To appear in SIOPT, accepted November 2006

FITZPATRICK FUNCTIONS AND CONTINUOUS LINEAR MONOTONE OPERATORS*

HEINZ H. BAUSCHKE†, JONATHAN M. BORWEIN‡, AND XIANFU WANG§

Abstract. The notion of a maximal monotone operator is crucial in optimization as it captures both the subdifferential operator of a convex, lower semicontinuous, and proper function and any (not necessarily symmetric) continuous linear positive operator. It was recently discovered that most fundamental results on maximal monotone operators allow simpler proofs utilizing Fitzpatrick functions.

In this paper, we study Fitzpatrick functions of continuous linear monotone operators defined on a Hilbert space. A novel characterization of skew operators is presented. A result by Brézis and Haraux is reproved using the Fitzpatrick function. We investigate the Fitzpatrick function of the sum of two operators, and we show that a known upper bound is actually exact in finite-dimensional and more general settings. Cyclic monotonicity properties are also analyzed, and closed forms of the Fitzpatrick functions of all orders are provided for all rotators in the Euclidean plane.

Key words. Cyclic monotonicity, Fitzpatrick family, Fitzpatrick function, linear operator, maximal monotone operator, Moore-Penrose inverse, paramonotone operator, rotator.

AMS subject classifications. 47H05, 47B25, 47B65, 90C25.

PII. XXXXX

1. Introduction.

Throughout this paper, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle \text{ and induced norm } \|\cdot\|. \tag{1.1}$$

Recall that a set-valued operator $A\colon X \to 2^X$ is *monotone* if

$$\left.\begin{array}{l}(x,u)\in \operatorname{gra} A\\ (y,v)\in \operatorname{gra} A\end{array}\right\} \quad\Rightarrow\quad \langle x-y, u-v\rangle \geq 0, \tag{1.2}$$

where $\operatorname{gra} A = \big\{(x,u) \in X\times X \mid u \in Ax\big\}$ denotes the *graph* of A. The notion of a monotone operator is central to modern optimization and analysis [9, 10, 33, 34, 35, 36, 43]. Of particular importance are *maximal monotone operators*, i.e., monotone operators with graphs that cannot be enlarged without destroying monotonicity. Recently, several fundamental results on monotone operators have found — sometimes dramatically simpler — new proofs by utilizing *Fitzpatrick functions* [8, 9, 29, 39, 41, 42]. The Fitzpatrick function was first introduced by S. Fitzpatrick to study monotone operators via convex analysis [17]; see also [2, 6, 12, 13, 14, 15, 18, 24, 25, 31, 37, 38, 40]. The key classes of maximal monotone operators are subdifferential operators of proper,

*SIOPT to appear
http://www.siam.org/journals/

†Mathematics, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada (heinz.bauschke@ubc.ca). Research partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program.

‡ Faculty of Computer Science, Dalhousie University, 6050 University Avenue, Halifax, Nova Scotia B3H 1W5, Canada (jborwein@cs.dal.ca). Research partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program.

§Mathematics, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada (shawn.wang@ubc.ca). Research partially supported by the Natural Sciences and Engineering Research Council of Canada.

lower semicontinuous, and convex functions [32] and continuous, linear, monotone operators. The former class is very well understood [36] while the latter class is the topic of this paper.

The aim of this paper is to study Fitzpatrick functions for continuous, linear, and monotone operators.

It is well known that such operators are automatically maximal monotone (see, e.g., [36, page 30]); see also [1, 5, 7, 30, 36] for additional works on monotone-operator-theoretic properties of linear operators. Let $A\colon X \to X$ be continuous and linear. Linearity and (1.2) yield

$$A \text{ is monotone} \quad \Leftrightarrow \quad (\forall x \in X)\ \langle x, Ax\rangle \geq 0. \tag{1.3}$$

Thus monotonicity is determined solely by the behaviour of the *symmetric part* of A. We now recall the relevant notions.

Definition 1.1 (symmetric and skew part). *Let $A\colon X \to X$ be continuous and linear. Then $A_+ = \frac{1}{2}A + \frac{1}{2}A^*$ is the* symmetric part *of A, and $A_\circ = A - A_+ = \frac{1}{2}A - \frac{1}{2}A^*$ is the* skew part *of A.*

The next result is clear.

Proposition 1.2. *Let $A\colon X \to X$ be continuous and linear. Then A is monotone if and only if A_+ is monotone.*

Let us define the Fitzpatrick function [17] for linear operators.

Definition 1.3 (Fitzpatrick function). *Let $A\colon X \to X$ be continuous and linear. The* Fitzpatrick function *of A is*

$$F_A\colon X \times X \to \left]-\infty, +\infty\right] : (x,u) \mapsto \sup_{y\in X} \big(\langle x, Ay\rangle + \langle y, u\rangle - \langle y, Ay\rangle\big). \tag{1.4}$$

Before we survey some fundamental results concerning the Fitzpatrick function of a linear operator, we need to briefly explain our notation. We shall utilize throughout this paper notation and results that are standard in convex analysis and monotone operator theory. See [9, 33, 35, 36, 43] for comprehensive references. The *Fenchel conjugate* and *domain* of a function f is denoted by f^* and $\operatorname{dom} f$, respectively. The *ball* of radius ρ centred at x is denoted by $B(x;\rho)$. The *closure*, the *interior*, and the *indicator function* of a set $S \subseteq X$ are written as $\overline{S}$, $\operatorname{int} S$, and ι_S, respectively. For a continuous and linear operator $A\colon X \to X$, the *kernel* (also known as null space) of A is denoted by $\ker A$ and the *range* by $\operatorname{ran} A$. The identity operator is written as Id. If A is monotone and symmetric, it will occasionally be convenient to use the notation

$$(\forall x \in X)(\forall y \in X) \quad \langle x, y\rangle_A = \langle x, Ay\rangle \ \text{ and } \ \|x\|_A = \sqrt{\langle x, x\rangle_A} = \|\sqrt{A}x\|, \tag{1.5}$$

where $\sqrt{A}$ denotes the *square root* of A [23, Section 9.4].

Fact 1.4. [17] *Let $A\colon X \to X$ be continuous, linear, and monotone. Then:*

(i) *F_A is convex, lower semicontinuous, and proper.*

(ii) *$F_A = \langle\cdot,\cdot\rangle$ on* $\operatorname{gra} A$, *and $F_A > \langle\cdot,\cdot\rangle$ outside* $\operatorname{gra} A$.

(iii) *$\big(\forall (x,u) \in X \times X\big)\ F_A(x,u) \leq F_A^*(u,x) = \big(\iota_{\operatorname{gra} A} + \langle\cdot,\cdot\rangle\big)^{**}(x,u)$.*

Fact 1.4.(ii) motivates the following definition (see also [14]).

Definition 1.5 (Fitzpatrick family). *Let $A\colon X \to X$ be continuous, linear, and monotone. The* Fitzpatrick family $\mathcal{F}_A$ *consists of all functions $F\colon X \times X \to \left]-\infty, +\infty\right]$ such that F is convex, lower semicontinuous, $F \geq \langle\cdot,\cdot\rangle$, and $F = \langle\cdot,\cdot\rangle$ on* $\operatorname{gra} A$.

FACT 1.6. [17] *Let $A\colon X \to X$ be continuous, linear, and monotone. Then for every $(x,u) \in X \times X$,*

$$F_A(x,u) = \min_{F\in\mathcal{F}_A} F(x,u) \quad \text{and} \quad F_A^*(u,x) = \max_{F\in\mathcal{F}_A} F(x,u). \tag{1.6}$$

The plan for the remainder of the paper is as follows.

- In Section 2, we describe completely the Fitzpatrick function and its conjugate (Theorem 2.3). Some examples and a new *characterization of skew operators* in terms of the Fitzpatrick family (Theorem 2.9) are provided.
- The *range* of a continuous, linear, and monotone operator is studied in Section 3 and compared to the range of the adjoint. The closures of these two ranges coincide; however, the Volterra integral operator (Example 3.3) illustrates that the ranges themselves can differ.
- Section 4 deals with *rectangular* — also known as *property* (∗) *monotone* — operators, a class of operators introduced by Brézis and Haraux [11]. We state their main result and discuss some useful consequences. We also provide a characterization of rectangular operators in terms of their symmetric and skew parts (Corollary 4.10). This allows us to make a connection with *paramonotone operators* (Remark 4.11). A result by Brézis and Haraux is reproved using the Fitzpatrick function (Theorem 4.12).
- We turn to the *Fitzpatrick function of the sum* in Section 5. No general formula is known; in fact, Fitzpatrick posed this as an open problem (see [17, Problem 5.4]). We present a partial solution to his problem by showing that a known upper bound is actually exact in finite-dimensional spaces (Corollary 5.7) as well as in more general settings (Theorems 5.3 and 5.4, and Corollary 5.6).
- *Cyclic monotonicity* is a quantitative refinement of monotonicity that can be captured with higher-order Fitzpatrick functions. We begin in Section 6 by reviewing known results about these functions. We then present a new closed form (Example 6.4), a novel recursion formula (Theorem 6.5), and a localization of the domain (Corollary 6.7).
- In the final Section 7, we study cyclic monotonicity properties and higher-order Fitzpatrick functions of *rotators* in the Euclidean plane. Complete characterizations of n-cyclic monotonicity and explicit formulas for the Fitzpatrick functions are provided in all possible cases (Theorem 7.8). This extends considerably previously known results [2, Section 4].

2. The Fitzpatrick function and skew operators. The Fitzpatrick function of a continuous linear operator will be formulated in terms of a quadratic function that we present next.

DEFINITION 2.1 (quadratic function). *Let $A\colon X \to X$ be continuous, linear, and symmetric. Then we set $q_A\colon X \to \mathbb{R}\colon x \mapsto \frac{1}{2}\langle x, Ax\rangle$.*

FACT 2.2. *Let $A\colon X \to X$ be continuous, linear, and symmetric. Then*

$$q_A \text{ is convex} \Leftrightarrow A \text{ is monotone.} \tag{2.1}$$

In this case, the following is true.

(i) $\nabla q_A = A$.
(ii) $q_A^* \circ A = q_A$.
(iii) $\operatorname{ran} A \subseteq \operatorname{dom} q_A^* \subseteq \overline{\operatorname{ran}}\, A$.

(iv) $q_A^* \geq 0$ *and* $(\forall u \in X)(\forall \rho \in \mathbb{R}\smallsetminus\{0\})$ $q_A^*(\rho u) = \rho^2 q_A^*(u)$. *Consequently,* $\operatorname{dom} q_A^*$ *is a subspace.*

(v) *If* $\operatorname{ran} A$ *is closed, then* $q_A^* = \iota_{\operatorname{ran} A} + q_{A^\dagger}$, *where* $A^\dagger$ *is the* Moore-Penrose inverse [20] *of* A.

(vi) *If* A *is bijective, then* $q_A^* = q_{A^{-1}}$.

