A convex-analytical approach to extension results for n-cyclically monotone operators

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Abstract

Results concerning extensions of monotone operators have a long history dating back to a classical paper by Debrunner and Flor from 1964. In 1999, Voisei obtained refinements of Debrunner and Flor's work for n-cyclically monotone operators. His proofs rely on von Neumann's minimax theorem as well as Kakutani's fixed point theorem.

In this note, we provide a new proof of the central case of Voisei's work. This proof is more elementary and rooted in convex analysis. It utilizes only Fitzpatrick functions and Fenchel-Rockafellar duality.

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1 Introduction

Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } p = \langle \cdot, \cdot \rangle \text{ and induced norm } \|\cdot\|. \tag{1}$$

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Let $A\colon X \to 2^X$, where 2^X denotes the *power set* of X, and let $n \in \{2,3,\ldots\}$. Then A is *n-monotone* (or *n-cyclically monotone*) if

$$\left.\begin{array}{c}(a_1,a_1^*) \in \operatorname{gra} A,\\ \vdots \\ (a_n,a_n^*) \in \operatorname{gra} A\\ a_{n+1} := a_1\end{array}\right\} \quad \Rightarrow \quad \sum_{i=1}^{n} \langle a_{i+1} - a_i, a_i^* \rangle \leq 0, \tag{2}$$

where $\operatorname{gra} A := \big\{(x,x^*) \in X \times X \mid x^* \in Ax\big\}$ denotes the *graph* of A. Any subset G of $X \times X$ can be uniquely identified with the graph of some operator $B\colon X \to 2^X$; it will convenient to say that the set G is *n-monotone* if the operator B is. Returning to A, we note that 2-monotonicity simplifies to

$$(\forall (x,x^*) \in \operatorname{gra} A)(\forall (y,y^*) \in \operatorname{gra} A) \quad \langle x-y, x^*-y^* \rangle \geq 0, \tag{3}$$

i.e., to ordinary monotonicity. The operator A is *maximal n-monotone* if A is n-monotone and no proper extension (in the sense of inclusion of graphs) of A is n-monotone. Thanks to Zorn's Lemma, every n-monotone operator admits a maximal n-monotone extension. If A is maximal 2-monotone, then A is *maximal monotone*. Maximal monotone operators play an important role in modern analysis and optimization; see, e.g., the books [6, 7, 20, 21, 22, 27]. The following function $F_{A,n}\colon X \times X \to [-\infty,+\infty]$ is useful in the study of n-monotonicity [1, 3]. The *Fitzpatrick function* of order n associated with A evaluated at $(x,x^*) \in X \times X$ is

$$F_{A,n}(x,x^*) := \sup_{\substack{(a_1,a_1^*) \in \operatorname{gra} A,\\ \vdots \\ (a_{n-1},a_{n-1}^*) \in \operatorname{gra} A}} \left(\sum_{i=1}^{n-2} \langle a_{i+1} - a_i, a_i^* \rangle\right) + \langle x - a_{n-1}, a_{n-1}^* \rangle + \langle a_1, x^* \rangle. \tag{4}$$

The classical Fitzpatrick function [12] is $F_A := F_{A,2}$, i.e.,

$$F_A\colon X \times X \to [-\infty,+\infty] : (x,x^*) \mapsto \sup_{(a,a^*) \in \operatorname{gra} A} \langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle. \tag{5}$$

Fitzpatrick functions have turned out be powerful tools in modern monotone operator theory; see, e.g., [2, 4, 5, 8, 9, 10, 13, 15, 16, 18, 19, 23, 24, 26].

A basic result on extending monotone operators was provided by Debrunner and Flor [11] in 1964. A central case of their work states that if $A\colon X \to 2^X$ is monotone and $\operatorname{gra} A \neq \varnothing$, then for every $w \in X$, there exits $x \in \overline{\operatorname{conv}}\operatorname{dom} A$ such that $\{(x, w-x)\} \cup \operatorname{gra} A$ is monotone. (As is customary, $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$ is the *domain* of A and $\overline{\operatorname{conv}}\, S$ is the *closed convex hull* of a subset S of X.) Extension results for general n-monotone operators were presented more recently by Voisei [25] in 1999. The proof of his main result is relatively intricate, and it requires not only von Neumann's minimax theorem but also Kakutani's fixed point theorem.

