HOW TO TRANSFORM ONE CONVEX FUNCTION CONTINUOUSLY INTO ANOTHER*

HEINZ H. BAUSCHKE[†], YVES LUCET[‡], AND MICHAEL TRIENIS[§]

Abstract. The proximal average operator provides a parametric family of convex functions that continuously transform one convex function into another even when the domains of the two functions do not intersect. We prove that the proximal average operator is a homotopy with respect to the epi-topology, study its properties, and present several explicit formulas for specific classes of functions. The parametric family inherits desirable properties such as differentiability and strict convexity from the given functions.

The results illustrate the powerful tools available in convex and variational analysis from both a theoretical and a computational point of view.

Key words. Attouch-Wets topology, continuous transformation, convex function, deformation, epi-convergence, epi-topology, Fenchel conjugate, homotopy, proximal average, proximal mapping

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1. Introduction.

The idea of potentially connecting mathematical objects in a continuous fashion is fundamental: in geometry, points might be connected through paths; in approximation theory, data points might be connected by some interpolant; in topology, functions might be transformed into each other via homotopies. For instance, consider the two functions

$$(1.1) \quad f_0(x) = x + 2 \quad \text{and} \quad f_1(x) = x^2.$$

How can we transform f_0 continuously into f_1 ? Perhaps the most natural approach is to use the pointwise *arithmetic average*

$$(1 - \lambda)f_0 + \lambda f_1,$$

where $\lambda \in [0, 1]$. This transformation has very nice properties: the intermediate functions $(1 - \lambda)f_0 + \lambda f_1$ are all continuous, and for every $x \in \mathbb{R}$, the mapping $\lambda \mapsto f_\lambda(x)$ is also continuous. See Fig. 1.1(a) for a visualization. This approach works very well as long as the two given functions f_0 and f_1 are continuous and they share the same domain; however, it strikingly breaks down if f_0 and f_1 have *disjoint*

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[†]Mathematics, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada (heinz.bauschke@ubc.ca). Research partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program.

[‡]Computer Science, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada (yves.lucet@ubc.ca). Research partially supported by the Natural Sciences and Engineering Research Council of Canada.

[§]Computer Science, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada (mjtrieni@interchange.ubc.ca). Research partially supported by the Pacific Institute for the Mathematical Sciences.

FIG. 1.1. Averages of the linear function $f_0(x) = x + 2$ and the quadratic function $f_1(x) = x^2$.

domains because then the functions values cannot be averaged arithmetically. At this point, it is not at all clear how one could transform

$$(1.2) \quad f_0(x) = -\ln(-x) \quad \text{and} \quad f_1(x) = f_0(-x) = -\ln(x)$$

continuously into each other because the domains of these functions are, respectively, the disjoint intervals $]-\infty, 0[$ and $]0, +\infty[$.

Quite surprisingly, convex analysis will pave the way to transform these two functions (and many other convex functions) into each other. The average which provides this transformation is called the *proximal average* and denoted by

$$\mathcal{P}(f_0, \lambda, f_1),$$

where $\lambda \in [0, 1]$.

Although we are not yet ready to describe its inner workings, let us nonetheless provide two visualizations of the proximal average. Fig. 1.1(b) illustrates the proximal average for the two functions f_0 and f_1 from (1.1), while Fig. 1.2 illustrates the proximal average for the two logarithms defined in (1.2).

As will become clear in the following sections, the proximal average has various desirable properties that the arithmetic average generally lacks. For instance, if one of the given functions is differentiable, then so is the proximal average. Similarly, if one of the given functions is defined on the entire real line, then so is the proximal average.

The goal of this note is to introduce and study the proximal average. This transformation requires some basic convex calculus and it is thus accessible to students taking advanced undergraduate or beginning graduate courses in convex optimization and analysis. The material presented here provides a nice motivation for convex and variational analysis. It can also be used as a new topic in a corresponding course and — as we have not seen the main results elsewhere — possibly as a starting point for further research.

The remainder of the paper is organized as follows. In Section 2, we set the stage by reviewing notions from convex optimization and analysis. The proximal average is defined in terms of Fenchel conjugates, which are introduced in Section 3. We then present in Section 4 the formula for the proximal average as well as basic properties and examples. Section 5 introduces epi-convergence of functions. The associated topology,

FIG. 1.2. The proximal average of f_0 and f_1 , as defined in (1.2). Note that $\text{dom } f_0 \cap \text{dom } f_1 = \emptyset$ — which shows that the arithmetic average is of no use in such settings — and that for $\lambda \in]0, 1[$, the proximal average $\mathcal{P}(f_0, \lambda, f_1)$ has full domain.

the epi-topology, turns out to be exactly the right notion to establish continuity of the proximal average operator (Theorem 5.4). As demonstrated in Section 6, the proximal average inherits good properties — such as smoothness or strict convexity — from the functions it averages. As an application, we present a renorming result (Theorem 6.5). Additional examples of proximal averages are provided in Section 7, which can be viewed as a source of exercises. Some more technical results and proofs are relegated to the Appendices.

