# A Strongly Convergent Reflection Method for Finding the Projection onto the Intersection of Two Closed Convex Sets in a Hilbert Space

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**Abstract**

A new iterative method for finding the projection onto the intersection of two closed convex sets in a Hilbert space is presented. It is a Haugazeau-like modification of a recently proposed averaged alternating reflections method which produces a strongly convergent sequence.

**Keywords:** Best approximation problem, convex set, projection, strong convergence.

## 1 Introduction

Throughout this paper,

$$X \text{ is a real Hilbert space with inner product } \langle \cdot \mid \cdot \rangle \text{ and induced norm } \| \cdot \|, \tag{1}$$

and

$$A \text{ and } B \text{ are two closed convex sets in } X \text{ such that } C = A \cap B \neq \varnothing. \tag{2}$$

Given a point $x \in X$, the problem under consideration is the *best approximation problem*

$$\text{find } \; c \in C \quad \text{such that} \quad \|x - c\| = \inf \|x - C\|. \tag{3}$$

This problem, which was already studied by von Neumann in the 1930s in this general Hilbert space setting, is of fundamental importance in applied mathematics (see [5] for historical references, recent applications, algorithms, and further references).

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The aim of this note is to present a new strongly convergent method — termed *Haugazeau-like Averaged Alternating Reflections (HAAR)* — for finding the solution of (3) iteratively. This algorithm is a modification of the *Averaged Alternating Reflections (AAR)* scheme, which we recently introduced in [4]. To describe AAR, we require some notation from convex analysis. Given any nonempty closed convex set $S$ in $X$, denote the *projector* (best approximation operator) onto $S$ by $P_S$. Further, let $I$ be the identity operator on $X$ and let $R_S = 2P_S - I$ be the *reflector* with respect to $S$. We recall that the normal cone to $S$ at $x \in S$ is defined by $N_S(x) = \big\{x^* \in X \mid (\forall s \in S)\ \ \langle x^* \mid s - x\rangle \leq 0\big\}$. Both AAR and HAAR rely upon the operator

$$T = \tfrac{1}{2}R_A R_B + \tfrac{1}{2}I, \tag{4}$$

and their analyses require the nonempty closed convex cone

$$K = N_{B-A}(0). \tag{5}$$

We are now ready to describe AAR and its asymptotic behavior (see also [4] for background).

**Fact 1.1 (AAR)** *Suppose that $x \in X$. Then the sequence of averaged alternating reflections (AAR) $(T^n x)_{n\in\mathbb{N}}$ converges weakly to a point in*

$$\operatorname{Fix} T = \big\{z \in X \mid Tz = z\big\} = C + K. \tag{6}$$

*Moreover, the sequence $(P_B T^n x)_{n\in\mathbb{N}}$ is bounded and each of its weak cluster points lies in $C$.*

*Proof.* The identity (6) was proved in [4, Corollary 3.9]. The statements regarding weak convergence and weak cluster points follows from [8, Theorem 1] applied to the normal cone operators $N_A$ and $N_B$. (See also [3, Fact 5.9] and [4, Theorem 3.13(ii)].) ☐

Fact 1.1 implies that the weak cluster points of the sequence $(P_B T^n x)_{n\in\mathbb{N}}$ solve the *convex feasibility problem*

$$\text{find } \ c \in C. \tag{7}$$

Although such points solve (7), they may nonetheless be neither strong cluster points nor the solution of the best approximation problem (3) (see [4, Section 1] for a counterexample). These shortcomings of AAR motivated us to look for variants of AAR with better convergence properties. In Section 2, we investigate the relative geometry of the sets $A$ and $B$, culminating in the formula $P_B P_{C+K} = P_C$ (see Corollary 2.9). This identity, Fact 1.1, and a consequence of the weak-to-strong convergence principle [2] lead in Section 3 to the precise formulation of HAAR. A crucial ingredient of HAAR is Haugazeau's [7] explicit projector onto the intersection of two halfspaces. Our main result (Theorem 3.3) guarantees strong convergence to the nearest point in $C$, i.e., to the solution of (3).