The objective of this note is to provide a new, convex-analytical proof of the central case of Voisei's work. This novel proof utilizes Fitzpatrick functions and Fenchel-Rockafellar duality.

The remainder of the paper is organized as follows. Section 2 collects some known and some new auxiliary results, mainly on the Fitzpatrick function and its conjugate. These results will make the proof of our main result (Theorem 3.2) in Section 3 more transparent. Throughout, we utilize standard notation and results from convex analysis; see, e.g., [20, 27]. Specifically, the *interior*, *relative interior*, *closure*, *convex hull* of a set S is denoted by $\operatorname{int} S$, $\operatorname{ri} S$, $\overline{S}$, $\operatorname{conv} S$, respectively. The *inverse* of $A\colon X \to 2^X$ is denoted by A^{-1} and defined by $\operatorname{gra} A^{-1} := \big\{(x^*,x) \in X \times X \mid (x,x^*) \in \operatorname{gra} A\big\}$; the *range* of A, written $\operatorname{ran} A$, is precisely $\operatorname{dom} A^{-1}$. The *domain* of a function $f\colon X \to [-\infty,+\infty] = \mathbb{R}\cup\{-\infty,+\infty\}$ is $\operatorname{dom} f := \big\{x \in X \mid f(x) < +\infty\big\}$; $f^*\colon X \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}\langle x,x^*\rangle - f(x)$ is the *Fenchel conjugate* of f. Finally, the *indicator function* ι_S of a subset S of X maps points in S to 0, and points in $X \smallsetminus S$ to $+\infty$.

2 Auxiliary results

Recall that $]x,y] := \big\{(1-\lambda)x + \lambda y \mid 0 < \lambda \leq 1\big\}$ for any two vectors x and y in X.

Definition 2.1 *Let S be a subset of X. We define the* segmental hull of S *by*

$$\operatorname{seg} S := \big\{x \in X \mid (\exists\, y \in S)\ \ ]x,y] \subseteq S\big\}. \tag{6}$$

It is clear that if $S \subseteq X$, then $S \subseteq \operatorname{seg} S \subseteq \overline{S}$. The segmental hull of S, which is also known as the set of points that are linearly accessible from S [14, Section 2.D], will be convenient for the presentation of Corollary 2.8. We now provide conditions that guarantee that the segmental hull is as large as the closed convex hull.

Proposition 2.2 *Let S be a subset of X. Suppose that one of the following holds.*

(i) *S is closed and convex.*

(ii) $\varnothing \neq \operatorname{int}\operatorname{conv} S \subseteq S$.

(iii) *X is finite-dimensional and* $\operatorname{ri}\operatorname{conv} S \subseteq S$.

(iv) *X is finite-dimensional and S is convex.*

(v) *$S = C \smallsetminus \{c_1,\ldots,c_m\}$, where C is an open convex set and $\{c_1,\ldots,c_m\}$ is a finite subset of C.*

Then $\operatorname{seg} S = \overline{\operatorname{conv}}\, S$.

Proof. The inclusion $\operatorname{seg} S \subseteq \overline{\operatorname{conv}}\, S$ is trivial. If $S = \varnothing$, then $\operatorname{seg} S = \overline{\operatorname{conv}}\, S = \varnothing$. Thus, we assume that $S \neq \varnothing$. (i): In this case, $\operatorname{seg} S = \overline{\operatorname{conv}}\, S = S$. (ii): Take $x \in \overline{\operatorname{conv}}\, S$ and $y \in \operatorname{int}\operatorname{conv} S$. Then $]x,y] \subseteq \operatorname{int}\operatorname{conv} S$ by [14, Equation (11.1)]. Hence $]x,y] \subseteq S$ and thus $x \in \operatorname{seg} S$. (iii): Since $S \neq \varnothing$, we obtain $\operatorname{conv} S \neq \varnothing$. By [20, Theorem 6.2], $\operatorname{ri}\operatorname{conv} S \neq \varnothing$. Take $y \in \operatorname{ri}\operatorname{conv} S$ and $x \in \overline{\operatorname{conv}}\, S$. The assumption and [20, Theorem 6.1] imply that $]x,y] \subseteq \operatorname{ri}\operatorname{conv} S \subseteq S$. Thus $x \in \operatorname{seg} S$. (iv): Clear from (iii). (v): A consequence of [14, Lemma 11.A]. ■

