Abstract

A new iterative method for finding the projection onto the intersection of two closed convex sets in a Hilbert space is presented. It is a Haugazeau-like modification of a recently proposed averaged alternating reflections method which produces a strongly convergent sequence.

Keywords: Best approximation problem, convex set, projection, strong convergence.

1 Introduction

Throughout this paper,

\[ X \text{ is a real Hilbert space with inner product } \langle \cdot | \cdot \rangle \text{ and induced norm } \| \cdot \|, \]

and

\[ A \text{ and } B \text{ are two closed convex sets in } X \text{ such that } C = A \cap B \neq \emptyset. \]

Given a point \( x \in X \), the problem under consideration is the best approximation problem

\[ \text{find } c \in C \text{ such that } \| x - c \| = \inf \| x - C \|. \]

This problem, which was already studied by von Neumann in the 1930s in this general Hilbert space setting, is of fundamental importance in applied mathematics (see [5] for historical references, recent applications, algorithms, and further references).

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*Mathematics, Irving K. Barber School, UBC Okanagan, Kelowna, B.C. V1V 1V7, Canada. E-mail: heinz.bauschke@ubc.ca.

†Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie – Paris 6, 75005 Paris, France. E-mail: plc@math.jussieu.fr, 33+1 4427 6319 (Voice), 33+1 4427 7200 (Fax).

‡Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716-2553, U.S.A. E-mail: rluke@math.udel.edu.
The aim of this note is to present a new strongly convergent method — termed *Haugazeau-like Averaged Alternating Reflections* (HAAR) — for finding the solution of (3) iteratively. This algorithm is a modification of the *Averaged Alternating Reflections* (AAR) scheme, which we recently introduced in [4]. To describe AAR, we require some notation from convex analysis. Given any nonempty closed convex set $S$ in $X$, denote the projector (best approximation operator) onto $S$ by $P_S$. Further, let $I$ be the identity operator on $X$ and let $R_S = 2P_S - I$ be the reflector with respect to $S$. We recall that the normal cone to $S$ at $x \in S$ is defined by $N_S(x) = \{ x^* \in X \mid (\forall s \in S) \langle x^* | s - x \rangle \leq 0 \}$.

Both AAR and HAAR rely upon the operator

$$T = \frac{1}{2}R_AR_B + \frac{1}{2}I,$$

and their analyses require the nonempty closed convex cone

$$K = N_{B-A}(0).$$

We are now ready to describe AAR and its asymptotic behavior (see also [4] for background).

**Fact 1.1 (AAR)** Suppose that $x \in X$. Then the sequence of averaged alternating reflections (AAR) $(T^n x)_{n \in \mathbb{N}}$ converges weakly to a point in

$$\text{Fix } T = \{ z \in X \mid Tz = z \} = C + K.$$  

Moreover, the sequence $(P_BT^n x)_{n \in \mathbb{N}}$ is bounded and each of its weak cluster points lies in $C$.

**Proof.** The identity (6) was proved in [4, Corollary 3.9]. The statements regarding weak convergence and weak cluster points follows from [8, Theorem 1] applied to the normal cone operators $N_A$ and $N_B$. (See also [3, Fact 5.9] and [4, Theorem 3.13(ii)].) □

Fact 1.1 implies that the weak cluster points of the sequence $(P_BT^n x)_{n \in \mathbb{N}}$ solve the convex feasibility problem

$$\text{find } c \in C.$$  

Although such points solve (7), they may nonetheless be neither strong cluster points nor the solution of the best approximation problem (3) (see [4, Section 1] for a counterexample). These shortcomings of AAR motivated us to look for variants of AAR with better convergence properties.

In Section 2, we investigate the relative geometry of the sets $A$ and $B$, culminating in the formula

$$P_B P_{C+K} = P_C$$

(see Corollary 2.9). This identity, Fact 1.1, and a consequence of the weak-to-strong convergence principle [2] lead in Section 3 to the precise formulation of HAAR. A crucial ingredient of HAAR is Haugazeau’s [7] explicit projector onto the intersection of two halfspaces. Our main result (Theorem 3.3) guarantees strong convergence to the nearest point in $C$, i.e., to the solution of (3).

# 2 Relative geometry of two sets

We shall utilize the following notions from fixed point theory; see, e.g., [6].
**Definition 2.1** Suppose that \( R : X \to X \). Then:

(i) \( R \) is firmly nonexpansive, if

\[
(\forall x \in X)(\forall y \in X) \quad \|Rx - Ry\|^2 + \|(I - R)x - (I - R)y\|^2 \leq \|x - y\|^2.
\]  

(ii) \( R \) is nonexpansive, if

\[
(\forall x \in X)(\forall y \in X) \quad \|Rx - Ry\| \leq \|x - y\|.
\]

It is well known, for example, that the projector onto a nonempty closed convex set is firmly nonexpansive.