Proof. (See also [3, Proposition 12.3.6].) (i)&(ii): [5, Theorem 3.6.(i)]. See also [30, Theorem 5.1] for a considerably more general version of (i). (iii): [4, Fact 2.2(iii)]. (iv): Elementary. (v): See [6, Proposition 3.7(iv)]. (vi): Clear from (v). ☐

THEOREM 2.3. *Let* $A\colon X \to X$ *be continuous, linear, and monotone. Then:*

(i) $\big(\forall (x,u) \in X \times X\big)$ $\quad F_A(x,u) = 2q_{A_+}^*\big(\tfrac{1}{2}u + \tfrac{1}{2}A^*x\big) = F_{A_+}(x, u - A_\circ x)$.

(ii) $\operatorname{ran} A_+ \subseteq (A^* \oplus \operatorname{Id})(\operatorname{dom} F_A) = \operatorname{dom} q_{A_+}^* \subseteq \overline{\operatorname{ran}}\, A_+$.

(iii) $\big(\forall (u,x) \in X \times X\big)$ $\quad F_A^*(u,x) = \iota_{\operatorname{gra} A}(x,u) + \langle x, Ax\rangle$.

Proof. Fix $(x,u) \in X \times X$. (i): This follows from

(2.2)
$$\begin{aligned} F_A(x,u) &= \sup_{y\in X} \langle x, Ay\rangle + \langle y, u\rangle - \langle y, Ay\rangle = 2\sup_{y\in X} \langle y, \tfrac{1}{2}u + \tfrac{1}{2}A^*x\rangle - q_{A_+}(y) \\ &= 2q_{A_+}^*\big(\tfrac{1}{2}u + \tfrac{1}{2}A^*x\big) = 2q_{A_+}^*\big(\tfrac{1}{2}(u - A_\circ x) + \tfrac{1}{2}A_+x\big) = F_{A_+}(x, u - A_\circ x). \end{aligned}$$

(ii): The equality is a consequence of (i), and the inclusions are then clear from Fact 2.2.(iii). (iii): This follows from Fact 1.4.(iii) and the fact that the function $(u,x) \mapsto \iota_{\operatorname{gra} A}(x,u) + \langle x, u\rangle = \iota_{\operatorname{gra} A}(x,u) + \langle x, Ax\rangle$ is already convex, lower semicontinuous, and proper. ☐

The next two results play a role in the proof of Theorem 5.4 below.

EXAMPLE 2.4. *Let* $A\colon X \to X$ *be continuous, linear, monotone, and symmetric. Then*

$$\big(\forall x \in X\big)\big(\forall y \in X\big) \quad F_A(x, Ay) = \tfrac{1}{4}\langle x+y, A(x+y)\rangle. \tag{2.3}$$

Proof. Take $x \in X$ and $y \in X$. Using Theorem 2.3.(i) and Fact 2.2.(ii), we obtain

$$F_A(x, Ay) = 2q_A^*\big(\tfrac{1}{2}Ay + \tfrac{1}{2}Ax\big) = 2q_A\big(\tfrac{1}{2}x + \tfrac{1}{2}y\big) = \tfrac{1}{4}\langle x+y, A(x+y)\rangle, \tag{2.4}$$

as required. ☐

EXAMPLE 2.5 (closed range symmetric operator). *Let* $A\colon X \to X$ *be continuous, linear, monotone, and symmetric such that* $\operatorname{ran} A$ *is closed. Then*

$$\begin{aligned} \big(\forall (x,u) \in X \times X\big) \quad F_A(x,u) &= \iota_{\operatorname{ran} A}(u) + \tfrac{1}{4}\big(\langle x, Ax\rangle + 2\langle x, u\rangle + \langle A^\dagger u, u\rangle\big) \\ &= \iota_{\operatorname{ran} A}(u) + \tfrac{1}{4}\|x + A^\dagger u\|_A^2 \end{aligned} \tag{2.5}$$

and hence

$$\operatorname{dom} F_A = X \times \operatorname{ran} A. \tag{2.6}$$

Proof. Fix $(x,u) \in X \times X$. Using Theorem 2.3.(i), Fact 2.2.(v) and standard properties of the Moore-Penrose inverse [20], we deduce that

$$
\begin{aligned}
(2.7)\qquad F_A(x,u) &= 2q_A^*\big(\tfrac{1}{2}u+\tfrac{1}{2}Ax\big)\\
&= 2\iota_{\operatorname{ran} A}\big(\tfrac{1}{2}u+\tfrac{1}{2}Ax\big)+2q_{A^\dagger}\big(\tfrac{1}{2}u+\tfrac{1}{2}Ax\big)\\
&= \iota_{\operatorname{ran} A}(u)+\langle A^\dagger(\tfrac{1}{2}u+\tfrac{1}{2}Ax),\tfrac{1}{2}u+\tfrac{1}{2}Ax\rangle\\
&= \iota_{\operatorname{ran} A}(u)+\tfrac{1}{4}\big(\langle A^\dagger u,u\rangle+\langle A^\dagger u,Ax\rangle+\langle A^\dagger Ax,u\rangle+\langle A^\dagger Ax,Ax\rangle\big)\\
&= \iota_{\operatorname{ran} A}(u)+\tfrac{1}{4}\big(\langle A^\dagger u,u\rangle+\langle AA^\dagger u,x\rangle+\langle x,AA^\dagger u\rangle+\langle AA^\dagger Ax,x\rangle\big)\\
&= \iota_{\operatorname{ran} A}(u)+\tfrac{1}{4}\big(\langle x,Ax\rangle+2\langle x,u\rangle+\langle A^\dagger u,u\rangle\big)\\
&= \iota_{\operatorname{ran} A}(u)+\tfrac{1}{4}\big(\langle x,Ax\rangle+\langle x,AA^\dagger u\rangle+\langle A^\dagger u,Ax\rangle+\langle A^\dagger u,AA^\dagger u\rangle\big)\\
&= \iota_{\operatorname{ran} A}(u)+\tfrac{1}{4}\langle x+A^\dagger u,A(x+A^\dagger u)\rangle\\
&= \iota_{\operatorname{ran} A}(u)+\tfrac{1}{4}\|x+A^\dagger u\|_A^2,
\end{aligned}
$$

as desired. ☐

Remark 2.6. *Let A be as in Example 2.5. A referee pointed out that* (2.6) *can also be proved as follows. Take $(x,u)\in X\times X$. Utilizing Theorem 2.3.(i) and Fact 2.2.(iii), we have*

$$
(2.8)\qquad (x,u)\in\operatorname{dom}F_A \quad\Leftrightarrow\quad \tfrac{1}{2}u+\tfrac{1}{2}Ax\in\operatorname{dom}q_A^*=\operatorname{ran}A \quad\Leftrightarrow\quad u\in\operatorname{ran}A.
$$

Let us provide two further examples. The first one is related to [29, Example 1], while the second one generalizes [29, Example 3].

Example 2.7 (bijective symmetric operator). *Let $A\colon X\to X$ be continuous, linear, monotone, symmetric, and bijective. Then*

$$
\begin{aligned}
(2.9)\qquad \big(\forall (x,u)\in X\times X\big)\quad F_A(x,u) &= \tfrac{1}{4}\big(\langle x,Ax\rangle+2\langle x,u\rangle+\langle A^{-1}u,u\rangle\big)\\
&= \tfrac{1}{4}\|x+A^{-1}u\|_A^2.
\end{aligned}
$$

Proof. This is clear from Example 2.5. ☐

Example 2.8 (skew operator). *Let $A\colon X\to X$ be continuous, linear, and skew. Then*

$$
(2.10)\qquad \big(\forall (x,u)\in X\times X\big)\quad F_A(x,u)=F_A^*(u,x)=\iota_{\operatorname{gra} A}(x,u).
$$

Proof. Since A is skew, $A^*=-A$, $A_+=0$ and thus $\operatorname{dom}q_{A_+}^*=\operatorname{ran}A_+=\{0\}$ is closed (Fact 2.2.(iii)). Using Theorem 2.3.(i), Fact 2.2.(iv), and Theorem 2.3.(iii), we obtain that

$$
\begin{aligned}
(2.11)\qquad F_A(x,u) &= 2q_{A_+}^*\big(\tfrac{1}{2}u+\tfrac{1}{2}A^*x\big)=2\iota_{\{0\}}\big(\tfrac{1}{2}u+\tfrac{1}{2}A^*x\big)\\
&= 2\iota_{\{0\}}\big(\tfrac{1}{2}u-\tfrac{1}{2}Ax\big)=\iota_{\{0\}}(u-Ax)=\iota_{\operatorname{gra} A}(x,u)\\
&= \iota_{\operatorname{gra} A}(x,u)+\langle x,Ax\rangle=F_A^*(u,x),
\end{aligned}
$$

which completes the proof. ☐

We now present a new characterization of skew operators using the Fitzpatrick family.

THEOREM 2.9. *Let $A\colon X \to X$ be a continuous, linear, and monotone. Then A is skew $\Leftrightarrow$ $\mathcal{F}_A$ is a singleton. In this case, $\mathcal{F}_A = \{\iota_{\operatorname{gra} A}\}$.*

Proof. Fix $(x,u) \in X \times X$. "$\Leftarrow$": If $u - Ax \notin \operatorname{ran} A_+$, then $u - Ax \neq 0$. Now suppose that $u - Ax \neq 0$. Then $(x,u) \notin \operatorname{gra} A$ and hence $F_A^*(u,x) = +\infty$ by Theorem 2.3.(iii). Fact 1.6 implies that $F_A(x,u) = +\infty$, i.e., $(x,u) \notin \operatorname{dom} F_A$. If $u + A^*x$ belonged to $\operatorname{ran} A_+$, then $q_{A_+}^*(u + A^*x) < +\infty$ (by Fact 2.2.(iii)) and hence $(x,u) \in \operatorname{dom} F_A$ (by Theorem 2.3.(i)), which is absurd. Thus $u + A^*x \notin \operatorname{ran} A_+$. Now $u + A^*x = u - Ax + 2A_+x$, which implies $u - Ax \notin \operatorname{ran} A_+$. Altogether, we have verified the equivalence

$$\big(\forall (x,u) \in X \times X\big) \quad u - Ax \neq 0 \quad \Leftrightarrow \quad u - Ax \notin \operatorname{ran} A_+. \tag{2.12}$$

Since $(\forall u \in X \smallsetminus \{0\})$ $u - A0 = u \neq 0$, (2.12) yields $u = u - A0 \notin \operatorname{ran} A_+$. Hence $\operatorname{ran} A_+ = \{0\}$; equivalently, $A_+ = 0$ and therefore $A = A_\circ$. "$\Rightarrow$": Example 2.8 and Fact 1.6. ∎

REMARK 2.10. *Loosely speaking, Theorem 2.9 states that a Fitzpatrick family with only one element corresponds to a "bad" (here, skew) monotone operator. The situation is similar for subdifferential operators: $\mathcal{F}_{\partial f}$ reduces to the singleton $\{f \oplus f^*\}$ when f is sublinear or an indicator function (see [12, 13] and also [2]).*

3. Range. In this section, we compare the range of a continuous linear monotone operator to the range of its adjoint.

PROPOSITION 3.1. *Let $A\colon X \to X$ be continuous, linear, and monotone. Then $\ker A = \ker A^*$ and $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, A^*$.*

Proof. Take $x \in \ker A$ and $v \in \operatorname{ran} A$, say $v = Ay$. Then $(\forall \alpha \in \mathbb{R})$ $0 \leq \langle \alpha x + y, A(\alpha x + y)\rangle = \alpha\langle x, v\rangle + \langle y, Ay\rangle$. Hence $\langle x, v\rangle = 0$ and thus $\ker A \subset (\operatorname{ran} A)^\perp = \ker A^*$. Since A^* is also continuous, linear, and monotone, we obtain $\ker A^* \subset \ker A^{**} = \ker A$. Altogether, $\ker A = \ker A^*$ and therefore $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, A^*$. ∎

REMARK 3.2.