Remark 2.3 For each of the items (ii), (iii), and (v) of Proposition 2.2, it is possible to construct a *nonconvex* set satisfying the condition of that item.

The following two results are known.

Fact 2.4 [1, Proposition 2.4] *Let* $A\colon X \to 2^X$ *and let* $n \in \{2, 3, \ldots\}$. *Then*

$$A \text{ is } n\text{-monotone} \Leftrightarrow F_{A,n} = p \text{ on } \operatorname{gra} A. \tag{7}$$

Fact 2.5 [1, Proposition 2.7] *Let* $A\colon X \to 2^X$ *be* n*-monotone for some* $n \in \{2, 3, \ldots\}$, *let* $(x, x^*) \in X \times X$, *and define* $B\colon X \to 2^X$ *via* $\operatorname{gra} B = \operatorname{gra} A \cup \{(x, x^*)\}$. *Then*

$$B \text{ is } n\text{-monotone} \Leftrightarrow F_{A,n}(x, x^*) \leq \langle x, x^*\rangle. \tag{8}$$

The next result localizes the domain of the Fenchel conjugate of the Fitzpatrick function.

Proposition 2.6 *Let* $A\colon X \to 2^X$. *Then*

$$\operatorname{conv} \operatorname{gra} A^{-1} \subseteq \operatorname{dom} F^*_{A,2} \subseteq \overline{\operatorname{conv}} \operatorname{gra} A^{-1} \subseteq \overline{\operatorname{conv}} \operatorname{ran} A \times \overline{\operatorname{conv}} \operatorname{dom} A \tag{9}$$

and

$$(\forall n \in \{3, 4, \ldots\}) \quad \operatorname{conv} \operatorname{ran} A \times \operatorname{conv} \operatorname{dom} A \subseteq \operatorname{dom} F^*_{A,n} \subset \overline{\operatorname{conv}} \operatorname{ran} A \times \overline{\operatorname{conv}} \operatorname{dom} A. \tag{10}$$

Proof. The inclusions (9) follow since $F_{A,2} = (p + \iota_{\operatorname{gra} A^{-1}})$ and hence $F^*_{A,2} = (p + \iota_{\operatorname{gra} A^{-1}})^{**}$; see also [12, Theorem 4.3]. Now fix $n \in \{3, 4, \ldots\}$, define $p_n\colon (X \times X)^{n-1} \to \mathbb{R}$ by

$$(x_1, x_1^*, \cdots, x_{n-1}, x^*_{n-1}) \mapsto \Big(\sum_{i=1}^{n-2} \langle x_i - x_{i+1}, x_i^*\rangle\Big) + \langle x_{n-1}, x^*_{n-1}\rangle, \tag{11}$$

set $G_n := (\operatorname{gra} A)^{n-1}$ and $L\colon X \times X \to (X \times X)^{n-1}\colon (y, y^*) \mapsto (y^*, 0, \ldots, 0, y)$. Then

$$F_{A,n} = (p_n + \iota_{G_n})^* \circ L \tag{12}$$

and $L^*\colon (X \times X)^{n-1} \to X \times X\colon (x_1, x_1^*, \ldots, x_{n-1}, x^*_{n-1}) \mapsto (x^*_{n-1}, x_1)$. It follows from [27, Theorem 2.3.1(ix)] that

$$F^*_{A,n} = \big((p_n + \iota_{G_n})^* \circ L\big)^* = \big(L^*(p_n + \iota_{G_n})^{**}\big)^{**}. \tag{13}$$