**Fact 2.2** Suppose that \( R : X \to X \). Then \( R \) is firmly nonexpansive if and only if \( 2R - I \) is nonexpansive.

*Proof.* See [6, Theorem 12.1]. \( \square \)

**Fact 2.3** Suppose that \( S \) is a nonempty closed convex set in \( X \) and that \( x \in X \). Then there exists a unique point \( P_S x \in S \) such that \( \|x - P_S x\| = \inf \|x - S\| \). The point \( P_S x \) is characterized by

\[
P_S x \in S \quad \text{and} \quad (\forall s \in S) \quad \langle s - P_S x \mid x - P_S x \rangle \leq 0.
\]

The induced operator \( P_S : X \to S \) : \( x \mapsto P_S x \) is called the projector onto \( S \); it is firmly nonexpansive and consequently, the reflector \( R_S = 2P_S - I \) is nonexpansive.

The following property will be utilized repeatedly.

**Fact 2.4** Suppose that \( S \) is a nonempty closed convex set in \( X \) and that \( z \in X \). Then for every \( x \in X \), we have \( P_z + S x = z + P_S(x - z) \).

*Proof.* Use (10). \( \square \)

We record two additional auxiliary results.

**Fact 2.5** Suppose that \( U \) and \( V \) are two nonempty closed convex sets in \( X \). Suppose further that \( u \in U \) and that \( v \in V \). Then \( N_{U + V}(u + v) = N_U(u) \cap N_V(v) \).

*Proof.* See, e.g., [1, Section 4.6]. \( \square \)

**Proposition 2.6** Suppose that \( U \) and \( V \) are two nonempty closed convex sets in \( X \) such that \( U \perp V \). Then \( U + V \) is closed and \( P_{U + V} = P_U + P_V \).
Proof. Suppose that \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\) are sequences in \(U\) and \(V\), respectively, such that \(u_n + v_n\) converges. For every \(\{m, n\} \subset \mathbb{N}\), we have 
\[
\|u_n + v_n - (u_m + v_m)\|^2 = \|u_n - u_m\|^2 + \|v_n - v_m\|^2.
\]
Hence \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\) are both Cauchy sequences, since \((u_n + v_n)_{n \in \mathbb{N}}\) is. Thus \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\) are both convergent, which implies that \(\lim_{n \to \infty} u_n + v_n \in U + V\).

Now let \(x \in X\), \(u \in U\), and \(v \in V\). Since \(\{u - P_U x, -P_U x\} \perp \{v - P_V x, -P_V x\}\), Fact 2.3 implies that
\[
\langle u + v - P_U x - P_V x \mid x - P_U x - P_V x \rangle = \langle u - P_U x \mid x - P_U x \rangle + \langle v - P_V x \mid x - P_V x \rangle
\]
\[
= \langle u - P_U x \mid x - P_U x \rangle + \langle v - P_V x \mid x - P_V x \rangle
\]
\[
\leq 0.
\]
Using Fact 2.3 again, it follows that \(P_{U+V} x = P_U x + P_V x\). □

**Proposition 2.7** Suppose that \(c \in C\). Then \(K = N_B(c) \cap (-N_A(c)) \subset (C - C)\perp\).

**Proof.** Using (5) and Fact 2.5, we deduce that
\[
K = N_{B-A}(0) = N_{B+(-A)}(c + (-c)) = N_B(c) \cap N_A(-c) = N_B(c) \cap (-N_A(c)).
\]
Let \(x \in K\). By (12), \(\sup \langle x \mid B - c \rangle \leq 0\) and \(\sup \langle -x \mid A - c \rangle \leq 0\). Since \(C = A \cap B\), it follows that \(\sup \langle x \mid C - c \rangle \leq 0\) and that \(\sup \langle -x \mid C - c \rangle \leq 0\). Therefore, \(x \in (C - c)\perp = (C - C)\perp\). □

**Theorem 2.8** Suppose that \(x \in X\) and that \(c \in C\). Then \(P_{C+K} x = P_C x + P_K(x - c)\).

**Proof.** Set \(L = C - C\). Then \(C - c \subset L\) and, by Proposition 2.7, \(K \subset L\perp\). Corollary 2.4 and Proposition 2.6 yield
\[
P_{C+K} x = P_{c+((C-c)+K)} x
\]
\[
= c + P_{(C-c)+K}(x - c)
\]
\[
= c + P_{C-c}(x - c) + P_K(x - c)
\]
\[
= P_C x + P_K(x - c),
\]
which completes the proof. □