(i) *A referee pointed out that Proposition 3.1 also follows from [30, Corollary 3.5], which is more general.*
(ii) *Example 3.3 below illustrates that the closures in Proposition 3.1 are critical.*
(iii) *An operator $A\colon X \to X$ such that $\operatorname{ran} A = \operatorname{ran} A^*$ is called* range-symmetric *or* EP*; see [26, page 408]. Proposition 3.1 implies that every continuous, linear, and monotone operator with closed range is range-symmetric. See [16, Theorem 2.3] for equivalent properties in the matrix case.*
(iv) *Every* normal matrix *A (i.e., $AA^* = A^*A$) is range-symmetric: indeed, we then have $\operatorname{ran} A = \operatorname{ran} AA^* = \operatorname{ran} A^*A = \operatorname{ran} A^*$ (the first and the last equality follow, e.g., from [26, page 212]).*
(v) *However, a range-symmetric monotone matrix need not be normal:*

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \tag{3.1}$$

is monotone, but $AA^* \neq A^*A$.

EXAMPLE 3.3 (Volterra operator). *Set $X = L_2[0,1]$. The* Volterra integration operator *[21, Problem 148] is defined by*

$$V\colon X \to X\colon x \mapsto Vx, \quad \text{where} \quad Vx\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_0^t x. \tag{3.2}$$

Fix $x \in X$. Then

$$(V^*x)(t) = \int_t^1 x. \tag{3.3}$$

and $\ker V = \ker V^* = \{0\}$, *so V and V^* have dense range. Set $e \equiv 1 \in X$. Now* (3.2) *and* (3.3) *imply $(V + V^*)x = \langle x, e\rangle e$ and thus $\langle x, (V + V^*)x\rangle = \langle x, e\rangle^2 \geq 0$. Hence*

$$V \text{ is monotone and } V_+x = \tfrac{1}{2}\langle x, e\rangle e. \tag{3.4}$$

Moreover, $q_{V_+}(x) = \frac{1}{2}\langle x, V_+x\rangle = \frac{1}{4}\langle x, e\rangle^2$ and $\operatorname{ran} V_+ = \mathbb{R}e$ is closed. Now Fact 2.2.(ii) and Theorem 2.3.(i)&(iii) result in

$$\begin{aligned} F_V \colon X \times X &\to \left]-\infty, +\infty\right] \\ (z, w) &\mapsto \begin{cases} \frac{1}{2}\langle w + V^*z, e\rangle^2, & \text{if } w + V^*z = \langle w + V^*z, e\rangle e; \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} F_V^* \colon X \times X &\to \left]-\infty, +\infty\right] \\ (w, z) &\mapsto \begin{cases} \frac{1}{2}\langle z, e\rangle^2, & \text{if } w = Vz; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.6}$$

Next, assume that $Vx = V^*y$, i.e., $(\forall t \in [0, 1])$ $\int_0^t x = \int_t^1 y$. Evaluating this at $t = 0$ and $t = 1$, we learn that $\langle y, e\rangle = \langle x, e\rangle = 0$. We thus have verified the implication

$$\left.\begin{array}{r} x \in X, \\ y \in X, \\ Vx = V^*y \end{array}\right\} \quad \Rightarrow \quad \langle x, e\rangle = \langle y, e\rangle = 0 \tag{3.7}$$

and the inclusion

$$\operatorname{ran} V \cap \operatorname{ran} V^* \subseteq \big\{Vx \mid x \in \{e\}^\perp\big\}. \tag{3.8}$$

Conversely, if $x \in \{e\}^\perp$, then

$$(\forall t \in [0, 1]) \quad (Vx)(t) = \langle x, e\rangle - \int_t^1 x = \big(V^*(-x)\big)(t) \tag{3.9}$$

and hence $Vx \in \operatorname{ran} V \cap \operatorname{ran} V^*$. Altogether,

$$\operatorname{ran} V \cap \operatorname{ran} V^* = \{Vx \colon x \in \{e\}^\perp\}. \tag{3.10}$$

Since $\langle e, e\rangle = 1 \neq 0$, the implication (3.7) *shows that $Ve \notin \operatorname{ran} V^*$ and that $V^*e \notin$* $\operatorname{ran} V$. *Therefore,*

$$\operatorname{ran} V \not\subseteq \operatorname{ran} V^* \text{ and } \operatorname{ran} V^* \not\subseteq \operatorname{ran} V. \tag{3.11}$$

4. Rectangular monotone operators. We now turn to a property related to the domain of the Fitzpatrick function.

DEFINITION 4.1 (rectangular). *Let $A\colon X \to X$ be continuous, linear, and monotone. Then A is* rectangular *if $X \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$.*

REMARK 4.2.

(i) *The property referred to in Definition 4.1 was first introduced by Brézis and Haraux [11]. In the literature it is also known as* property (*) *and as* 3*-monotone. *However, we follow here Simons' [39] more descriptive naming convention which is based on his observation that — since $\operatorname{dom} F_A \subseteq \overline{\operatorname{dom} A \times \operatorname{ran} A} = X \times \overline{\operatorname{ran}} A$ is always true — the operator A is rectangular if and only if $\overline{\operatorname{dom}}\, F_A$ is the "rectangle" $X \times \overline{\operatorname{ran}} A$.*

(ii) *In the context of general monotone operators, the subdifferential operator is known to be rectangular [11].*

(iii) *As a consequence of (ii), we note that every continuous, linear, monotone, and symmetric operator is rectangular (Fact 2.2.(i)). This will be reproved in Corollary 4.9 below.*

The importance of rectangularity stems from a powerful result due to Brézis-Haraux [11], which we state next in the present context of linear operators.

FACT 4.3 (Brézis-Haraux). *Let A and B be continuous, linear, and monotone operators from X to X, and suppose that A or B is rectangular. Then $\overline{\operatorname{ran}}\,(A+B) = \overline{\operatorname{ran} A + \operatorname{ran} B}$ and $\operatorname{int} \operatorname{ran}(A+B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$.*

Proof. See [11], and also [36, 39] for different proofs. ☐

It is worthwhile to list some of the most important consequences of Fact 4.3.

COROLLARY 4.4. *Let A and B be continuous, linear, and monotone operators from X to X. Suppose that A or B is rectangular, and that A or B is surjective. Then $A+B$ is surjective.*

Proof. Fact 4.3 yields $X = \operatorname{int} X = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B) = \operatorname{int} \operatorname{ran}(A+B)$. Therefore, $X = \operatorname{ran}(A+B)$ and $A+B$ is surjective. ☐

COROLLARY 4.5. *Let A and B be continuous, linear, and monotone operators from X to X such that A or B is rectangular. Then $\ker(A+B) = \ker A \cap \ker B$.*

Proof. Using Proposition 3.1 and Fact 4.3, we obtain

$$
\begin{aligned}
(\ker A \cap \ker B)^{\perp} &= \overline{(\ker A)^{\perp} + (\ker B)^{\perp}} = \overline{\operatorname{ran} A^* + \operatorname{ran} B^*} \\
&= \overline{\operatorname{ran} A + \operatorname{ran} B} = \overline{\operatorname{ran}}\,(A+B) \\
&= \big(\ker(A+B)\big)^{\perp}.
\end{aligned} \tag{4.1}
$$

The result follows by taking orthogonal complements. ☐

COROLLARY 4.6. *Let A and B be continuous, linear, and monotone operators from X to X. Suppose that A or B is rectangular, and that A or B is injective. Then $A+B$ is injective.*

COROLLARY 4.7. *Let A and B be continuous, linear, and monotone operators from X to X. Suppose that A or B is rectangular, and that A or B is bijective. Then $A+B$ is bijective.*

PROPOSITION 4.8. *Let $A\colon X \to X$ be continuous, linear, and monotone. Then the following are equivalent.*

(i) *A is rectangular.*

(ii) $\operatorname{ran} A + \operatorname{ran} A^* \subseteq \operatorname{dom} q_{A_+}^*$.

(iii) $\operatorname{ran} A_\circ \subseteq \operatorname{dom} q_{A_+}^*$.

Proof. "(i)⇔(ii)": This is a direct consequence of Theorem 2.3.(i). "(ii)⇒(iii)": $\operatorname{ran} A_\circ = \operatorname{ran}(A - A^*) \subseteq \operatorname{ran} A - \operatorname{ran} A^* = \operatorname{ran} A + \operatorname{ran} A^* \subseteq \operatorname{dom} q^*_{A_+}$. "(ii)⇐(iii)": Fact 2.2.(iii)&(iv) and the fact that $A^* = A_+ - A_\circ$ yield $\operatorname{ran} A + \operatorname{ran} A^* = \operatorname{ran}(A_+ + A_\circ) + \operatorname{ran}(A_+ - A_\circ) \subseteq \operatorname{ran} A_+ + \operatorname{ran} A_\circ \subseteq \operatorname{dom} q^*_{A_+} + \operatorname{dom} q^*_{A_+} = \operatorname{dom} q^*_{A_+}$. ◻

COROLLARY 4.9. *Let $A\colon X \to X$ be continuous, linear, monotone, and symmetric. Then A is rectangular.*

Proof. Utilizing Fact 2.2.(iii), we see that $\operatorname{ran} A + \operatorname{ran} A^* = \operatorname{ran} A_+ \subseteq \operatorname{dom} q^*_{A_+}$. The result follows from Proposition 4.8. ◻

COROLLARY 4.10. *Let $A\colon X \to X$ be continuous, linear, and monotone, and suppose that* $\operatorname{ran} A_+$ *is closed. Then A is rectangular if and only if* $\operatorname{ran} A_\circ \subseteq \operatorname{ran} A_+$.

Proof. Fact 2.2.(iii) shows that $\operatorname{dom} q^*_{A_+} = \operatorname{ran} A_+$. Now apply Proposition 4.8. ◻

REMARK 4.11 (paramonotone operators). *Let $X = \mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$ be monotone. By [22, Proposition 3.2.(ii)], A is* paramonotone $\Leftrightarrow \ker A_+ \subseteq \ker A$. *On the other hand, using Corollary 4.5 (applied to A_+ and $A_\circ$) and Corollary 4.10, we have the equivalences* $\ker A_+ \subseteq \ker A \Leftrightarrow \ker A_+ \subseteq \ker A_+ \cap \ker A_\circ \Leftrightarrow \ker A_+ \subseteq \ker A_\circ$ $\Leftrightarrow \operatorname{ran} A_\circ \subseteq \operatorname{ran} A_+ \Leftrightarrow$ *A is rectangular. Altogether,*

$$A \text{ is paramonotone if and only if } A \text{ is rectangular.} \tag{4.2}$$

See [22] for further information on paramonotone operators.