On the one hand, $\operatorname{dom}(L^*(p_n + \iota_{G_n})^{**}) = L^* \operatorname{dom}((p_n + \iota_{G_n})^{**})$ and $\operatorname{conv} G_n \subseteq \operatorname{dom}((p_n + \iota_{G_n})^{**}) \subseteq \overline{\operatorname{conv}}\, G_n$. On the other hand, $\operatorname{dom}(L^*(p_n + \iota_{G_n})^{**}) \subseteq \operatorname{dom}(L^*(p_n + \iota_{G_n})^{**})^{**} \subseteq \overline{\operatorname{dom}}\,(L^*(p_n + \iota_{G_n})^{**})$. Using also the continuity of L^*, we deduce altogether that

$$L^* \operatorname{conv} G_n \subseteq \operatorname{dom} F^*_{A,n} \subseteq \overline{L^* \operatorname{conv} G_n}. \tag{14}$$

Note that $\operatorname{conv} G_n = \operatorname{conv}(\operatorname{gra} A)^{n-1} = (\operatorname{conv} \operatorname{gra} A)^{n-1}$. Since $n - 1 \geq 2$, we obtain that $L^* \operatorname{conv} G_n = L^*((\operatorname{conv} \operatorname{gra} A)^{n-1}) = \operatorname{conv} \operatorname{ran} A \times \operatorname{conv} \operatorname{dom} A$. This and (14) imply (10). ■

Proposition 2.7 *Let $A\colon X \to 2^X$ be n-monotone for some $n \in \{2, 3, \ldots\}$. Then*

$$p \leq F^*_{A,n} \quad \text{on} \quad X \times \operatorname{dom} A. \tag{15}$$

Proof. Take $(y^*, x) \in X \times \operatorname{dom} A$. Then there exists x^* such that $(x, x^*) \in \operatorname{gra} A$. Then Fenchel-Young inequality yields $F_{A,n}(x, x^*) + F^*_{A,n}(y^*, x) \geq \langle (x, x^*), (y^*, x) \rangle = \langle x, y^* \rangle + \langle x, x^* \rangle$. On the other hand, Fact 2.4 implies $F_{A,n}(x, x^*) = \langle x, x^* \rangle$. Altogether, $F^*_{A,n}(y^*, x) \geq \langle x, y^* \rangle$. ■

Corollary 2.8 *Let $A\colon X \to 2^X$ be n-monotone for some $n \in \{2, 3, \ldots\}$, and suppose that* $\operatorname{seg} \operatorname{dom} A = \overline{\operatorname{conv}} \operatorname{dom} A$. *Then*

$$p \leq F^*_{A,n}. \tag{16}$$

Proof. Fix $(x^*, x) \in X \times X$. We must show that $p(x^*, x) \leq F^*_{A,n}(x^*, x)$. In view of Proposition 2.6, we assume that $x \in \overline{\operatorname{conv}} \operatorname{dom} A$. By hypothesis, there exists $y \in \operatorname{dom} A$ such that $]x, y] \subseteq \operatorname{dom} A$. Since $y \in \operatorname{dom} A$, there exists $y^* \in X$ such that $(y, y^*) \in \operatorname{gra} A$. We deduce from Proposition 2.6 that $(y^*, y) \in \operatorname{gra} A^{-1} \subseteq \operatorname{dom} F^*_{A,n}$. It follows that $](x^*, x), (y^*, y)] \subseteq X \times \operatorname{dom} A$. Using Proposition 2.7, we estimate

$$\begin{aligned} p(x^*, x) &= \lim_{\lambda \to 0^+} p\big((1-\lambda)(x^*, x) + \lambda(y^*, y)\big) \\ &\leq \varlimsup_{\lambda \to 0^+} F^*_{A,n}\big((1-\lambda)(x^*, x) + \lambda(y^*, y)\big) \\ &\leq \varlimsup_{\lambda \to 0^+} (1-\lambda) F^*_{A,n}(x^*, x) + \lambda F^*_{A,n}(y^*, y) \\ &= F^*_{A,n}(x^*, x). \end{aligned} \tag{17}$$