**Corollary 2.9** Suppose that \(x \in X\). Then \(P_B P_{C+K} x = P_C x\).

**Proof.** Since \(P_C x \in C\), Theorem 2.8 implies that \(P_{C+K} x = P_C x + P_K(x - P_C x)\). Hence, using Proposition 2.7, we deduce that
\[
P_{C+K} x - P_C x = P_K(x - P_C x) \in K \subset N_B(P_C x).
\]
As \(P_C x \in B\), this shows that \(P_B P_{C+K} x = P_C x\). □
3 Main result

Definition 3.1 Suppose that \((x, y, z) \in X^3\) satisfies

\[
\{ w \in X \mid \langle w - y \mid x - y \rangle \leq 0 \} \cap \{ w \in X \mid \langle w - z \mid y - z \rangle \leq 0 \} \neq \emptyset.
\] \hspace{1cm} (15)

Set

\[
\pi = \langle x - y \mid y - z \rangle, \quad \mu = \|x - y\|^2, \quad \nu = \|y - z\|^2, \quad \rho = \mu \nu - \pi^2,
\] \hspace{1cm} (16)

and further

\[
Q(x, y, z) = \begin{cases} 
z, & \text{if } \rho = 0 \text{ and } \pi \geq 0; \\
x + (1 + \pi/\nu)(z - y), & \text{if } \rho > 0 \text{ and } \pi \nu \geq \rho; \\
y + (\nu/\rho)(\pi(x - y) + \mu(z - y)), & \text{if } \rho > 0 \text{ and } \pi \nu < \rho. 
\end{cases}
\] \hspace{1cm} (17)

In [7], Haugazeau introduced the operator \(Q\) as an explicit description of the projector onto the intersection of the two halfspaces defined in (15). He proved in [7, Théorème 3-2] that the sequence \((y_n)_{n \in \mathbb{N}}\) defined by \(y_0 = x\) and

\[
(\forall n \in \mathbb{N}) \quad y_{n+1} = Q(x, y_n, (1 - \lambda_n)y_n + \lambda_n R y_n)
\] \hspace{1cm} (18)

converges strongly to \(P_{C\times} x\). The next result is a particular application of the weak-to-strong convergence principle of [2], which will be used to reach the same conclusion for the proposed HAAR method.

Fact 3.2 Suppose that \(R: X \to X\) is nonexpansive and that \(\text{Fix} R \neq \emptyset\). Suppose further that \(x \in X\) and that \((\lambda_n)_{n \in \mathbb{N}}\) is a sequence in \([0, 1/2]\) such that \(\inf_{n \in \mathbb{N}} \lambda_n > 0\). Set \(y_0 = x\) and define \((y_n)_{n \in \mathbb{N}}\) by

\[
(\forall n \in \mathbb{N}) \quad y_{n+1} = Q(x, y_n, (1 - \lambda_n)y_n + \lambda_n R y_n).
\] \hspace{1cm} (19)

Then \((y_n)_{n \in \mathbb{N}}\) converges strongly to \(P_{\text{Fix} R} x\).

Proof. This follows from [2, Corollary 6.6(ii)]. ∎

We are now in a position to introduce HAAR and to establish its convergence properties.

Theorem 3.3 (HAAR) Suppose that \(x \in X\) and that \((\mu_n)_{n \in \mathbb{N}}\) is a sequence in \([0, 1]\) such that \(\inf_{n \in \mathbb{N}} \mu_n > 0\). Define the sequence \((y_n)_{n \in \mathbb{N}}\) generated by Haugazeau-like averaged alternating reflections by \(y_0 = x\) and

\[
(\forall n \in \mathbb{N}) \quad y_{n+1} = Q(x, y_n, (1 - \mu_n)y_n + \mu_n T y_n).
\] \hspace{1cm} (20)

Then \((y_n)_{n \in \mathbb{N}}\) converges strongly to \(P_{C + K} x\). Moreover, \((P_{B y_n})_{n \in \mathbb{N}}\) converges strongly to \(P_{C\times} x\).
Several comments on Theorem 3.3 are in order.