The next result can be deduced from [11, Proposition 2]. The proof provided here is somewhat simpler and based on the Fitzpatrick function, and the result is stated in a more applicable form.

THEOREM 4.12. *Let $A\colon X \to X$ be continuous, linear, and monotone. Then the following are equivalent.*

(i) *A is rectangular.*
(ii) *For some $\gamma > 0$, $\|\gamma A - \operatorname{Id}\| \le 1$.*
(iii) *A^* is rectangular.*

Proof. The conditions all hold when $A = 0$, so assume that $A \neq 0$. "(i)⇒(ii)": Consider the function

$$f\colon X \to \,]-\infty, +\infty] : x \mapsto F_A(x, 0). \tag{4.3}$$

Then f is convex, lower semicontinuous, and proper by Fact 1.4.(i)&(ii). Since A is rectangular, $X \times \{0\} \subseteq X \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$. Hence $\operatorname{dom} f = X$. It follows, e.g., from [43, Theorem 2.2.20] that there exists $\delta > 0$ and $\beta > 0$ such that $(\forall x \in B(0;\delta))$ $f(x) = F_A(x, 0) = \sup_{y\in X} \langle x, Ay\rangle - \langle y, Ay\rangle \le \beta$. Fix $x \in B(0;\delta)$ and $y \in X$. Then

$$(\forall \rho \in \mathbb{R}) \quad 0 \le \beta + \langle \rho y, A(\rho y)\rangle - \langle x, A(\rho y)\rangle = \beta + \rho^2\langle y, Ay\rangle - \rho\langle x, Ay\rangle. \tag{4.4}$$

We claim that

$$(\forall x \in B(0;\delta))(\forall y \in X) \quad \langle x, Ay\rangle^2 \le 4\beta\langle y, Ay\rangle. \tag{4.5}$$

If $\langle y, Ay\rangle = 0$, then (4.4) shows that $\langle x, Ay\rangle = 0$ and hence (4.5) holds. Now assume that $\langle y, Ay\rangle \neq 0$. In terms of ρ, the right side of (4.4) is a nonnegative quadratic function. Substituting the minimizer $\langle x, Ay\rangle/(2\langle y, Ay\rangle)$ of this quadratic function into (4.4) yields an inequality that is equivalent to (4.5). In turn, (4.5) leads to

$$(\forall y \in X) \quad \delta^2\|Ay\|^2 \le 4\beta\langle y, Ay\rangle. \tag{4.6}$$

Set $\alpha = \delta^2/(4\beta)$. We deduce that $(\forall y \in X)$ $\langle y, \alpha Ay\rangle \ge \|\alpha Ay\|^2$, i.e., αA is firmly nonexpansive. This (see [19]) is equivalent to the nonexpansivity of $2\alpha A - \operatorname{Id}$, i.e., to

$\|2\alpha A - \mathrm{Id}\,\| \leq 1$. "(ii)⇒(i)": Set $\alpha = \gamma/2$. Fix x and y in X and take $z \in X$. Utilizing the equivalences αA is firmly nonexpansive ⇔ $\|2\alpha A - \mathrm{Id}\,\| \leq 1$ ⇔ $\|2\alpha A^* - \mathrm{Id}\,\| \leq 1$ ⇔ αA^* is firmly nonexpansive, we estimate

(4.7)
$$\begin{aligned}\langle x, Az\rangle + \langle z, Ay\rangle - \langle z, Az\rangle &= \big(\langle x, Az\rangle - \tfrac{1}{2}\langle z, Az\rangle\big) + \big(\langle A^*z, y\rangle - \tfrac{1}{2}\langle z, A^*z\rangle\big)\\ &\leq \big(\|x\|\,\|Az\| - \tfrac{1}{2}\alpha\|Az\|^2\big) + \big(\|A^*z\|\,\|y\| - \tfrac{1}{2}\alpha\|A^*z\|^2\big)\\ &\leq \frac{1}{2\alpha}\big(\|x\|^2 + \|y\|^2\big),\end{aligned}$$

where the last inequality was obtained by computing the maxima of the quadratic functions $\rho \mapsto \|x\|\rho - \frac{1}{2}\alpha\rho^2$ and $\rho \mapsto \|y\|\rho - \frac{1}{2}\alpha\rho^2$, respectively. It follows from (4.7) that

$$(\forall x \in X)(\forall y \in X) \quad F_A(x, Ay) \leq \frac{1}{\gamma}\big(\|x\|^2 + \|y\|^2\big), \tag{4.8}$$

hence $X \times \operatorname{ran} A \subset \operatorname{dom} F_A$. "(ii)⇔(iii)": Apply the equivalence (i)⇔(ii) to A^*. □

Corollary 4.13. *The continuous, linear, monotone, and rectangular operators form a convex cone.*

Proof. It is clear that they form a cone. Suppose A and B are continuous, linear, monotone, and rectangular. Then there exist $\gamma_A > 0$ and $\gamma_B > 0$ such that $\|\gamma_A A - \mathrm{Id}\,\| \leq 1$ and $\|\gamma_B B - \mathrm{Id}\,\| \leq 1$. Set $\gamma = \frac{1}{2}\min\{\gamma_A, \gamma_B\}$ and estimate $\|\gamma(A + B) - \mathrm{Id}\,\| \leq \frac{1}{2}\|2\gamma A - \mathrm{Id}\,\| + \frac{1}{2}\|2\gamma B - \mathrm{Id}\,\| \leq 1$. Hence $A + B$ is rectangular and the proof is complete. □

The next example was established by direct computation in [4]; however, Theorem 4.12 yields a very transparent and simple proof.

Example 4.14. *Let* $R\colon X^n \to X^n\colon (x_1, x_2, \dots, x_n) \mapsto (x_n, x_1, \dots, x_{n-1})$ *be the* right-shift operator *on* X^n. *Then* $\mathrm{Id} - R$ *is rectangular.*

Proof. Since $\|1 \cdot (\mathrm{Id} - R) - \mathrm{Id}\,\| = \|-R\| = 1$, the result is a consequence of Theorem 4.12. □

We conclude this section by providing a novel nonsmooth proof of a result on the domain of the Fitzpatrick function of the subdifferential operator (see also [6, Theorem 2.6]).

Theorem 4.15. *Let* $f\colon X \to \,]-\infty, +\infty]$ *be convex, lower semicontinuous, and proper. Then*

$$\operatorname{dom} f \times \operatorname{dom} f^* \subseteq \operatorname{dom} F_{\partial f} \subseteq \overline{\operatorname{dom}}\, f \times \overline{\operatorname{dom}}\, f^*. \tag{4.9}$$

Proof. The first inclusion is elementary (see also [6, Proposition 2.1]). Now take $(x, u) \in \operatorname{dom} F_{\partial f}$ and set $C = \overline{\operatorname{dom}}\, f$. Assume to the contrary that $x \notin C$, hence $f(x) = +\infty$ and $d_C(x) = \inf \|x - C\| > 0$. Fix $x_0 \in \operatorname{dom} f$ and define the family of nonconvex but lower semicontinuous functions

$$(\forall \rho > 0) \quad f_\rho\colon X \to \,]-\infty, +\infty] : y \mapsto \begin{cases} f(y), & \text{if } y \neq x;\\ f(x_0) + \rho, & \text{if } y = x.\end{cases} \tag{4.10}$$

The *Approximate Mean Value Theorem* of Mordukhovich and Shao (see [27, Theorem 3.49] or [28, Theorem 8.2]), applied to f_ρ and the points x_0 and x, shows that for every $\rho > 0$, there exist $y_\rho \in [x_0, x[$ and a sequence $(y_{\rho,n}, v_{\rho,n})_{n\in\mathbb{N}}$ in $\operatorname{gra}\partial f$ such that $y_{\rho,n} \to y_\rho$ and

$$\underline{\lim}_{n\in\mathbb{N}} \Big\langle \frac{x-y_{\rho,n}}{\|x-y_{\rho,n}\|}, v_{\rho,n}\Big\rangle \geq \frac{f_\rho(x)-f_\rho(x_0)}{\|x-x_0\|} = \frac{\rho}{\|x-x_0\|}. \tag{4.11}$$

Therefore, there exists a sequence $((z_n, w_n))_{n\in\mathbb{N}}$ in $\operatorname{gra}\partial f$ such that

$$\Big\langle \frac{x-z_n}{\|x-z_n\|}, w_n\Big\rangle \to +\infty. \tag{4.12}$$

By definition of $F_{\partial f}$, the Cauchy-Schwarz inequality, and (4.12), we obtain

$$\begin{aligned} F_{\partial f}(x,u) &= \sup_{(y,v)\in\operatorname{gra}\partial f} \big(\langle x,v\rangle + \langle y,u\rangle - \langle y,v\rangle\big) \\ &= \sup_{(y,v)\in\operatorname{gra}\partial f} \big(\langle x-y,v\rangle + \langle y-x,u\rangle + \langle x,u\rangle\big) \\ &\geq \sup_{(y,v)\in\operatorname{gra}\partial f} \Big(\|x-y\|\Big(\Big\langle \frac{x-y}{\|x-y\|}, v\Big\rangle - \|u\|\Big) + \langle x,u\rangle\Big) \\ &\geq \overline{\lim_{n\in\mathbb{N}}} \Big(\|x-z_n\|\Big(\Big\langle \frac{x-z_n}{\|x-z_n\|}, w_n\Big\rangle - \|u\|\Big) + \langle x,u\rangle\Big) \\ &\geq \overline{\lim_{n\in\mathbb{N}}} \Big(d_C(x)\Big(\Big\langle \frac{x-z_n}{\|x-z_n\|}, w_n\Big\rangle - \|u\|\Big) + \langle x,u\rangle\Big) \\ &= +\infty. \end{aligned} \tag{4.13}$$

This contradicts the assumption that $F_{\partial f}(x,u) < +\infty$. Therefore, $x \in \overline{\operatorname{dom}} f$. An analogous argument (applied to f^*) implies that $u \in \overline{\operatorname{dom}} f^*$. ☐

5. The Fitzpatrick function of the sum. One of Simon Fitzpatrick's open problems [17, Problem 5.4] is to find the Fitzpatrick function of the sum of two operators. This has proven to be a difficult problem. However, an upper bound is always readily available.

