Fenchel duality, Fitzpatrick functions and the extension of firmly nonexpansive mappings

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Abstract

Recently, S. Reich and S. Simons provided a novel proof of the Kirszbraun-Valentine extension theorem using Fenchel duality and Fitzpatrick functions. In the same spirit, we provide a new proof of an extension result for firmly nonexpansive mappings with an optimally localized range.

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Throughout this paper, we assume that X is a real Hilbert space, with inner product $p = \langle \cdot \mid \cdot \rangle$ and induced norm $\|\cdot\|$, and we denote the identity mapping on X by Id. A mapping T from a subset D of X to X is called *firmly nonexpansive*, if

$$(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2; \tag{1}$$

equivalently [13, 14], if $2T - \mathrm{Id}$ is *nonexpansive* (Lipschitz continuous with constant 1), i.e.,

$$(\forall x \in D)(\forall y \in D) \quad \|(2T - \mathrm{Id})x - (2T - \mathrm{Id})y\| \leq \|x - y\|, \tag{2}$$

or if

$$(\forall x \in D)(\forall y \in D) \quad 0 \leq \langle Tx - Ty \mid (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle . \tag{3}$$

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Firmly nonexpansive mappings play an important role in various contexts, see, e.g., [1, 2, 3, 7, 8, 9, 10, 15, 17, 21, 22, 25]. The Kirszbraun-Valentine theorem (see, e.g., [5, 13, 16, 20, 26]) states that any nonexpansive mapping can be extended to a nonexpansive mapping defined on the whole space. A beautiful proof of this result, based on Fenchel duality and Fitzpatrick functions, was recently provided by Reich and Simons [23]. (For further applications of Fitzpatrick functions, see, e.g., [4, 24].) In this note, we refine their technique to obtain a new proof of an extension theorem for firmly nonexpansive mappings where the range of the extension is optimally localized. This extension theorem easily implies the Kirszbraun-Valentine result. Notation not explicitly defined in the following is standard in convex analysis, see, e.g., [27].

Definition 1 *Let D be a nonempty subset of X and let $T\colon D \to X$ be firmly nonexpansive. Then the associated* Fitzpatrick function [12] $\varphi = \varphi_T$ *is*

$$X \times X \to \left]-\infty, +\infty\right] : (x, y) \mapsto \sup_{d \in D} \langle x \,|\, d - Td\rangle + \langle y \,|\, Td\rangle - \langle Td \,|\, d - Td\rangle\,, \tag{4}$$

and we also set $G = G_T = \big\{(d - Td, Td) \mid d \in D\big\}$.

Proposition 2 *Let D be a nonempty subset of X, let $T\colon D \to X$ be firmly nonexpansive, and let x and y be in X. Then:*

(i) $\varphi = (\iota_G + p)^*$.

(ii) *The extension $\widetilde{T}\colon D \cup \{y\} \to X$ of T which maps y to x is still firmly nonexpansive if and only if $\varphi(y - x, x) \le p(y - x, x)$.*

(iii) *φ is convex, lower semicontinuous and proper.*

(iv) $\operatorname{conv} G \subset \operatorname{dom} \varphi^* \subset \overline{\operatorname{conv}}\, G \subset \overline{\operatorname{conv}}\, (\operatorname{Id} - T)(D) \times \overline{\operatorname{conv}}\, T(D)$.

(v) $p \le \varphi^*$.

Proof. Fix x and y in X. (i): For every $d \in D$, we have

$$\langle x \,|\, d - Td\rangle + \langle y \,|\, Td\rangle - \langle d - Td \,|\, Td\rangle = \langle (d - Td, Td) \,|\, (x, y)\rangle - (p + \iota_G)(d - Td, Td), \tag{5}$$

from which the identity follows by supremizing over $d \in D$. (ii): This is a consequence of (3). (iii): By (ii), $\varphi \le p$ on G. Hence φ is proper. The function φ is convex and lower semicontinuous, as it is a Fenchel conjugate by (i). (iv): This is clear, since $G = \operatorname{dom}(\iota_G + p)$ and $\varphi = (\iota_G + p)^*$. (v): In view of (3), $p(G - G) \subset [0, +\infty[$. Suppose that $(x, y) \in \operatorname{conv} G$, say it is a finite convex combination $(x, y) = \sum_{i \in I} \lambda_i (x_i, y_i)$ of elements in G. Then $\sum_{i \in I} \lambda_i p(x_i, y_i) = p(x, y) + \frac{1}{2} \sum_{i,j \in I} p((x_i, y_i) - (x_j, y_j)) \ge p(x, y)$, hence $p \le \operatorname{conv}(\iota_G + p)$. Since p is continuous, it follows that $p \le \overline{\operatorname{conv}}\, (\iota_G + p) = (\iota_G + p)^{**} = \varphi^*$. ■