This completes the proof. ■

Remark 2.9 Let $A\colon X \to 2^X$, let $z \in X$, let $w \in X$, and let $n \in \{2, 3, \ldots\}$. Define $A_1\colon X \to 2^X\colon x \mapsto -w + Ax$ and $A_2\colon X \to 2^X\colon x \mapsto A(z + x)$. Then A is n-monotone $\Leftrightarrow$ A_1 is n-monotone $\Leftrightarrow$ A_2 is n-monotone, and $p \leq F^*_{A,n} \Leftrightarrow p \leq F^*_{A_1,n} \Leftrightarrow p \leq F^*_{A_2,n}$.

3 Extension results

The proof of our main result relies on the following fundamental fact.

Fact 3.1 (Fenchel-Rockafellar duality) [27, Corollary 2.8.5] *Let Z be a real Hilbert space, let $L\colon Z \to X$ be a continuous and linear operator, let $f\colon Z \to\ ]-\infty, +\infty]$ and $g\colon X \to\ ]-\infty, +\infty]$ be two proper, convex, and lower semicontinuous functions. Suppose that* $0 \in \operatorname{int}(\operatorname{dom} g - L \operatorname{dom} f)$. *Then*

$$\inf_{z \in Z} f(z) + g(Lz) = -\min_{x \in X} f^*(-L^* x) + g^*(x). \tag{18}$$

Theorem 3.2 *Let $A\colon X \to 2^X$ be n-monotone for some $n \in \{2,3,\ldots\}$. Suppose that* $\operatorname{gra} A \neq \varnothing$ *and that*

$$p \leq F^*_{A,n}. \tag{19}$$

Then for every $w \in X$, there exists $x \in \overline{\operatorname{conv}}\operatorname{dom} A$ such that $\{(x, w-x)\} \cup \operatorname{gra} A$ is n-monotone.

Proof. Set $C := \overline{\operatorname{conv}}\operatorname{dom} A$. In view of Fact 2.5, we must show that $(\forall w \in X)$ $(\exists\, x \in C)$ $F_{A,n}(x, w-x) \leq \langle x, w-x\rangle$, i.e., that

$$\min_{x\in X} F_{A,n}(x, w-x) + \|x\|^2 - \langle x, w\rangle + \iota_C(x) \leq 0. \tag{20}$$

First, assume that $w = 0$. In this case, it suffices to show that

$$\min_{x\in X} \tfrac{1}{2}F_{A,n}(x,-x) + \big(\tfrac{1}{2}\|x\|^2 + \iota_C(x)\big) \leq 0. \tag{21}$$

Set $q := \frac{1}{2}\|\cdot\|^2$, $d_C := \iota_C \,\Box\, \|\cdot\|$, $f\colon X \times X \to \left]-\infty,+\infty\right] : (y^*, y) \mapsto \frac{1}{2}F^*_{A,n}(2y^*, 2y)$, $g = (q + \iota_C)^* = q \,\Box\, \iota_C^* = q - (q \,\Box\, \iota_C) = q - \frac{1}{2}d_C^2$, and $L\colon X \times X \to X : (y^*, y) \mapsto y - y^*$. (Here "$\Box$" stands for the *infimal convolution* of two functions; see, e.g., [27].) We claim that

$$\inf_{z\in X\times X} f(z) + g(Lz) \geq 0. \tag{22}$$

To see this, fix $(y^*, y) \in X \times X$. By Proposition 2.6, $\operatorname{dom} F^*_{A,n} \subseteq \overline{\operatorname{conv}}\operatorname{ran} A \times \overline{\operatorname{conv}}\operatorname{dom} A$. We thus only have to consider the case when $2y \in C$. In view of (19), we obtain

$$\begin{aligned}
2\big(f(y^*,y) + g(L(y^*,y))\big) &= 2\big(f(y^*,y) + g(y-y^*)\big)\\
&= F^*_{A,n}(2y^*,2y) + \|y-y^*\|^2 - d_C^2(y-y^*)\\
&\geq F^*_{A,n}(2y^*,2y) + \|y-y^*\|^2 - \|(y-y^*)-2y\|^2\\
&\geq \langle 2y, 2y^*\rangle + \|y-y^*\|^2 - \|(y-y^*)-2y\|^2\\
&= 0.
\end{aligned} \tag{23}$$