Definition 5.1. *Let $A\colon X \to X$ and $B\colon X \to X$ be continuous, linear, and monotone operators, and set*

$$\big(\forall (x,u) \in X\times X\big) \quad \Phi_{\{A,B\}}(x,u) = \big(F_A(x,\cdot) \,\square\, F_B(x,\cdot)\big)(u) = \inf_{v+w=u} F_A(x,v) + F_B(x,w). \tag{5.1}$$

Proposition 5.2 (upper bound). *Let $A\colon X \to X$ and $B\colon X \to X$ be continuous, linear, and monotone operators. Then $F_{A+B} \leq \Phi_{\{A,B\}}$.*

Proof. See [6, Proposition 4.2]. ☐

In [6, Section 4] it is shown that in the context of subdifferential operators, this upper bound is sometimes — but not always — tight. In the remainder of this section we investigate the upper bound in the present context of continuous, linear, and monotone operators.

Theorem 5.3. *Let $A\colon X \to X$ and $B\colon X \to X$ be continuous, linear, and monotone operators. Suppose that one of the following conditions is satisfied.*

(i) *A is skew, and B is skew.*
(ii) *A is symmetric, and B is skew.*

Then $F_{A+B} = \Phi_{\{A,B\}}$.

Proof. Fix $(x,u) \in X \times X$. (i): Repeated application of Example 2.8 yields

$$\begin{aligned} F_{A+B}(x,u) &= \iota_{\operatorname{gra}(A+B)}(x,u) \\ &= \inf_{v+w=u} \iota_{\{Ax\}}(v) + \iota_{\{Bx\}}(w) \\ &= \inf_{v+w=u} \iota_{\operatorname{gra} A}(x,v) + \iota_{\operatorname{gra} B}(x,w) \\ &= \inf_{v+w=u} F_A(x,v) + F_B(x,w) \\ &= \Phi_{\{A,B\}}(x,u). \end{aligned} \tag{5.2}$$

(ii): Theorem 2.3.(i) and Example 2.8 result in

$$\begin{aligned} F_{A+B}(x,u) &= F_A(x,u-Bx) \\ &= \inf_{v\in Bx} F_A(x,u-v) \\ &= \inf_{v\in X} F_A(x,u-v) + \iota_{\operatorname{gra} B}(x,v) \\ &= \inf_{v\in X} F_A(x,u-v) + F_B(x,v) \\ &= \Phi_{\{A,B\}}(x,u). \end{aligned} \tag{5.3}$$

The proof is complete. ☐

The "purely symmetric" counterpart to Theorem 5.3 seems to require a closedness assumption. We are grateful to a referee for providing us with a simpler and more powerful proof.

THEOREM 5.4. *Let* $A\colon X \to X$ *and* $B\colon X \to X$ *be continuous, linear, monotone, and symmetric. Then* $F_{A+B} = \Phi_{\{A,B\}}$ *on* $\operatorname{ran}(A+B)$. *Consequently, if* $\operatorname{ran}(A+B)$ *is closed, then* $F_{A+B} = \Phi_{\{A,B\}}$.

Proof. Fix $x \in X$ and $y \in X$. Utilizing Example 2.4 thrice, we obtain

$$\begin{aligned} \Phi_{\{A,B\}}\big(x,(A+B)y\big) &\leq F_A(x,Ay) + F_B(x,By) \\ &= \tfrac{1}{4}\langle x+y, A(x+y)\rangle + \tfrac{1}{4}\langle x+y, B(x+y)\rangle \\ &= \tfrac{1}{4}\langle x+y, (A+B)(x+y)\rangle \\ &= F_{A+B}\big(x,(A+B)y\big). \end{aligned} \tag{5.4}$$

Thus

$$\Phi_{\{A,B\}} \leq F_{A+B} \;\text{ on } X \times \operatorname{ran}(A+B). \tag{5.5}$$

On the other hand, by Proposition 5.2, $F_{A+B} \leq \Phi_{\{A,B\}}$. Altogether, $F_{A+B} = \Phi_{\{A,B\}}$ on $X \times \operatorname{ran}(A+B)$. If $\operatorname{ran}(A+B)$ is closed, we deduce from (2.6) that $F_{A+B} = \Phi_{\{A,B\}}$ everywhere. ☐

REMARK 5.5. *We do not know whether or not the conclusion of Theorem 5.4 remains true when the assumption on the closedness of the range of the sum of the operators is omitted. Indeed, we do not know whether or not two continuous, linear, and monotone operators* $A\colon X \to X$ *and* $B\colon X \to X$ *exist for which* $F_{A+B} \neq \Phi_{\{A,B\}}$.

COROLLARY 5.6. *Let* $A\colon X \to X$ *and* $B\colon X \to X$ *be continuous, linear, and monotone operators such that* $\operatorname{ran}(A_+ + B_+)$ *is closed. Then* $F_{A+B} = \Phi_{\{A,B\}}$.

Proof. Fix $(x,u) \in X \times X$. Using Theorem 5.3.(ii), Theorem 5.4, Theorem 5.3.(i), and Theorem 5.3.(ii) again, we obtain

(5.6)
$$\begin{aligned}
&F_{A+B}(x,u)\\
&= F_{A_+ + A_\circ + B_+ + B_\circ}(x,u) = F_{(A_+ + B_+)+(A_\circ + B_\circ)}(x,u)\\
&= \inf_{v+w=u} F_{A_+ + B_+}(x,v) + F_{A_\circ + B_\circ}(x,w)\\
&= \inf_{v+w=u} \big(\inf_{v_1+v_2=v} F_{A_+}(x,v_1) + F_{B_+}(x,v_2) + \inf_{w_1+w_2=w} F_{A_\circ}(x,w_1) + F_{B_\circ}(x,w_2) \big)\\
&= \inf_{v_1+v_2+w_1+w_2=u} F_{A_+}(x,v_1) + F_{A_\circ}(x,w_1) + F_{B_+}(x,v_2) + F_{B_\circ}(x,w_2)\\
&= \inf_{u_1+u_2=u} \big(\inf_{v_1+w_1=u_1} F_{A_+}(x,v_1) + F_{A_\circ}(x,w_1) + \inf_{v_2+w=u_2} F_{B_+}(x,v_2) + F_{B_\circ}(x,w_2) \big)\\
&= \inf_{u_1+u_2=u} F_A(x,u_1) + F_B(x,u_2)\\
&= \Phi_{\{A,B\}}(x,u),
\end{aligned}$$

as required. □

COROLLARY 5.7. *Suppose that X is finite-dimensional, and let $A\colon X \to X$ and $B\colon X \to X$ be continuous, linear, and monotone operators. Then $F_{A+B} = \Phi_{\{A,B\}}$.*

6. Cyclic monotonicity. An interesting quantitative grading of monotonicity is the notion of cyclic monotonicity of order n. As demonstrated in [2], this property is captured with a Fitzpatrick function of the corresponding order. In this section, we study these notions for continuous linear operators. Let us start with the relevant definitions.

DEFINITION 6.1 (n-cyclic monotonicity). *Let $A\colon X \to X$ be continuous and linear. Then A is* n-cyclically monotone *if $n \in \{2,3,\ldots\}$ and*

$$(\forall (x_1,\ldots,x_n) \in X^n) \quad \left(\sum_{i=1}^{n-1} \langle x_{i+1} - x_i, Ax_i \rangle \right) + \langle x_1 - x_n, Ax_n \rangle \leq 0. \tag{6.1}$$

The operator A is cyclically monotone *if A is n-cyclically monotone for every $n \in \{2,3,\ldots\}$.*

Note that an operator is monotone if and only if it is 2-cyclically monotone.

DEFINITION 6.2 (Fitzpatrick function of order n). *Let $A\colon X \to X$. For every $n \in \{2,3,\ldots\}$, the* Fitzpatrick function of A of order n *is*

(6.2)
$$F_{A,n}(x,u) = \sup_{(x_1,\ldots,x_{n-1}) \in X^{n-1}} \left(\sum_{i=1}^{n-2} \langle x_{i+1} - x_i, Ax_i \rangle \right) + \langle x - x_{n-1}, Ax_{n-1} \rangle + \langle x_1, u \rangle.$$

We set $F_{A,\infty} = \sup_{n \in \{2,3,\ldots\}} F_{A,n}$.

Note that $F_{A,2} = F_A$. We refer the reader to [2], where it is shown that $F_{A,n}$ is well suited to study n-cyclic monotonicity of A. Most relevant for our current setting is the following result.

FACT 6.3. [2, Theorem 2.9] *Let $A\colon X \to X$ be maximal monotone, and let $n \in \{2,3,\ldots\}$. Then A is n-cyclically monotone $\Leftrightarrow$ $\operatorname{gra} A = \big\{(x,u) \in X \times X \mid F_{A,n}(x,u) = \langle x,u \rangle\big\}$.*

Let us compute the Fitzpatrick functions of an arbitrary continuous, linear, symmetric, and positive definite operator. This result generalizes [2, Example 4.4].

EXAMPLE 6.4. *Let $A\colon X \to X$ be continuous, linear, symmetric, and positive definite, and let $n \in \{2,3,\ldots\}$. Then*

$$F_{A,n}\colon X\times X \to \mathbb{R}\colon (x,u) \mapsto \tfrac{n-1}{2n}\big(\|x\|_A^2 + \|u\|_{A^{-1}}^2\big) + \tfrac{1}{n}\langle x,u\rangle \tag{6.3}$$

and

$$F_{A,\infty} = \tfrac{1}{2}\|\cdot\|_A^2 \oplus \tfrac{1}{2}\|\cdot\|_{A^{-1}}^2. \tag{6.4}$$

Proof. By [2, Example 4.4], we have

$$F_{\mathrm{Id},n}\colon X\times X \to \mathbb{R}\colon (x,u) \mapsto \tfrac{n-1}{2n}\big(\|x\|^2 + \|u\|^2\big) + \tfrac{1}{n}\langle x,u\rangle \tag{6.5}$$

and

$$F_{\mathrm{Id},\infty} = \tfrac{1}{2}\|\cdot\|^2 \oplus \tfrac{1}{2}\|\cdot\|^2. \tag{6.6}$$

Fix $(x,u) \in X\times X$. By definition, $F_{A,n}(x,u)$ is equal to

$$\begin{aligned} &\sup_{(x_1,\ldots,x_{n-1})\in X^{n-1}} \Big(\sum_{i=1}^{n-2}\langle x_{i+1}-x_i, Ax_i\rangle\Big) + \langle x - x_{n-1}, Ax_{n-1}\rangle + \langle x_1, u\rangle \\ &= \sup_{(x_1,\ldots,x_{n-1})\in X^{n-1}} \Big(\sum_{i=1}^{n-2}\langle x_{i+1}-x_i, x_i\rangle_A\Big) + \langle x - x_{n-1}, x_{n-1}\rangle_A + \langle x_1, A^{-1}u\rangle_A. \end{aligned} \tag{6.7}$$

The result now follows by applying (6.5)&(6.6) to Id, viewed as an operator on $(X, \langle\cdot,\cdot\rangle_A)$. ∎

We now provide a simple, yet powerful, recursion formula.