2. Convex Optimization: Basic Set Up.

From now on, we assume that

$$X = \mathbb{R}^d \text{ with inner product } \langle x, y \rangle = x^T y \text{ and norm } \|x\| = \sqrt{\langle x, x \rangle}.$$

The most basic constrained optimization problem is to

$$(2.1) \quad \text{minimize } f \text{ over } C,$$

where f is called the objective function and C is the constraint set. The *indicator function* corresponding to C is

$$\iota_C: X \rightarrow]-\infty, +\infty] = \mathbb{R} \cup \{+\infty\}: x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

Working with $+\infty$ and indicator functions allows us to express (2.1) as the equivalent unconstrained optimization problem

$$(2.2) \quad \text{minimize } f + \iota_C \text{ over } X.$$

This is a common theme in modern convex analysis and optimization; see, e.g., [8, 9, 13, 20, 21]. These books also serve as reference material for notation and results not explicitly defined here. Now fix a function $f: X \rightarrow]-\infty, +\infty]$. Then f is *convex* if

$$(\forall x \in X)(\forall y \in X)(\forall \lambda \in]0, 1[) \quad f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y);$$

and f is *lower semicontinuous* if

$$x_n \rightarrow x \Rightarrow f(x) \leq \underline{\lim} f(x_n),$$

where $\underline{\lim}$ denotes the limit inferior. We also say that f is *proper* if f is somewhere finite. A more geometric view of these three concepts rests on the notion of the *epigraph* of f , which is defined by

$$(2.3) \quad \text{epi } f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$$

and which consists of all points lying *on or above* the graph of f . Recall that a set C is convex if $(1 - \lambda)x + \lambda y$ belongs to C whenever $x \in C$, $y \in C$, and $\lambda \in [0, 1]$. Then one may check that f is convex, lower semicontinuous, and proper if and only if the set $\text{epi } f$ is convex, closed, and nonempty, respectively. The (effective) *domain* of f is

$$\text{dom } f := \{x \in X \mid f(x) \in \mathbb{R}\};$$

it coincides with the projection of $\text{epi } f$ onto the X -component. The function f has *full domain* if $\text{dom } f = X$. We shall work with the following class of functions.

$$(2.4) \quad \mathcal{F} := \{f: X \rightarrow]-\infty, +\infty] \mid f \text{ is convex, lower semicontinuous, and proper}\}.$$

EXAMPLE 2.1. *Let C be a subset of X . Then the indicator function ι_C belongs to $\mathcal{F}$ if and only if the set C is convex, closed, and nonempty.*

The next example is the multi-dimensional analogue of the function f_1 from (1.2). It lies at the heart of interior point methods in linear optimization.

EXAMPLE 2.2 (logarithmic barrier). *Set $\mathbb{R}_{++}^d := \{x \in \mathbb{R}^d \mid \text{each } x_i > 0\}$ and*

$$\text{lb}: \mathbb{R}^d \rightarrow]-\infty, +\infty]: x \mapsto \begin{cases} -\sum_{i=1}^d \ln(x_i), & \text{if } x \in \mathbb{R}_{++}^d; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then lb belongs to $\mathcal{F}$.

3. Convex Optimization: Fenchel Conjugates.

We require one more notion from convex optimization before we are able to write down the formula for the proximal average. The *Fenchel conjugate* of a proper function $f: X \rightarrow]-\infty, +\infty]$ is defined by

$$f^*: X \rightarrow]-\infty, +\infty]: x^* \mapsto \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

Let us briefly comment on a geometric interpretation. Fix $x^* \in X$, thought of as a “slope”, and consider the family $(H_s)_{s \in \mathbb{R}}$ of all hyperplanes in $X \times \mathbb{R}$ with normal vector $(x^*, -1)$, where

$$H_s := \{(x, r) \in X \times \mathbb{R} \mid \langle x, x^* \rangle - r = s\} = \{(x, \langle x, x^* \rangle - s) \mid x \in X\}.$$

Given a fixed $s \in \mathbb{R}$, one of the following two mutually exclusive cases occurs.

Case 1: The graph of f (and hence $\text{epi } f$) lies *above or on* the hyperplane H_s , i.e., $(\forall x \in \text{dom } f) \ f(x) \geq \langle x, x^* \rangle - s$; equivalently,

$$s \geq f^*(x^*).$$

Case 2: There exists a point $(x, f(x))$ on the graph of f lying *strictly below* the hyperplane H_s , i.e., $f(x) < \langle x, x^* \rangle - s$; equivalently,

$$s < f^*(x^*).$$

The two cases together imply that $f^*(x^*)$ is the minimum of all values s for which the graph of f lies above or on the hyperplane H_s . (If $f^*(x^*) = +\infty$, then no hyperplane H_s lies below or on the graph of f .) Thus, the hyperplane $H_{f^*(x^*)}$ intersects the vertical $\{0\} \times \mathbb{R}$ axis at the largest possible height $-f^*(x^*)$. (See also [13, Chapter X] for another geometric interpretation.) If f is differentiable everywhere, then f^* coincides with the classical Legendre transformation.