## 2 Relative geometry of two sets

We shall utilize the following notions from fixed point theory; see, e.g., [6].

**Definition 2.1** *Suppose that* $R\colon X \to X$. *Then:*

(i) $R$ *is* firmly nonexpansive, *if*

$$(\forall x \in X)(\forall y \in X) \ \ \|Rx - Ry\|^2 + \|(I-R)x - (I-R)y\|^2 \leq \|x - y\|^2. \tag{8}$$

(ii) $R$ *is* nonexpansive, *if*

$$(\forall x \in X)(\forall y \in X) \ \ \|Rx - Ry\| \leq \|x - y\|. \tag{9}$$

It is well known, for example, that the projector onto a nonempty closed convex set is firmly nonexpansive.

**Fact 2.2** *Suppose that* $R\colon X \to X$. *Then* $R$ *is firmly nonexpansive if and only if* $2R - I$ *is nonexpansive.*

*Proof.* See [6, Theorem 12.1]. ☐

**Fact 2.3** *Suppose that* $S$ *is a nonempty closed convex set in* $X$ *and that* $x \in X$. *Then there exists a unique point* $P_S x \in S$ *such that* $\|x - P_S x\| = \inf \|x - S\|$. *The point* $P_S x$ *is characterized by*

$$P_S x \in S \quad \text{and} \quad (\forall s \in S) \quad \langle s - P_S x \,|\, x - P_S x\rangle \leq 0. \tag{10}$$

*The induced operator* $P_S\colon X \to S\colon x \mapsto P_S x$ *is called the* projector *onto* $S$*; it is firmly nonexpansive and consequently, the reflector* $R_S = 2P_S - I$ *is nonexpansive.*

The following property will be utilized repeatedly.

**Fact 2.4** *Suppose that* $S$ *is a nonempty closed convex set in* $X$ *and that* $z \in X$. *Then for every* $x \in X$, *we have* $P_{z+S}\, x = z + P_S(x - z)$.

*Proof.* Use (10). ☐

We record two additional auxiliary results.

**Fact 2.5** *Suppose that* $U$ *and* $V$ *are two nonempty closed convex sets in* $X$. *Suppose further that* $u \in U$ *and that* $v \in V$. *Then* $N_{U+V}(u + v) = N_U(u) \cap N_V(v)$.

*Proof.* See, e.g., [1, Section 4.6]. ☐

**Proposition 2.6** *Suppose that* $U$ *and* $V$ *are two nonempty closed convex sets in* $X$ *such that* $U \perp V$. *Then* $U + V$ *is closed and* $P_{U+V} = P_U + P_V$.

*Proof.* Suppose that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are sequences in $U$ and $V$, respectively, such that $(u_n + v_n)_{n\in\mathbb{N}}$ converges. For every $\{m,n\} \subset \mathbb{N}$, we have $\|(u_n+v_n)-(u_m+v_m)\|^2 = \|u_n-u_m\|^2+\|v_n-v_m\|^2$. Hence $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are both Cauchy sequences, since $(u_n + v_n)_{n\in\mathbb{N}}$ is. Thus $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are both convergent, which implies that $\lim_{n\in\mathbb{N}} u_n + v_n \in U + V$.

Now let $x \in X$, $u \in U$, and $v \in V$. Since $\{u - P_U x, -P_U x\}\perp\{v - P_V x, -P_V x\}$, Fact 2.3 implies that

$$\begin{aligned}\langle u + v - P_U x - P_V x \,|\, x - P_U x - P_V x\rangle &= \langle u - P_U x \,|\, x - P_U x\rangle + \langle u - P_U x \,|\, -P_V x\rangle \\ &\quad + \langle v - P_V x \,|\, x - P_V x\rangle + \langle v - P_V x \,|\, -P_U x\rangle \\ &= \langle u - P_U x \,|\, x - P_U x\rangle + \langle v - P_V x \,|\, x - P_V x\rangle \\ &\leq 0. \end{aligned} \tag{11}$$