THEOREM 6.5 (recursion). *Let $A\colon X \to X$ be monotone, and let $n \in \{2,3,\ldots\}$. Then*

$$\big(\forall (x,u) \in X\times X\big) \quad F_{A,n+1}(x,u) = \sup_{y\in X}\big(F_{A,n}(y,u) + \langle x-y, Ay\rangle\big). \tag{6.8}$$

Proof. Fix $(x,u)\in X\times X$. Using the definition, we see that $F_{A,n+1}(x,u)$ is equal to

$$\begin{aligned} &\sup_{(x_1,\ldots,x_n)\in X^n} \Big(\sum_{i=1}^{n-1}\langle x_{i+1}-x_i, Ax_i\rangle\Big) + \langle x - x_n, Ax_n\rangle + \langle x_1, u\rangle \\ &= \sup_{x_n\in X}\left(\sup_{(x_1,\ldots,x_{n-1})\in X^{n-1}} \Big(\sum_{i=1}^{n-2}\langle x_{i+1}-x_i, Ax_i\rangle\Big) + \langle x_n - x_{n-1}, Ax_{n-1}\rangle + \langle x_1, u\rangle\right) \\ &\qquad + \langle x - x_n, Ax_n\rangle \\ &= \sup_{x_n\in X}\Big(F_{A,n}(x_n,u) + \langle x - x_n, Ax_n\rangle\Big). \end{aligned} \tag{6.9}$$

The proof is complete. ∎

This section is concluded with two results on the domain of the Fitzpatrick function of order n.

THEOREM 6.6. *Let* $f\colon X \to \left]-\infty,+\infty\right]$ *be convex, lower semicontinuous, and proper, and let* $n \in \{2,3,\ldots\}$*. Then*

$$\operatorname{dom} f \times \operatorname{dom} f^* \subseteq \operatorname{dom} F_{\partial f,n} \subseteq \operatorname{dom} F_{\partial f} \subseteq \overline{\operatorname{dom}}\, f \times \overline{\operatorname{dom}}\, f^*. \tag{6.10}$$

Proof. By [2, Theorem 3.5], we know that $F_{\partial f,n} \leq f \oplus f^*$, which implies the first inequality of (6.10). The second inequality is clear since $(F_{\partial f,n})_{n\in\{2,3,\ldots\}}$ is an increasing sequence. The third inequality follows from Theorem 4.15. ☐

COROLLARY 6.7. *Let* $A\colon X \to X$ *be continuous, linear, monotone, and symmetric, and let* $n \in \{2,3,\ldots,\}$*. Then*

$$X \times \operatorname{ran} A \subseteq \operatorname{dom} F_{A,n} \subseteq X \times \overline{\operatorname{ran}}\, A. \tag{6.11}$$

7. Rotators in the Euclidean plane. This section covers rotators in the Euclidean plane. We characterize their cyclic monotonicity properties, and we provide formulas for the Fitzpatrick function of any order.

From now on, $X = \mathbb{R}^2$ and

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{where } \theta \in [0,\pi/2]. \tag{7.1}$$

The main result of this section will be stated at the end. For clarity of presentation, we break up the proof into several propositions. The first proposition characterizes n-cyclic monotonicity. See also Asplund's paper [1] for characterizations for general matrices.

PROPOSITION 7.1. *Let* $n \in \{2,3,\ldots\}$*. Then* A_θ *is* n*-cyclically monotone* $\Leftrightarrow$ $\theta \in [0,\pi/n]$.

Proof. If $n = 2$, then the symmetric part of A_θ is $\cos\theta\,\mathrm{Id}$ and the equivalence is clear. Thus, we assume that $n \in \{3,4,\ldots\}$. We shall characterize the n-cyclic monotonicity of A_θ in terms of the positive semidefiniteness of an associated Hermitian matrix. Take n points $x_1 = (\xi_1,\eta_1),\ldots,x_n = (\xi_n,\eta_n)$ in X, and set $x_{n+1} = x_1$. We must show that

$$0 \geq \sum_{i=1}^{n} \langle x_{i+1} - x_i, A_\theta x_i\rangle. \tag{7.2}$$

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way: $x = (\xi,\eta)$ in $\mathbb{R}^2$ corresponds to $\xi + \mathrm{i}\eta$ in $\mathbb{C}$, where $\mathrm{i} = \sqrt{-1}$, and $\langle x,y\rangle = \operatorname{Re}(\overline{x}y)$ for x and y in $\mathbb{C}$. The operator A_θ corresponds to complex multiplication by

$$\omega = \exp(\mathrm{i}\theta). \tag{7.3}$$

Thus we aim to show that $0 \geq \operatorname{Re}\big(\sum_{i=1}^n (\overline{x_{i+1} - x_i})\omega x_i\big) = \sum_{i=1}^n \operatorname{Re}\big(\overline{(x_{i+1} - x_i)}\omega x_i\big)$, an inequality which we now reformulate in $\mathbb{C}^n$. Denote the $n \times n$-identity matrix by

$\mathbf{I}$, and set

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \\ & & & & 0 \\ 0 & & & & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix} \in \mathbb{C}^{n\times n} \quad \text{and} \quad \mathbf{R} = \omega\mathbf{I} \in \mathbb{C}^{n\times n}. \tag{7.4}$$

Identifying $\mathbf{x} \in \mathbb{C}^n$ with $(x_1,\ldots,x_n) \in X^n$, we note that (7.2) means $0 \geq \operatorname{Re}\big(((\mathbf{B}-\mathbf{I})\mathbf{x})^*\mathbf{R}\mathbf{x}\big)$; equivalently, $0 \geq \mathbf{x}^*(\mathbf{B}^*-\mathbf{I})\mathbf{R}\mathbf{x} + \mathbf{x}^*\mathbf{R}^*(\mathbf{B}-\mathbf{I})\mathbf{x}$. Set

$$\begin{aligned} \mathbf{C}_n &= (\mathbf{I}-\mathbf{B}^*)\mathbf{R} + \mathbf{R}^*(\mathbf{I}-\mathbf{B}) \\ &= \begin{pmatrix} (\omega+\overline{\omega}) & -\overline{\omega} & 0 & \cdots & 0 & -\omega \\ -\omega & (\omega+\overline{\omega}) & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & (\omega+\overline{\omega}) & -\overline{\omega} \\ -\overline{\omega} & 0 & \cdots & 0 & -\omega & (\omega+\overline{\omega}) \end{pmatrix} \in \mathbb{C}^{n\times n}. \end{aligned} \tag{7.5}$$

Then

$$A_\theta \text{ is } n\text{-cyclically monotone} \Leftrightarrow \mathbf{C}_n \text{ is positive semidefinite on } \mathbb{C}^n. \tag{7.6}$$

Note that the matrix $\mathbf{C}_n$ is a circulant Toeplitz matrix. E.g., by [26, Exercise 5.8.12], the set of eigenvalues of $\mathbf{C}_n$ is

$$\Lambda_n = \big\{p(1), p(\zeta), \ldots, p(\zeta^{n-1})\big\}, \quad \text{where} \quad p\colon t \mapsto (\omega+\overline{\omega}) - \omega t - \overline{\omega}t^{n-1}, \tag{7.7}$$

and where ζ is an arbitrary n^{th} root of unity. It will be convenient to work with

$$\zeta_n = \exp(-2\pi \mathrm{i}/n). \tag{7.8}$$

Then

$$\begin{aligned} \big(\forall k \in \{0,1,\ldots,n-1\}\big) \quad p(\zeta_n^k) &= \omega + \overline{\omega} - \omega\zeta_n^k - \overline{\omega}(\zeta_n^k)^{n-1} \\ &= \omega + \overline{\omega} - \omega\zeta_n^k - \overline{\omega}(\zeta_n^{n-1})^k \\ &= \omega + \overline{\omega} - \omega\zeta_n^k - \overline{\omega}(\overline{\zeta_n})^k \\ &= \omega + \overline{\omega} - (\omega\zeta_n^k + \overline{\omega\zeta_n^k}) \\ &= 2\big(\cos(\theta) - \cos(2k\pi/n - \theta)\big). \end{aligned} \tag{7.9}$$

"$\Leftarrow$": Assume that $\theta \in [0, \pi/n]$. If $k \in \{1,2,\ldots,n-1\}$, then $\theta \leq 2k\pi/n - \theta < 2\pi - \theta$ and (7.9) implies that $p(\zeta_n^k) \geq 0$. On the other hand, $p(1) = 0$. Altogether, every eigenvalue in Λ_n is nonnegative and the Hermitian matrix $\mathbf{C}_n$ is thus positive semidefinite. Therefore, by (7.6), A_θ is n-cyclically monotone.

"$\Rightarrow$": Assume that $\theta \in\,]\pi/(n+1), \pi/n]$. It suffices to show that A_θ is not $(n+1)$-cyclically monotone. Now (7.9) implies that $p(\zeta_{n+1}) = 2\big(\cos(\theta) - \cos(2\pi/(n+1) -$

$\theta)\big) < 0$ since $0 < 2\pi/(n+1) - \theta < \theta$. In view of (7.7)&(7.6), we deduce that Λ_{n+1} contains a strictly negative eigenvalue, i.e., the matrix $\mathbf{C}_{n+1}$ is not positive semidefinite, and therefore A_θ is not $(n+1)$-cyclically monotone. ☐

REMARK 7.2. *The symmetric part of every continuous linear monotone operator is a subdifferential and hence cyclically monotone. Hence, higher order n-cyclic monotonicity properties are not captured in the symmetric part. In other words, the analog of Proposition 1.2 for n-cyclically monotone operators, where $n \in \{3, 4, \ldots\}$, is false: $A_{\pi/2}$ is not 3-cyclically monotone (by Proposition 7.1), yet its symmetric part $\big(A_{\pi/2}\big)_+ = 0$ is cyclically monotone.*

PROPOSITION 7.3. *Let $n \in \{2, 3, \ldots\}$ and suppose that $\theta \in\,]\pi/(n+1), \pi/n]$. Then $F_{A_\theta, n+1} \equiv +\infty$.*

Proof. We shall utilize the following result on tridiagonal Toeplitz matrices, see [26, Example 7.2.5].

