C_1 \times C_2 \times C_3 \times C_4 := H_1 \times H_2 \times H_3 \times H_4. \tag{6.29}$$

Then $L = \{0\}$, $\mathcal{L} = \{(\frac{1}{2}, \frac{1}{2})\}$, and $F_\lambda = \{f_\lambda\}$, where

$$f_\lambda = \left(\frac{1}{2-\lambda}, \frac{1}{2-\lambda}\right). \tag{6.30}$$

Similarly, the fixed points corresponding to the three remaining cyclic permutations are:

$$f_\lambda = \left(\frac{1}{2-\lambda}, \frac{1-\lambda}{2-\lambda}\right), \text{ for } C_1 \times C_2 \times C_3 \times C_4 := H_2 \times H_3 \times H_4 \times H_1; \tag{6.31}$$

$$f_\lambda = \left(\frac{1-\lambda}{2-\lambda}, \frac{1-\lambda}{2-\lambda}\right), \text{ for } C_1 \times C_2 \times C_3 \times C_4 := H_3 \times H_4 \times H_1 \times H_2; \tag{6.32}$$

$$f_\lambda = \left(\frac{1-\lambda}{2-\lambda}, \frac{1-\lambda}{2-\lambda}\right), \text{ for } C_1 \times C_2 \times C_3 \times C_4 := H_4 \times H_1 \times H_2 \times H_3. \tag{6.33}$$

Notice that the four curves corresponding to (6.30)–(6.33), depicted in Figure 3, are all *linear*. In contrast, the four fixed point curves given by

$$f_\lambda = \left(\frac{2(4-3\lambda+\lambda^2)}{16-16\lambda+5\lambda^2}, \frac{2(4-\lambda)}{16-16\lambda+5\lambda^2}\right), \tag{6.34}$$
$$\text{for } C_1 \times C_2 \times C_3 \times C_4 := H_1 \times H_3 \times H_2 \times H_4;$$

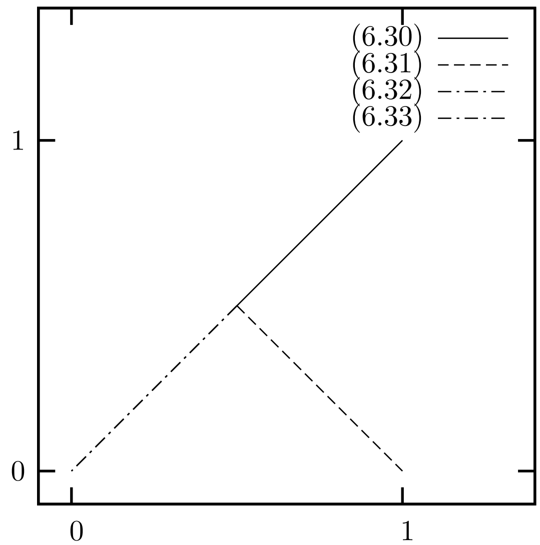

FIGURE 3. Four linear fixed point curves $(f_\lambda)_{\lambda\in]0,1]}$.

$$f_\lambda = \left(\frac{2(4-3\lambda+\lambda^2)}{16-16\lambda+5\lambda^2}, \frac{2(4-5\lambda+\lambda^2)}{16-16\lambda+5\lambda^2}\right), \quad \text{for } C_1\times C_2\times C_3\times C_4 := H_3\times H_2\times H_4\times H_1; \tag{6.35}$$

$$f_\lambda = \left(\frac{2(4-3\lambda)}{16-16\lambda+5\lambda^2}, \frac{2(4-5\lambda+2\lambda^2)}{16-16\lambda+5\lambda^2}\right), \quad \text{for } C_1\times C_2\times C_3\times C_4 := H_2\times H_4\times H_1\times H_3; \tag{6.36}$$

$$f_\lambda = \left(\frac{2(4-7\lambda+3\lambda^2)}{16-16\lambda+5\lambda^2}, \frac{2(4-5\lambda+2\lambda^2)}{16-16\lambda+5\lambda^2}\right), \quad \text{for } C_1\times C_2\times C_3\times C_4 := H_4\times H_1\times H_3\times H_2. \tag{6.37}$$

are all *nonlinear*; see Figure 4.

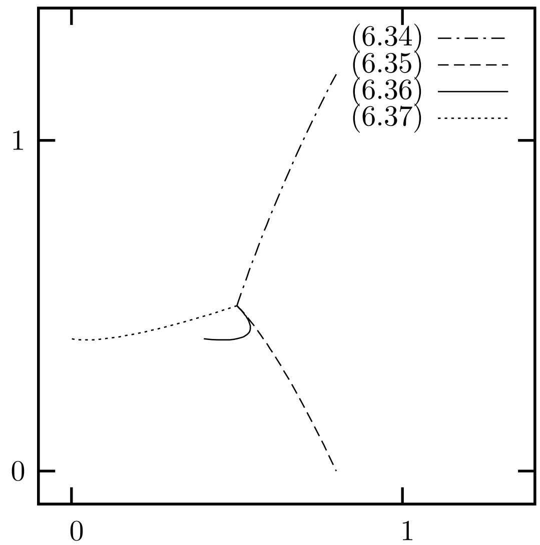

FIGURE 4. Four nonlinear fixed point curves $(f_\lambda)_{\lambda\in]0,1]}$.

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