This verifies (22). Since $\operatorname{dom} g = X$, Fact 3.1 implies that

$$\min_{x\in X} f^*(-L^*x) + g^*(x) \leq 0. \tag{24}$$

Because $f^* = \frac{1}{2}F_{A,n}$, $g^* = q + \iota_C$, and $L^*\colon X \to X \times X : x \mapsto (-x, x)$, we note that (24) is precisely (21). Now assume that $w \neq 0$ and set $B\colon X \to 2^X : x \mapsto -w + Ax$. By Remark 2.9, (19) holds for B. The above yields a point $(x,-x) \in C \times X$ such that $\{(x,-x)\} \cup \operatorname{gra} B$ is n-monotone. Therefore, $\{(x, w-x)\} \cup \operatorname{gra} A$ is n-monotone. ■

The proof of Theorem 3.2 is a refinement of the proof of [2, Theorem 4]. Let us record two important consequence of Theorem 3.2. The first one is classical and the central case of a result due to Debrunner and Flor [11].

Corollary 3.3 (Debrunner-Flor) (See also [7, Theorem 2.1] and [11].) *Let* $A\colon X \to 2^X$ *be monotone such that* $\operatorname{gra} A \neq \varnothing$. *Then*

$$(\forall w \in X)(\exists\, x \in \overline{\operatorname{conv}}\ \operatorname{dom} A) \quad 0 \leq \inf_{(y,y^*)\in \operatorname{gra} A} \langle x-y, (w-x)-y^*\rangle. \tag{25}$$

Proof. Let B be a maximal monotone extension of A. Then $F_A \leq F_B$ and hence

$$F_B^* \leq F_A^*. \tag{26}$$

On the other hand, Fitzpatrick proved that

$$p \leq F_B \tag{27}$$

and that

$$(\forall (y,y^*) \in X \times X) \quad F_B(y,y^*) \leq F_B^*(y^*,y), \tag{28}$$

see [12, Corollary 3.9 and Proposition 4.2], respectively. Altogether, $p \leq F_A^*$ and the result now follows from Theorem 3.2. ■

The second consequence is the central case of Voisei's paper.

Corollary 3.4 (Voisei) (See also [25, Theorem 3.1].) *Suppose that* X *is finite-dimensional and let* $A\colon X \to 2^X$ *be* n*-monotone for some* $n \in \{3,4,\ldots\}$. *Suppose further that* $\operatorname{gra} A \neq \varnothing$ *and that* $\operatorname{dom} A$ *is convex. Then for every* $w \in X$, *there exists* $x \in \overline{\operatorname{dom}}\, A$ *such that* $\{(x, w-x)\} \cup \operatorname{gra} A$ *is* n*-monotone.*

Proof. Proposition 2.2(iv) implies that $\operatorname{seg} \operatorname{dom} A = \overline{\operatorname{conv}}\ \operatorname{dom} A = \overline{\operatorname{dom}}\, A$. Hence $p \leq F_{A,n}^*$ by Corollary 2.8. The conclusion therefore follows from Theorem 3.2. ■

Remark 3.5 Several comments are in order.

(i) The proof of Corollary 3.3 shows that the hypothesis that $p \leq F_{A,n}^*$ is redundant when $n = 2$.

(ii) The work by Debrunner and Flor [11] and by Voisei is more general in that it features a single-valued, coercive, continuous, and monotone operator that is "hidden" in our work. Specifically, Voisei's [25, Theorem 3.1] states the following.