Using Fact 2.3 again, it follows that $P_{U+V} x = P_U x + P_V x$. ☐

**Proposition 2.7** *Suppose that $c \in C$. Then $K = N_B(c) \cap \big( - N_A(c)\big) \subset (C - C)^{\perp}$.*

*Proof.* Using (5) and Fact 2.5, we deduce that

$$K = N_{B-A}(0) = N_{B+(-A)}\big(c + (-c)\big) = N_B(c) \cap N_{-A}(-c) = N_B(c) \cap \big( - N_A(c)\big). \tag{12}$$

Let $x \in K$. By (12), $\sup \langle x \,|\, B - c\rangle \leq 0$ and $\sup \langle -x \,|\, A - c\rangle \leq 0$. Since $C = A \cap B$, it follows that $\sup \langle x \,|\, C - c\rangle \leq 0$ and that $\sup \langle -x \,|\, C - c\rangle \leq 0$. Therefore, $x \in (C - c)^{\perp} = (C - C)^{\perp}$. ☐

**Theorem 2.8** *Suppose that $x \in X$ and that $c \in C$. Then $P_{C+K} x = P_C x + P_K (x - c)$.*

*Proof.* Set $L = C - C$. Then $C - c \subset L$ and, by Proposition 2.7, $K \subset L^{\perp}$. Corollary 2.4 and Proposition 2.6 yield

$$\begin{aligned} P_{C+K} x &= P_{c+((C-c)+K)} x \\ &= c + P_{(C-c)+K}(x - c) \\ &= c + P_{C-c}(x - c) + P_K(x - c) \\ &= P_C x + P_K(x - c), \end{aligned} \tag{13}$$

which completes the proof. ☐

**Corollary 2.9** *Suppose that $x \in X$. Then $P_B P_{C+K} x = P_C x$.*

*Proof.* Since $P_C x \in C$, Theorem 2.8 implies that $P_{C+K} x = P_C x + P_K(x - P_C x)$. Hence, using Proposition 2.7, we deduce that

$$P_{C+K} x - P_C x = P_K(x - P_C x) \in K \subset N_B(P_C x). \tag{14}$$

As $P_C x \in B$, this shows that $P_B P_{C+K} x = P_C x$. ☐

## 3 Main result

**Definition 3.1** *Suppose that* $(x, y, z) \in X^3$ *satisfies*

$$\big\{w \in X \mid \langle w - y \,|\, x - y\rangle \leq 0\big\} \cap \big\{w \in X \mid \langle w - z \,|\, y - z\rangle \leq 0\big\} \neq \varnothing. \tag{15}$$

*Set*

$$\pi = \langle x - y \,|\, y - z\rangle\,, \quad \mu = \|x - y\|^2, \quad \nu = \|y - z\|^2, \quad \rho = \mu\nu - \pi^2, \tag{16}$$

*and further*

$$Q(x, y, z) = \begin{cases} z, & \text{if } \rho = 0 \text{ and } \pi \geq 0; \\ x + (1 + \pi/\nu)(z - y), & \text{if } \rho > 0 \text{ and } \pi\nu \geq \rho; \\ y + (\nu/\rho)\big(\pi(x - y) + \mu(z - y)\big), & \text{if } \rho > 0 \text{ and } \pi\nu < \rho. \end{cases} \tag{17}$$

In [7], Haugazeau introduced the operator $Q$ as an explicit description of the projector onto the intersection of the two halfspaces defined in (15). He proved in [7, Théorème 3-2] that the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_0 = x$ and

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = Q\big(x, Q(x, y_n, P_B y_n), P_A Q(x, y_n, P_B y_n)\big) \tag{18}$$

converges strongly to $P_C x$. The next result is a particular application of the weak-to-strong convergence principle of [2], which will be used to reach the same conclusion for the proposed HAAR method.