If $\alpha \in \mathbb{C} \smallsetminus \{0\}$, $\beta \in \mathbb{C}$, and $\gamma \in \mathbb{C} \smallsetminus \{0\}$, then the eigenvalues and the eigenvectors of the $n \times n$ matrix

$$\begin{pmatrix} \beta & \alpha & 0 & \cdots & 0 \\ \gamma & \beta & \alpha & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \gamma & \beta & \alpha \\ 0 & \cdots & 0 & \gamma & \beta \end{pmatrix} \tag{7.10}$$

are given by

(7.11)

$$\lambda_k = \beta + 2\alpha\rho\cos\big(k\pi/(n+1)\big) \quad \text{and} \quad \mathbf{x}_k = \begin{pmatrix} \rho\sin\big(k\pi/(n+1)\big) \\ \rho^2\sin\big(2k\pi/(n+1)\big) \\ \rho^3\sin\big(3k\pi/(n+1)\big) \\ \vdots \\ \rho^n\sin\big(nk\pi/(n+1)\big) \end{pmatrix},$$

respectively, where

$$k \in \{1, 2, \ldots, n\} \quad \text{and} \quad \rho = \sqrt{\gamma/\alpha}. \tag{7.12}$$

We identify $\mathbb{R}^2$ with $\mathbb{C}$ as in the proof of Proposition 7.1, where we set $\omega = \exp(\mathrm{i}\theta)$. By (6.2), for an arbitrary $(x, u) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

(7.13)

$$\begin{aligned} F_{A_\theta, n+1}(x, u) &= \sup_{a_1, \ldots, a_n} \Big(\sum_{i=1}^{n-1} \langle a_{i+1} - a_i, A_\theta a_i \rangle \Big) + \langle x - a_n, A_\theta a_n \rangle + \langle a_1 - x, u \rangle + \langle x, u \rangle \\ &= \sup_{a_1, \ldots, a_n} \mathrm{Re}\bigg(\Big(\sum_{i=1}^{n-1} \overline{(a_{i+1} - a_i)}\omega a_i \Big) + \overline{(-a_n)}\omega a_n + \overline{x}\omega a_n + \overline{a_1} u \bigg) \\ &= \sup_{\mathbf{a} \in \mathbb{C}^n} \tfrac{1}{2}\big(\mathbf{a}^*\mathbf{H}\mathbf{a} + (\overline{x}\omega a_n + x\overline{\omega a_n}) + (\overline{a_1}u + a_1\overline{u}) \big), \end{aligned}$$

where $\mathbf{a} = (a_1, \ldots, a_n)^T \in \mathbb{C}^n$ and

$$\mathbf{H} = \begin{pmatrix} -(\omega+\overline{\omega}) & \overline{\omega} & 0 & \cdots & 0 \\ \omega & -(\omega+\overline{\omega}) & \overline{\omega} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & \omega & -(\omega+\overline{\omega}) & \overline{\omega} \\ 0 & \cdots & 0 & \omega & -(\omega+\overline{\omega}) \end{pmatrix} \in \mathbb{C}^{n\times n}. \tag{7.14}$$

By (7.11), the n eigenvalues of the Hermitian matrix $\mathbf{H}$ are given by

$$\begin{aligned} \big(\forall k \in \{1,\ldots,n\}\big) \quad \lambda_k &= -(\omega+\overline{\omega}) + 2\,\overline{\omega}\sqrt{\omega/\overline{\omega}}\cos\big(k\pi/(n+1)\big) \\ &= 2\big(\cos\big(k\pi/(n+1)\big) - \cos(\theta)\big). \end{aligned} \tag{7.15}$$

Since $0 < \pi/(n+1) < \theta \leq \pi/2$, we deduce that

$$\lambda_1 = 2\big(\cos(\pi/(n+1)) - \cos(\theta)\big) > 0. \tag{7.16}$$

Furthermore, since $\mathbf{H}$ is Hermitian, it can be unitarily diagonalized. There exists a unitary matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ such that $\mathbf{U}^*\mathbf{H}\mathbf{U} = \mathbf{D}$ is a diagonal matrix, with eigenvalues $\lambda_1, \ldots, \lambda_n$ on its diagonal. On one hand, changing variables via $\mathbf{a} = \mathbf{U}\mathbf{y}$, where $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{C}^n$, we have

$$\mathbf{a}^*\mathbf{H}\mathbf{a} = \lambda_1|y_1|^2 + \cdots + \lambda_n|y_n|^2. \tag{7.17}$$

Note that if $\mathbf{y} = \tau(1, 0, \ldots, 0)^T$, then $\mathbf{a}^*\mathbf{H}\mathbf{a} = \lambda_1\tau^2$ is a convex quadratic in τ. On the other hand,

$$(\overline{x}\omega a_n + x\overline{\omega a_n}) + (\overline{a_1}x^* + a_1\overline{x^*}) \tag{7.18}$$

is $\mathbb{R}$-linear in $\mathbf{a}$, in $\mathbf{y}$, and in τ. Altogether, the supremum in (7.13) is equal to $+\infty$. This completes the proof. ☐

Proposition 7.4. *Let $n \in \{2, 3, \ldots, \}$ and suppose that $\theta = \pi/n$. Then $F_{A_\theta,n} = \iota_{\operatorname{gra} A_\theta} + \langle\cdot,\cdot\rangle$.*

Proof. Fix $(x, u) \in X \times X$. If $u = A_\theta x$, then $F_{A_\theta,n}(x, u) = \langle x, u\rangle$ by Fact 6.3. Thus assume that $u \neq A_\theta x$. Arguing as in the proof of Proposition 7.3, we see that

$$F_{A_\theta,n}(x,u) = \sup_{\mathbf{a}\in\mathbb{C}^{n-1}} \tfrac{1}{2}\big(\mathbf{a}^*\mathbf{H}\mathbf{a} + (\overline{x}\omega a_{n-1} + x\overline{\omega a_{n-1}}) + (\overline{a_1}u + a_1\overline{u})\big), \tag{7.19}$$

where $\omega = \exp(\mathrm{i}\theta) = \exp(\pi\mathrm{i}/n)$, $\mathbf{a} = (a_1, \ldots, a_{n-1})^T \in \mathbb{C}^{n-1}$ and

$$\mathbf{H} = \begin{pmatrix} -(\omega+\overline{\omega}) & \overline{\omega} & 0 & \cdots & 0 \\ \omega & -(\omega+\overline{\omega}) & \overline{\omega} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & \omega & -(\omega+\overline{\omega}) & \overline{\omega} \\ 0 & \cdots & 0 & \omega & -(\omega+\overline{\omega}) \end{pmatrix} \in \mathbb{C}^{(n-1)\times(n-1)}, \tag{7.20}$$

and where the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ are given by (this is the counterpart of (7.15))

$$\big(\forall k \in \{1,\ldots,n-1\}\big) \quad \mu_k = 2\big(\cos(k\pi/n) - \cos(\theta)\big) \leq 0. \tag{7.21}$$

Note that $\mu_1 = 0$ and that, by (7.11),

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} \exp(\pi \mathrm{i}/n)\sin(\pi/n) \\ \exp(2\pi \mathrm{i}/n)\sin(2\pi/n) \\ \vdots \\ \exp((n-1)\pi \mathrm{i}/n)\sin((n-1)\pi/n) \end{pmatrix} \tag{7.22}$$

is a corresponding eigenvector. Then $\mathbf{Hb} = \mathbf{0} \in \mathbb{C}^{n-1}$. Using $z\mathbf{b}$, where $z \in \mathbb{C}$, rather than the general vector $\mathbf{a}$ in (7.19), we estimate

(7.23)

$$\begin{aligned} & F_{A_\theta,n}(x,u) \\ \geq\ & \sup_{z\in\mathbb{C}} \mathrm{Re}\,(\overline{x}\omega z b_{n-1} + \overline{z b_1} u) \\ =\ & \sup_{z\in\mathbb{C}} \mathrm{Re}\,\big(\overline{x}\exp(\pi \mathrm{i}/n) z \exp((n-1)\pi \mathrm{i}/n)\sin((n-1)\pi/n) + \overline{z}\exp(-\pi \mathrm{i}/n)\sin(\pi/n)u\big) \\ =\ & \sin(\pi/n)\sup_{z\in\mathbb{C}} \mathrm{Re}\,\big(\overline{z}(u\exp(-\pi \mathrm{i}/n) - x)\big). \end{aligned}$$

Because $u \neq A_\theta x$, i.e., $u \neq \exp(\pi \mathrm{i}/n)x$ viewed in $\mathbb{C}$, we see that $u\exp(-\pi \mathrm{i}/n) - x \neq 0$. Thus, the last supremum is equal to $+\infty$. ☐

The following example will be utilized in Proposition 7.6.

Example 7.5. *Suppose that $\theta \in [0, \pi/2[$. Then*

$$F_{A_\theta,2}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\colon (x,u) \mapsto \frac{1}{4\cos\theta}\|u + A_\theta^* x\|^2. \tag{7.24}$$

Proof. The symmetric part of A_θ is equal to $\cos(\theta)\,\mathrm{Id}$, and hence invertible. The result follows by combining Theorem 2.3.(i) and Fact 2.2.(vi). ☐

Proposition 7.6. *Let $n \in \{2, 3, \ldots\}$ and suppose that $\theta \in \,]0, \pi/n[$. Then*

(7.25)

$$F_{A_\theta,n}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$(x,u) \mapsto \frac{\sin(n-1)\theta}{2\sin n\theta}\big(\|x\|^2 + \|u\|^2\big) + \frac{\sin\theta}{\sin n\theta}\langle x, A_\theta^{n-1}u\rangle \tag{7.26}$$

$$= \frac{\sin\theta}{2\sin n\theta}\bigg(\Big(\frac{\sin(n-1)\theta}{\sin\theta} - 1\Big)\big(\|x\|^2 + \|A_\theta^{n-1}u\|^2\big) + \|x + A_\theta^{n-1}u\|^2\bigg). \tag{7.27}$$

Proof. Observe that (7.27) is a direct consequence of (7.26). It suffices to verify (7.25)–(7.26), and we do this by induction on n. Fix $(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2$. Consider the case when $n = 2$. Using Example 7.5 and the trigonometric identity $(\sin\theta)/(\sin 2\theta) = 1/(2\cos\theta)$, we obtain

$$F_{A_\theta,2}(x,u) = \frac{1}{4\cos\theta}\|u + A_\theta^* x\|^2 = \frac{\sin\theta}{2\sin 2\theta}\big(\|x\|^2 + \|u\|^2 + 2\langle u, A_\theta^* x\rangle\big), \tag{7.28}$$

which yields (7.26). We now assume that (7.26) holds for some $n \in \{2, 3, \ldots\}$, and we shall show that it also holds for $n+1$, provided that $\theta \in \,]0, \pi/(n+1)[$. Utilizing

Theorem 6.5 and trigonometric identities, we obtain

(7.29)

$$\begin{aligned}
&F_{A_\theta,n+1}(x,u)\\
&=\sup_{y\in X} F_{A_\theta,n}(y,u)+\langle x-y,A_\theta y\rangle\\
&=\sup_{y\in X} \frac{\sin(n-1)\theta}{2\sin n\theta}\big(\|y\|^2+\|u\|^2\big)+\frac{\sin\theta}{\sin n\theta}\langle y,A_\theta^{n-1}u\rangle+\langle A_\theta^* x,y\rangle-\langle y,A_\theta y\rangle\\
&=\sup_{y\in X} \Big(\frac{\sin(n-1)\theta}{2\sin n\theta}-\cos\theta\Big)\|y\|^2+\frac{\sin(n-1)\theta}{2\sin n\theta}\|u\|^2+\frac{\sin\theta}{\sin n\theta}\langle y,A_\theta^{n-1}u\rangle+\langle A_\theta^* x,y\rangle
\end{aligned}$$

(7.30)

$$=\sup_{y\in X} \frac{-\sin(n+1)\theta}{2\sin n\theta}\|y\|^2+\frac{\sin(n-1)\theta}{2\sin n\theta}\|u\|^2+\frac{\sin\theta}{\sin n\theta}\langle y,A_\theta^{n-1}u\rangle+\langle A_\theta^* x,y\rangle.$$

Since $\theta\in\,]0,\pi/(n+1)[$, the coefficient of $\|y\|^2$ is strictly negative, which shows that the quadratic function of y we take the supremum of in (7.30) is strictly concave. Setting the derivative of this quadratic function equal to 0, we find that the unique maximizer in (7.30) is

$$\frac{\sin n\theta}{\sin(n+1)\theta}\Big(\frac{\sin\theta}{\sin n\theta}A_\theta^{n-1}u+A_\theta^* x\Big). \tag{7.31}$$

Combining this with (7.29)&(7.30), followed by simplification and utilization of trigonometric identities, we deduce that

$$F_{A_\theta,n+1}(x,u)=\frac{\sin n\theta}{2\sin(n+1)\theta}\big(\|x\|^2+\|u\|^2\big)+\frac{\sin\theta}{\sin(n+1)\theta}\langle x,A_\theta^n u\rangle, \tag{7.32}$$

and this completes the proof. ☐

Remark 7.7. *Consider the setting of Proposition 7.6. Since $n\in\{2,3,\ldots\}$ and since $\theta\in\,]0,\pi/n[$, we have $\theta\leq(n-1)\theta<\pi-\theta$ and thus $\sin(n-1)\theta\geq\sin\theta$. While it is clear from the definition that $F_{A_\theta,n}$ is convex (see (6.2)), we see this also directly from (7.27).*

We have obtained complete knowledge of all Fitzpatrick functions. Let us summarize our findings.