> *Suppose that* X *is finite-dimensional and let* $A\colon X \to 2^X$ *be* n*-monotone for some* $n \in \{2,3,\ldots\}$. *Suppose that* $0 \in \operatorname{dom} A$, *that* $\operatorname{dom} A$ *is convex, and that* $B\colon X \to X$ *is coercive, continuous, and monotone such that* $\overline{\operatorname{dom}}\, A \subseteq \operatorname{dom} B$. *Then there exists some* $x \in \overline{\operatorname{dom}}\, A$ *such that* $\{(x, -Bx)\} \cup \operatorname{gra} A$ *is* n*-monotone.*

(Voisei states his result in a finite-dimensional Banach space. However, after renorming if necessary, one may work in a finite-dimensional Hilbert space.) We now see that Corollary 3.4 corresponds to the case when $B\colon X \to X : x \mapsto x - w$. An analogous comment can be made for the work of Debrunner and Flor [11].

(iii) Following [5, Section 2.3], we referred to the case when $B\colon X \to X\colon x \mapsto x - w$ in item (ii) as the "central case". Let us now explain why. Firstly, we are unaware of any application of extension results where B is utilized in its generality. Secondly, the case is central since all applications we are aware of boil down to invoking a Minty-like [17] characterization of maximal monotonicity. Here is a typical application (see [25, Proposition 3.1]):

> *Let $A\colon X \to 2^X$ be such that $\operatorname{dom} A$ is convex, and let $n \in \{2, 3, \ldots\}$. Then A is maximal n-monotone $\Leftrightarrow$ A is n-monotone and maximal monotone.*

See also [1, Corollary 2.15] and [5, Section 2.3].

(iv) It is the special structure of the central case (i.e., $B\colon X \to X\colon x \mapsto x - w$ in item (ii)) that allows us to provide the new, convex-analytical proof of Theorem 3.2. Our approach appears to be limited to this central case (since the function $x \mapsto F_{A,n}(x, Bx)$ does not have to be convex in the general case). Voisei requires in the proofs of his more general work not only von Neumann's minimax theorem but also Kakutani's fixed point theorem. The latter result is outside the realm of convex analysis. However, Theorem 3.2 is applicable even when $\operatorname{dom} A$ is (mildly) nonconvex (see Proposition 2.2, Remark 2.3, and Corollary 2.8).

(v) The hypotheses that $\operatorname{dom} A$ be convex (in Corollary 3.4) and that $p \le F^*_{A,n}$ (in Theorem 3.2) are important. Indeed, [1, Example 2.16] guarantees the existence of a maximal 3-monotone operator $A\colon X \to 2^X$ such that A is not maximal monotone and such that $X = \mathbb{R}^2$. In view of Minty's characterization of maximal monotonicity [17], we deduce that the conclusion of Corollary 3.4 does not hold. Hence $\operatorname{dom} A$ is not convex, a fact verified directly in the proof of [1, Example 2.16]. Moreover, [1, Corollary 2.15] implies that $p \not\le F^*_{A,3}$.

(vi) We do not know whether or not it is possible to adapt the proof of Theorem 3.2 to the more general setting when X is a (possibly even reflexive) Banach space. The definition of the counterparts of f, g, and L in the proof of Theorem 3.2 does not appear to be obvious. For instance, L would have to be defined on $X^* \times X$; hence, it cannot be the simple subtraction $y - y^*$ we saw in the Hilbert space case. It is tempting to set $L(y^*, y) = Jy - y^*$, where J is the duality mapping of X. However, since J is in general *not* linear, it would then be impossible to utilize the Banach space version of Fact 3.1.

The proof of the following extension result utilizes a technique borrowed from Voisei's proof of [25, Theorem 3.2].

Theorem 3.6 *Let $A\colon X \to 2^X$ be n-monotone for some $n \in \{2, 3, \ldots\}$. Suppose that $0 \in \operatorname{dom} A$. Denote the family of finite-dimensional subspaces of X by $\mathcal{Y}$. For every $Y \in \mathcal{Y}$, let P_Y be the projector onto Y and set $A_Y := P_Y \circ A\colon Y \to 2^Y$. Furthermore, suppose that*

$$(\forall\, Y \in \mathcal{Y}) \quad p|_{Y\times Y} \le F^*_{A_Y,n}. \tag{29}$$

Then for every $w \in X$, there exists $x \in \overline{\operatorname{conv}}\, \operatorname{dom} A$ such that $\{(x, w - x)\} \cup \operatorname{gra} A$ is n-monotone.