**Fact 3.2** *Suppose that* $R\colon X \to X$ *is nonexpansive and that* $\operatorname{Fix} R \neq \varnothing$. *Suppose further that* $x \in X$ *and that* $(\lambda_n)_{n\in\mathbb{N}}$ *is a sequence in* $\left]0, \frac{1}{2}\right]$ *such that* $\inf_{n\in\mathbb{N}} \lambda_n > 0$. *Set* $y_0 = x$ *and define* $(y_n)_{n\in\mathbb{N}}$ *by*

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = Q\big(x, y_n, (1 - \lambda_n)y_n + \lambda_n R y_n\big). \tag{19}$$

*Then* $(y_n)_{n\in\mathbb{N}}$ *converges strongly to* $P_{\operatorname{Fix} R}\, x$.

*Proof.* This follows from [2, Corollary 6.6(ii)]. ☐

We are now in a position to introduce HAAR and to establish its convergence properties.

**Theorem 3.3 (HAAR)** *Suppose that* $x \in X$ *and that* $(\mu_n)_{n\in\mathbb{N}}$ *is a sequence in* $]0, 1]$ *such that* $\inf_{n\in\mathbb{N}} \mu_n > 0$. *Define the sequence* $(y_n)_{n\in\mathbb{N}}$ *generated by Haugazeau-like averaged alternating reflections by* $y_0 = x$ *and*

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = Q\big(x, y_n, (1 - \mu_n)y_n + \mu_n T y_n\big). \tag{20}$$

*Then* $(y_n)_{n\in\mathbb{N}}$ *converges strongly to* $P_{C+K}\, x$. *Moreover,* $(P_B y_n)_{n\in\mathbb{N}}$ *converges strongly to* $P_C x$.

*Proof.* Since the reflectors $R_A$ and $R_B$ are both nonexpansive (see Fact 2.3), so is their composition $R = R_A R_B$. Consequently, Fact 2.2 implies that $T$ is firmly nonexpansive. Moreover, by Fact 1.1, $\operatorname{Fix} R = \operatorname{Fix}\big(\frac{1}{2}R + \frac{1}{2}I\big) = \operatorname{Fix} T = C + K$. The statement about strong convergence of $(y_n)_{n\in\mathbb{N}}$ follows from Fact 3.2 (with $\lambda_n = \mu_n/2$). Since $y_n \to P_{C+K}x$ and $P_B$ is continuous, we further deduce that $(P_B y_n)_{n\in\mathbb{N}}$ converges strongly to $P_B P_{C+K}x$, which is equal to $P_C x$ by Corollary 2.9. ☐

**Remark 3.4** Several comments on Theorem 3.3 are in order.

(i) While a detailed numerical study of HAAR lies outside the scope of this paper, we nonetheless briefly discuss a numerical example demonstrating the potential of HAAR. As in [4, Section 1] for AAR, we consider the case when $X = \mathbb{R}^2$, $A = \big\{(\xi_1, \xi_2) \in X \mid \xi_2 \leq 0\big\}$, and $B = \big\{(\xi_1, \xi_2) \in X \mid \xi_1 \leq \xi_2\big\}$. Let $x = (8, 4)$ so that $P_C x = (0, 0)$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence constructed as in Theorem 3.3 with $\mu_n \equiv 1$. Then $y_0 = x = (8, 4)$, $y_1 = (6, -2)$, and $y_n = (0, 0)$, for every $n \in \{2, 3, \ldots\}$. Therefore, $P_B y_0 = (6, 6)$, $P_B y_1 = (2, 2)$, and $P_B y_n = (0, 0)$, for every $n \in \{2, 3, \ldots\}$. Thus HAAR converges to the solution $P_C x = (0, 0)$ in just two steps. On the other hand, Dykstra's algorithm, which is a popular best approximation method (see, e.g., [5, Chapter 9]), requires infinitely many steps in this setting.

(ii) It is important to monitor the sequence $(P_B y_n)_{n\in\mathbb{N}}$ rather than $(y_n)_{n\in\mathbb{N}}$ in order to approximate $P_C x$. Indeed, let $A = B = \{0\}$ and $x \in X \smallsetminus \{0\}$. Then $K = X$ and thus $(y_n)_{n\in\mathbb{N}}$ converges to $P_{C+K}x = P_X x = x$ but not to $P_C x = \{0\}$.