Fact 3 (Fenchel duality) *Let Y be a real Hilbert space and $L\colon Y \to X$ be linear and continuous. Let $f\colon Y \to\ ]-\infty,+\infty]$ and $g\colon X \to\ ]-\infty,+\infty]$ be convex, lower semicontinuous, and proper such that g is continuous and finite at some point in $L\,\mathrm{dom}\, f$. Then*

$$\inf_{y\in Y} \big(f(y) + g(Ly)\big) = -\min_{x\in X} \big(f^*(-L^*x) + g^*(x)\big). \tag{6}$$

Proof. See, e.g., [27, Corollary 2.8.5]. ■

Theorem 4 *Let D be a nonempty subset of X, let $T\colon D \to X$ be firmly nonexpansive, and let $y \in X$. Then T has a firmly nonexpansive extension $\widetilde{T}\colon D \cup \{y\} \to \overline{\mathrm{conv}}\, T(D)$.*

Proof. Set $\varphi = \varphi_T$ and $C = \overline{\mathrm{conv}}\, T(D)$, and assume first that $y = 0$. In view of Proposition 2(ii), we must show that

$$\min_{x\in X} \varphi(x,-x) + \|x\|^2 + \iota_C(x) \le 0. \tag{7}$$

Set $f\colon X\times X \to\ ]-\infty,+\infty] : (x^*,y^*) \mapsto \frac{1}{2}\varphi^*(2x^*,2y^*)$ so that $f^* = \frac{1}{2}\varphi$, and let $j = \frac{1}{2}\|\cdot\|^2$. Now set $g = (j+\iota_C)^*$ and observe (using [19]) that $g = j\Box\iota_C^* = j - (j\Box\iota_C) = j - \frac{1}{2}d_C^2$, where $\Box$ denotes the infimal convolution and d_C the distance function. Set further $L\colon X\times X \to X\colon (x^*,y^*) \mapsto y^*-x^*$. We claim that $\inf_{(x^*,y^*)\in X\times X} f(x^*,y^*)+g(L(x^*,y^*)) \ge 0$. Indeed, pick $(x^*,y^*) \in \mathrm{dom}\, f$. By Proposition 2(iv), $(2x^*,2y^*) \in \mathrm{dom}\,\varphi^* \subset X\times C$ and hence $2y^* \in C$. Using Proposition 2(v), we deduce that

$$\begin{aligned}
0 &= 4\,\langle x^* \mid y^*\rangle + \|y^*-x^*\|^2 - \|x^*+y^*\|^2 && (8)\\
&= p(2x^*,2y^*) + \|y^*-x^*\|^2 - \|(y^*-x^*) - 2y^*\|^2 && (9)\\
&\le \varphi^*(2x^*,2y^*) + \|y^*-x^*\|^2 - d_C^2(y^*-x^*) && (10)\\
&= 2\big(f(x^*,y^*) + g(y^*-x^*)\big) && (11)\\
&= 2\big(f(x^*,y^*) + g(L(x^*,y^*))\big). && (12)
\end{aligned}$$

Hence $\inf(f+gL)(X\times X) \ge 0$ and, since $\mathrm{dom}\, g = X$, Fact 3 now implies that

$$\min_{x\in X} f^*(-L^*x) + g^*(x) \le 0. \tag{13}$$

Since $f^* = \frac{1}{2}\varphi$, $g^* = j + \iota_C$, and $L^*\colon X \to X\times X\colon x \mapsto (-x,x)$, we see that (13) clearly yields (7).

Now assume that $y \neq 0$. Let $E = D - y$ and define $U\colon E \to X\colon z \mapsto T(z+y)$. Then U is firmly nonexpansive and $U(E) = T(D)$. By what we just proved, there exists an extension $\widetilde{U}\colon E\cup\{0\} \to \overline{\mathrm{conv}}\, U(E) = \overline{\mathrm{conv}}\, T(D)$. Therefore, $\widetilde{T}\colon D\cup\{y\} \to \overline{\mathrm{conv}}\, T(D)\colon z \mapsto \widetilde{U}(z-y)$ is as required. ■

Corollary 5 *Let D be a nonempty subset of X and let $T\colon D \to X$ be firmly nonexpansive. Then T has a firmly nonexpansive extension $\widetilde{T}\colon X \to \overline{\operatorname{conv}}\, T(D)$.*