Proof. After translating if necessary, we assume that $w = 0$. By Fact 2.5, it suffices to show that

$$(\exists\, x \in \overline{\text{conv}}\ \text{dom}\, A) \quad F_{A,n}(x, -x) \le -\|x\|^2. \tag{30}$$

Note that $\mathcal{Y}$ is partially ordered and directed by "$\subseteq$". Since $0 \in \text{dom}\, A$ and in view of (29), we deduce from Theorem 3.2 that for every $Y \in \mathcal{Y}$, there exists $x_Y \in \overline{\text{conv}}\ \text{dom}\, A_Y = \overline{\text{conv}}\,(Y \cap \text{dom}\, A) \subseteq Y \cap \overline{\text{conv}}\ \text{dom}\, A$ such that $\{(x_Y, -x_Y)\} \cup \text{gra}\, A_Y$ is n-monotone. This yields a net $(x_Y)_{Y\in\mathcal{Y}}$ which satisfies (see Fact 2.5)

$$(\forall\, Y \in \mathcal{Y})(\forall Z \in \mathcal{Y}) \quad Y \subseteq Z \;\Rightarrow\; F_{A_Y,n}(x_Z, -x_Z) \le F_{A_Z,n}(x_Z, -x_Z) \le \langle x_Z, -x_Z\rangle. \tag{31}$$

Take $a_0^* \in A0$. Using $(a_1, a_1^*) = \cdots = (a_{n-1}, a_{n-1}^*) = (0, a_0^*)$ in (4) implies that

$$(\forall\, Y \in \mathcal{Y})(\forall (y, y^*) \in X \times X) \quad \langle y, a_0^*\rangle \le F_{A_Y,n}(y, y^*) \tag{32}$$

Combining (31) and (32) yields $(\forall\, Y \in \mathcal{Y})$ $\langle x_Y, a_0^*\rangle \le F_{A_Y,n}(x_Y, -x_Y) \le \langle x_Y, -x_Y\rangle$ and hence $\|x_Y\|^2 = \langle x_Y, x_Y\rangle \le \langle x_Y, -a_0^*\rangle \le \|x_Y\|\|a_0^*\|$. Therefore, $\sup_{Y\in\mathcal{Y}} \|x_Y\| \le \|a_0^*\|$ and the net $(x_Y)_{Y\in\mathcal{Y}}$ is bounded. After passing to a subnet and relabeling if necessary, we assume that $(x_Y)_{Y\in\mathcal{Y}}$ converges weakly to some point $x \in \overline{\text{conv}}\ \text{dom}\, A$. Using (31), we estimate for every $Y \in \mathcal{Y}$

$$F_{A_Y,n}(x, -x) \le \varliminf_{Z\in\mathcal{Z}} F_{A_Y,n}(x_Z, -x_Z) \le \varliminf_{Z\in\mathcal{Z}} \langle x_Z, -x_Z\rangle \le -\varliminf_{Z\in\mathcal{Z}} \|x_Z\|^2 \le -\|x\|^2. \tag{33}$$

Since $\sup_{Y\in\mathcal{Y}} F_{A_Y,n} = F_{A,n}$, we obtain (30). ■

Analogously to the proof of Corollary 3.4, with Remark 2.9 in mind, we may deduce the following result from Theorem 3.6.

Corollary 3.7 (See also [25, Theorem 3.2]) *Let $A\colon X \to 2^X$ be n-monotone for some $n \in \{2, 3, \ldots\}$. Suppose that* $\text{gra}\, A \neq \varnothing$ *and that* $\text{dom}\, A$ *is convex. Then for every $w \in X$, there exists $x \in \overline{\text{dom}}\, A$ such that $\{(x, w - x)\} \cup \text{gra}\, A$ is n-monotone.*

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