(iii) Theorem 3.3 can be utilized to handle best approximation problems with more than two sets. Suppose that $C_1, \ldots, C_J$ are finitely many closed convex sets in $X$ such that

$$C = C_1 \cap \cdots \cap C_J \neq \varnothing. \tag{21}$$

As in our corresponding discussion for AAR in [4, Section 4], we employ Pierra's product space technique [9]. Let us take $(\omega_j)_{1\leq j\leq J}$ in $]0, 1]$ such that $\sum_{j=1}^J \omega_j = 1$, and let us denote by $\mathbf{X}$ the Hilbert space $X^J$ with the inner product $\big((x_j)_{1\leq j\leq J}, (y_j)_{1\leq j\leq J}\big) \mapsto \sum_{j=1}^J \omega_j \langle x_j, y_j\rangle$. Set

$$\mathbf{A} = \big\{(x, \ldots, x) \in \mathbf{X} \colon x \in X\big\} \quad \text{and} \quad \mathbf{B} = C_1 \times \cdots \times C_J, \tag{22}$$

and observe that the set $C = \bigcap_{j=1}^J C_j$ in $X$ corresponds to the set $\mathbf{C} = \mathbf{A} \cap \mathbf{B}$ in $\mathbf{X}$. The projections of $\mathbf{x} = (x_j)_{1\leq j\leq J} \in \mathbf{X}$ onto $\mathbf{A}$ and $\mathbf{B}$ are given by

$$P_{\mathbf{A}}\mathbf{x} = \big(\textstyle\sum_{j=1}^J \omega_j x_j, \ldots, \sum_{j=1}^J \omega_j x_j\big) \quad \text{and} \quad P_{\mathbf{B}}\mathbf{x} = (P_{C_1}x_1, \ldots, P_{C_J}x_J), \tag{23}$$

respectively. Thus we have explicit formulae for $R_{\mathbf{A}} = 2P_{\mathbf{A}} - \mathbf{I}$ and $R_{\mathbf{B}} = 2P_{\mathbf{B}} - \mathbf{I}$, where $\mathbf{I}$ denotes the identity operator on $\mathbf{X}$. Let

$$\mathbf{T} = \tfrac{1}{2}(R_{\mathbf{A}} R_{\mathbf{B}} + \mathbf{I}), \tag{24}$$

let $x \in X$, and set $\mathbf{y}_0 = (x, x, \ldots, x) \in \mathbf{X}$. Define the sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ recursively by

$$\mathbf{y}_{n+1} = \mathbf{Q}(\mathbf{y}_0, \mathbf{y}_n, \mathbf{T}\mathbf{y}_n), \tag{25}$$

where $\mathbf{Q}$ is defined on $\mathbf{X}^3$ analogously to how $Q$ is defined on $X^3$ in Definition 3.1. Then Theorem 3.3 (with $\mu_n \equiv 1$) implies that $(P_{\mathbf{B}}\mathbf{y}_n)_{n\in\mathbb{N}}$ converges strongly $P_{\mathbf{C}}\mathbf{y}_0 = (P_C x, \ldots, P_C x)$. Consequently, $(P_{\mathbf{A}}P_{\mathbf{B}}\mathbf{y}_n)_{n\in\mathbb{N}}$ converges strongly to $P_{\mathbf{C}}\mathbf{y}_0$ as well. Since this last sequence lies in $\mathbf{A}$, we identify it with some sequence $(a_n)_{n\in\mathbb{N}}$ in $X$ via $(P_{\mathbf{A}}P_{\mathbf{B}}\mathbf{y}_n)_{n\in\mathbb{N}} = (a_n, \ldots, a_n)_{n\in\mathbb{N}}$. Altogether, the sequence $(a_n)_{n\in\mathbb{N}}$ converges strongly to $P_C x$.

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