Proof. Let $\mathcal{M}$ be the set of all pairs (U, E), where $D \subset E \subset X$ and $U\colon E \to \overline{\operatorname{conv}}\, T(D)$ is a firmly nonexpansive extension of T. Partially order $\mathcal{M}$ via $(U_1, E_1) \preceq (U_2, E_2)$ if $E_1 \subset E_2$ and U_2 extends U_1. Zorn's lemma guarantees the existence of a maximal element $(\widetilde{T}, \widetilde{D})$. Now Theorem 4 shows that $\widetilde{D} = X$. ■

Remark 6 (range localization is optimal) The conclusion that the range of the extension $\widetilde{T}$ lie in the *closed convex hull* of $T(D)$ cannot be improved upon in general. Indeed, let D be a nonempty subset of X, let T be $\operatorname{Id}|_D$, and let $\widehat{T}\colon X \to X$ be any firmly nonexpansive extension of T. Then $D = \operatorname{Fix} T \subset \operatorname{Fix} \widehat{T}$, and the last set is closed and convex [13, 14]. Hence $C = \overline{\operatorname{conv}}\, T(D) = \overline{\operatorname{conv}}\, D \subset \operatorname{Fix} \widehat{T} \subset \widehat{T}(X)$. In particular, let $\widetilde{T}\colon X \to C$ be any firmly nonexpansive extension of T as in Corollary 5. Then $\widetilde{T}(X) = C$ and $\widetilde{T}|C = \operatorname{Id}|_C$; therefore, $\widetilde{T}$ is the projector onto C.

Corollary 7 (Kirszbraun-Valentine) *Let D be a nonempty subset of X and let $N\colon D \to X$ be nonexpansive. Then N has a nonexpansive extension $\widetilde{N}\colon X \to \overline{\operatorname{conv}}\, N(D)$.*

Proof. (See also [13, 16, 20, 26] for different proofs and related results.) Let $T = \frac{1}{2}\operatorname{Id}|_D + \frac{1}{2}N$, which is firmly nonexpansive. Corollary 5 guarantees a firmly nonexpansive extension $\widetilde{T}\colon X \to \overline{\operatorname{conv}}\, T(D)$. Let P be the (firmly) nonexpansive projector onto $\overline{\operatorname{conv}}\, N(D)$. Then $\widetilde{N} = P \circ (2\widetilde{T} - \operatorname{Id})$ is as required. ■

Remark 8 We do not know whether it is possible to deduce Corollary 5 from Corollary 7. The following technique of going back and forth between firmly nonexpansive and nonexpansive mappings, utilized in the proof of Corollary 7, does not work in reverse. Let D be a nonempty subset of X and $T\colon D \to X$ be firmly nonexpansive. Then $N = 2T - \operatorname{Id}|_D\colon D \to X$ is nonexpansive, and hence (by Corollary 7) it has an extension $\widetilde{N}\colon X \to \overline{\operatorname{conv}}\, N(D)$. It is tempting to conjecture that $\widetilde{T} = \frac{1}{2}\operatorname{Id} + \frac{1}{2}\widetilde{N}$ would be an extension of T as in Corollary 5. However, let us concretely consider $D = \{0\} \subset X$ and $T\colon D \to X\colon 0 \mapsto 0$. Then $N = 2T - \operatorname{Id}_D = T$ and so $\widetilde{N} \equiv 0$. Hence $\widetilde{T} = \frac{1}{2}\operatorname{Id}$, which does not satisfy $\widetilde{T}(X) \subset \overline{\operatorname{conv}}\, T(D) = \{0\}$.

Remark 9 The correspondence revealed by Minty [18] between (maximal) monotone operators and firmly nonexpansive mappings (with full domain) provides a reformulation of Theorem 4 in terms of monotone operators (see, e.g., [6, Theorem 2.1]), which in turn relates to the work of Debrunner and Flor [11]. The new proof presented here provides a convex-analytical handle on these results (see also [4]). Furthermore, in the present Hilbert space setting, Reich's [21, Lemma 2.1] shows that Corollary 5 is equivalent to the following result. *Let A be a monotone operator on X with nonempty graph. Then A has a maximal monotone extension $\widetilde{A}$ such that* $\overline{\text{conv}}\,\text{dom}\,A = \overline{\text{conv}}\,\text{dom}\,\widetilde{A}$. (In fact, his result is about *accretive* operators in general Banach spaces.) Using [21, Proposition 2.2], it follows that Corollary 5 actually characterizes Hilbert spaces among all Banach spaces of dimension not less than three.

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