Remark 3.4

(i) While a detailed numerical study of HAAR lies outside the scope of this paper, we nonetheless briefly discuss a numerical example demonstrating the potential of HAAR. As in [4, Section 1] for AAR, we consider the case when \( X = \mathbb{R}^2 \), \( A = \{(\xi_1, \xi_2) \in X \mid \xi_2 \leq 0\} \), and \( B = \{(\xi_1, \xi_2) \in X \mid \xi_1 \leq \xi_2\} \). Let \( x = (8, 4) \) so that \( P_C x = (0, 0) \). Let \((y_n)_{n \in \mathbb{N}} \) be a sequence constructed as in Theorem 3.3 with \( \mu_n = 1 \). Then \( y_0 = x = (8, 4), y_1 = (6, -2) \), and \( y_n = (0, 0) \), for every \( n \in \{2, 3, \ldots\} \). Therefore, \( P_{BY_0} = (6, 6), P_{BY_1} = (2, 2) \), and \( P_{BY_n} = (0, 0) \), for every \( n \in \{2, 3, \ldots\} \). Thus HAAR converges to the solution \( P_C x = (0, 0) \) in just two steps. On the other hand, Dykstra’s algorithm, which is a popular best approximation method (see, e.g., [5, Chapter 9]), requires infinitely many steps in this setting.

(ii) It is important to monitor the sequence \((P_{BY_n})_{n \in \mathbb{N}} \) rather than \((y_n)_{n \in \mathbb{N}} \) in order to approximate \( P_C x \). Indeed, let \( A = B = \{0\} \) and \( x \in X \setminus \{0\} \). Then \( K = X \) and thus \((y_n)_{n \in \mathbb{N}} \) converges to \( P_{C+K} x = P_X x = x \) but not to \( P_C x = \{0\} \).

(iii) Theorem 3.3 can be utilized to handle best approximation problems with more than two sets. Suppose that \( C_1, \ldots, C_J \) are finitely many closed convex sets in \( X \) such that \[
C = C_1 \cap \cdots \cap C_J \neq \emptyset.
\] (21)

As in our corresponding discussion for AAR in [4, Section 4], we employ Pierra’s product space technique [9]. Let us take \( (\omega_j)_{1 \leq j \leq J} \) in \([0, 1] \) such that \( \sum_{j=1}^{J} \omega_j = 1 \), and let us denote by \( X \) the Hilbert space \( X^J \) with the inner product \( (x_{j1 \leq j \leq J}, y_{j1 \leq j \leq J}) \mapsto \sum_{j=1}^{J} \omega_j \langle x_j, y_j \rangle \). Set \[
A = \{(x, \ldots, x) \in X : x \in X\} \quad \text{and} \quad B = C_1 \times \cdots \times C_J,
\] (22)
and observe that the set \( C = \bigcap_{j=1}^{J} C_j \) in \( X \) corresponds to the set \( C = A \cap B \) in \( X \). The projections of \( x = (x_{j1 \leq j \leq J}) \in X \) onto \( A \) and \( B \) are given by

\[
P_A x = (\sum_{j=1}^{J} \omega_j x_j, \ldots, \sum_{j=1}^{J} \omega_j x_J) \quad \text{and} \quad P_B x = (P_{C_1} x_1, \ldots, P_{C_J} x_J),
\] (23)

respectively. Thus we have explicit formulae for \( R_A = 2P_A - I \) and \( R_B = 2P_B - I \), where \( I \) denotes the identity operator on \( X \). Let \[
T = \frac{1}{2}(R_A R_B + I),
\] (24)

let \( x \in X \), and set \( y_0 = (x, x, \ldots, x) \in X \). Define the sequence \((y_n)_{n \in \mathbb{N}} \) recursively by \[
y_{n+1} = Q(y_0, y, Ty_n),
\] (25)

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where \( Q \) is defined on \( X^3 \) analogously to how \( Q \) is defined on \( X^3 \) in Definition 3.1. Then Theorem 3.3 (with \( \mu_n \equiv 1 \)) implies that \((PB_y^n)_{n\in\mathbb{N}}\) converges strongly \( P_{C}y_0 = (P_Cx, \ldots, P_Cx) \). Consequently, \((PA_PB_y^n)_{n\in\mathbb{N}}\) converges strongly to \( P_{C}y_0 \) as well. Since this last sequence lies in \( A \), we identify it with some sequence \((a_n)_{n\in\mathbb{N}}\) in \( X \) via \((PA_PB_y^n)_{n\in\mathbb{N}} = (a_n, \ldots, a_n)_{n\in\mathbb{N}}\). Altogether, the sequence \((a_n)_{n\in\mathbb{N}}\) converges strongly to \( P_{C}x \).

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References