Theorem 7.8. *Let $\theta\in[0,\pi/2]$ and let A_θ be the rotator by θ in the Euclidean plane, i.e.,*

$$A_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}. \tag{7.33}$$

(i) *Case $\theta=0$. Then $A_\theta=\mathrm{Id}=\nabla\frac{1}{2}\|\cdot\|^2$ is cyclically monotone, $F_{\mathrm{Id},\infty}=\frac{1}{2}\|\cdot\|^2\oplus\frac{1}{2}\|\cdot\|^2$, and*

(7.34)

$$\big(\forall n\in\{2,3,\ldots\}\big)\quad F_{\mathrm{Id},n}\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}\colon(x,u)\mapsto\frac{n-1}{2n}\big(\|x\|^2+\|u\|^2\big)+\frac{1}{n}\langle x,u\rangle.$$

(ii) *Case $\theta\in\,]0,\pi/2]$. If $n\in\{2,3,\ldots\}\cap[2,\pi/\theta[$, then A_θ is n-cyclically monotone and*

(7.35)

$$F_{A_\theta,n}\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}\colon(x,u)\mapsto\frac{\sin(n-1)\theta}{2\sin n\theta}\big(\|x\|^2+\|u\|^2\big)+\frac{\sin\theta}{\sin n\theta}\langle x,A_\theta^{n-1}u\rangle.$$

If π/θ is an integer, then A_θ is (π/θ)-cyclically monotone and

$$F_{A_\theta,\pi/\theta} = \iota_{\operatorname{gra} A_\theta} + \langle\cdot,\cdot\rangle. \tag{7.36}$$

If $n \in \{2,3,\ldots,\} \cap \,]\pi/\theta, +\infty[$, then A_θ is not n-cyclically monotone and

$$F_{A_\theta,n} \equiv +\infty. \tag{7.37}$$

Proof. (i): This follows from Example 6.4 with $A = \mathrm{Id}$. (ii): A direct consequence of Propositions 7.1, 7.3, 7.4, and 7.6. ☐

Remark 7.9. *Theorem 7.8 greatly expands the knowledge about rotators and their Fitzpatrick functions. In previous work [2], only rotators by* 0 *or by π/n, where $n \in \{2,3,\ldots\}$, were considered. In that restricted setting, item (i) of Theorem 7.8 was known [2, Example 4.4]. It was also known that $A_{\pi/n}$ is n-cyclically monotone but not $(n+1)$-cyclically monotone [2, Example 4.6]. The formula* (7.36) *was only known for $\theta = \pi/2$ [2, Example 4.5], and formula* (7.37) *was only known for $n \in \{2,3,4\}$ [2, Remark 4.7].*

Acknowledgment. The authors wish to thank Sedi Bartz, Simeon Reich, and two anonymous referees for their pertinent comments which led to considerable improvements over the original version of this manuscript.

REFERENCES

[1] E. Asplund, "A monotone convergence theorem for sequences of nonlinear mappings," *Nonlinear Functional Analysis*, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, vol. XVIII Part 1, Chicago, pp. 1–9, 1970.

[2] S. Bartz, H. H. Bauschke, J. M. Borwein, S. Reich, and X. Wang, "Fitzpatrick functions, cyclic monotonicity, and Rockafellar's antiderivative," to appear in *Nonlinear Analysis*.

[3] H. H. Bauschke, *Projection Algorithms and Monotone Operators*, Ph.D. Thesis, Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, Canada, August 1996. Available at `http://www.cecm.sfu.ca/preprints/1996pp.html`

[4] H. H. Bauschke, "The composition of finitely many projections onto closed convex sets in Hilbert space is asymptotically regular," *Proceedings of the American Mathematical Society,* vol. 131, pp. 141–146, 2003.

[5] H. H. Bauschke and J. M. Borwein, "Maximal monotonicity of dense type, local maximal monotonicity, and monotonicity of the conjugate are all the same for continuous linear operators," *Pacific Journal of Mathematics,* vol. 189, pp. 1–20, 1999.

[6] H. H. Bauschke, D. A. McLaren, and H. S. Sendov, "Fitzpatrick functions: inequalities, examples and remarks on a problem by S. Fitzpatrick," to appear in *Journal of Convex Analysis*.

[7] H. H. Bauschke and S. Simons, "Stronger maximal monotonicity properties of linear operators," *Bulletin of the Australian Mathematical Society,* vol. 60, pp. 163–174, 1999.

[8] J. M. Borwein, "Maximal monotonicity via convex analysis," to appear in *Journal of Convex Analysis*.

[9] J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis,* Springer-Verlag, 2005.

[10] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert,* North-Holland, 1973.

[11] H. Brézis and A. Haraux, "Image d'une somme d'opérateurs monotones et applications," *Israel Journal of Mathematics,* vol. 23, pp. 165–186, 1976.

[12] R. S. Burachik and S. Fitzpatrick, "On a family of convex functions associated to subdifferentials," *Journal of Nonlinear and Convex Analysis*, vol. 6, pp. 165–171, 2005.

[13] R. S. Burachik and S. Fitzpatrick, "Corrigendum to: On a family of convex functions associated to subdifferentials," *Journal of Nonlinear and Convex Analysis*, vol. 6, p. 535, 2005.

[14] R. S. Burachik and B. F. Svaiter, "Maximal monotone operators, convex functions and a special family of enlargements," *Set-Valued Analysis*, vol. 10, pp. 297–316, 2002.

[15] R. S. Burachik and B. F. Svaiter, "Maximal monotonicity, conjugation and the duality product," *Proceedings of the American Mathematical Society*, vol. 131, pp. 2379–2383, 2003.

[16] S. Cheng and Y. Tian, "Two sets of new characterizations for normal and EP matrices," *Linear Algebra and its Applications*, vol. 375, pp. 181–195, 2003.
[17] S. Fitzpatrick, "Representing monotone operators by convex functions," *Workshop/Miniconference on Functional Analysis and Optimization (Canberra 1988)*, Proceedings of the Centre for Mathematical Analysis, Australian National University vol. 20, Canberra, Australia, pp. 59–65, 1988.
[18] Y. García, M. Lassonde, and J. P. Revalski, "Extended sums and extended compositions of monotone operators," to appear in *Journal of Convex Analysis*.
[19] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings,* Marcel Dekker, 1984.
[20] C. W. Groetsch, *Generalized Inverses of Linear Operators,* Marcel Dekker, 1977.
[21] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand Reinbold, 1967.
[22] A. N. Iusem, "On some properties of paramonotone operators," *Journal of Convex Analysis,* vol. 5, pp. 269–278, 1998.
[23] E. Kreyszig, *Introductory Functional Analysis with Applications,* John Wiley & Sons, 1989.
[24] J.-E. Martínez-Legaz and B. F. Svaiter, "Monotone operators representable by l.s.c. convex functions," *Set-Valued Analysis,* vol. 13, pp. 21–46, 2005.
[25] J.-E. Martínez-Legaz and M. Théra, "A convex representation of maximal monotone operators," *Journal of Nonlinear and Convex Analysis,* vol. 2, pp. 243–247, 2001.
[26] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, 2000.
[27] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I*, Springer-Verlag, 2006.
[28] B. S. Mordukhovich and Y. Shao, "Nonsmooth sequential analysis in Asplund spaces," *Transactions of the American Mathematical Society,* vol. 348, pp. 1235–1280, 1996.
[29] J. P. Penot, "The relevance of convex analysis for the study of monotonicity," *Nonlinear Analysis,* vol. 58, pp. 855–871, 2004.
[30] R. R. Phelps and S. Simons, "Unbounded linear monotone operators on nonreflexive Banach spaces," *Journal of Convex Analysis,* vol. 5, pp. 303–328, 1998.
[31] S. Reich and S. Simons, "Fenchel duality, Fitzpatrick functions and the Kirszbraun-Valentine extension theorem," *Proceedings of the American Mathematical Society,* vol. 133, pp. 2657–2660, 2005.
[32] R. T. Rockafellar, "On the maximal monotonicity of subdifferential mappings," *Pacific Journal of Mathematics,* vol. 33, pp. 209–216, 1970.
[33] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.
[34] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization,* vol. 14, pp. 877–898, 1976.
[35] R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*, Springer-Verlag, 1998.
[36] S. Simons, *Minimax and Monotonicity*, Lecture Notes in Mathematics, vol. 1693, Springer-Verlag, 1998.
[37] S. Simons, "Dualized and scaled Fitzpatrick functions," *Proceedings of the American Mathematical Society,* vol. 134, pp. 2983–2987, 2006.
[38] S. Simons, "Positive sets and monotone sets," to appear in *Journal of Convex Analysis*.
[39] S. Simons, "LC-functions and maximal monotonicity," *Journal of Nonlinear and Convex Analysis,* vol. 7, pp. 123–137, 2006.
[40] S. Simons, "The Fitzpatrick function and nonreflexive spaces," to appear in *Journal of Convex Analysis*.
[41] S. Simons and C. Zălinescu, "A new proof for Rockafellar's characterization of maximal monotone operators," *Proceedings of the American Mathematical Society*, vol. 132, pp. 2969–2972, 2004.
[42] S. Simons and C. Zălinescu, "Fenchel duality, Fitzpatrick functions and maximal monotonicity," *Journal of Nonlinear and Convex Analysis,* vol. 6, pp. 1–22, 2005.
[43] C. Zălinescu, *Convex Analysis in General Vector Spaces,* World Scientific Publishing, 2002.