We now review basic results on Fenchel conjugates. The *Biconjugate Theorem* states that

$$(3.1) \quad (\forall f \in \mathcal{F}) \quad f^{**} = f.$$

The *energy* function $\frac{1}{2}\|\cdot\|^2$ is the only self-conjugate function, i.e.,

$$(3.2) \quad (\forall f \in \mathcal{F}) \quad f = f^* \Leftrightarrow f = \frac{1}{2}\|\cdot\|^2.$$

Fenchel conjugates naturally arise in convex optimization. For example, the *Fenchel-Rockafellar Duality Theorem* states that given $f \in \mathcal{F}$, $g \in \mathcal{F}$, if $\text{dom } f$ has a nonempty intersection with the interior of $\text{dom } g$, then the primal optimization problem

$$\inf_{x \in X} f(x) + g(x)$$

has the same value as the dual optimization problem

$$- \inf_{x^* \in X} f^*(-x^*) + g^*(x^*)$$

and the latter infimum is actually attained, i.e., a minimum.

In Fourier analysis, convolution is a fundamental operation because it is dual to multiplication via Fourier transforms. In convex analysis, the counterpart to the convolution is the so-called *infimal convolution*, defined by

$$f \square g: X \rightarrow [-\infty, +\infty] : x \mapsto \inf_{y \in X} f(y) + g(x - y).$$

Indeed, the infimal convolution is dual to addition via Fenchel conjugation in the sense that $(f \square g)^* = f^* + g^*$ and that $(f + g)^* = f^* \square g^*$ if $\text{dom } f$ meets the interior of $\text{dom } g$.

Let us present the conjugate of the so-called *negative entropy*.

EXAMPLE 3.1 (negative entropy). Set $\mathbb{R}_+^d := \{x \in X \mid \text{each } x_i \geq 0\}$. With the standard convention that $0 \ln(0) = 0$, define

$$\text{negent}: \mathbb{R}^d \rightarrow]-\infty, +\infty] : x \mapsto \begin{cases} \sum_{i=1}^d x_i \ln(x_i) - x_i, & \text{if } x \in \mathbb{R}_+^d; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then negent belongs to $\mathcal{F}$ and $\text{negent}^* = \exp$, where $\exp: \mathbb{R}^d \rightarrow \mathbb{R} : x \mapsto \sum_{i=1}^d \exp(x_i)$.

Proof. It is clear that $\text{negent} \in \mathcal{F}$. Let us compute its conjugate. To this end, it suffices to consider the case when $d = 1$ since in the general case the supremum of each coordinate can be taken separately. Fix $y \in \mathbb{R}$. Then $f^*(y) = \sup_x (yx - (x \ln x - x))$. We supremize a concave function, so every critical point is a maximizer. Solving the critical point equation $0 = \frac{d}{dx}(yx - (x \ln x - x)) = y - \ln x$, we see that $x = e^y$ is the unique maximizer and hence $f^*(y) = ye^y - (e^y \ln(e^y) - e^y) = e^y$. $\square$

To whet the reader's appetite further, we visualize in Fig. 3.1 the proximal averages from the negative entropy to its conjugate, and from a quadratic function to an indicator function.

FIG. 3.1. Proximal averages of two functions with different domains.

4. Proximal Average.

The formula for the proximal average $\mathcal{P}$ appeared, in an equivalent form, first in [4], which dealt with aspects of convergence of certain fixed point iterations. The proximal average was used there to provide an explicit constructive proof of Moreau's observation [18] that the set of all so-called proximal mappings is convex — this is the motivation for the term “proximal average”.

DEFINITION 4.1 (proximal average). *The proximal average operator is*

$$(4.1) \quad \mathcal{P}: \mathcal{F} \times [0, 1] \times \mathcal{F} \rightarrow \{f \mid f: X \rightarrow [-\infty, +\infty]\}$$

$$(f_0, \lambda, f_1) \mapsto \left((1 - \lambda)(f_0 + \tfrac{1}{2}\|\cdot\|^2)^* + \lambda(f_1 + \tfrac{1}{2}\|\cdot\|^2)^* \right)^* - \tfrac{1}{2}\|\cdot\|^2.$$

The next result is an immediate consequence of the definition and (3.1).

PROPOSITION 4.2. *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in [0, 1]$. Then*

$$\mathcal{P}(f_0, 0, f_1) = f_0, \quad \mathcal{P}(f_0, 1, f_1) = f_1, \quad \text{and} \quad \mathcal{P}(f_0, \lambda, f_1) = \mathcal{P}(f_1, 1 - \lambda, f_0).$$

For $\lambda \in]0, 1[$, it appears on first glance to be unlikely that the proximal average $\mathcal{P}(f_0, \lambda, f_1)$ is a convex function (since the difference between convex functions generally fails to be convex). However, $\mathcal{P}(f_0, \lambda, f_1)$ is not only convex, but *its conjugate is the corresponding proximal average of the conjugates!*

THEOREM 4.3. *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in [0, 1]$. Then $\mathcal{P}(f_0, \lambda, f_1)$ belongs to $\mathcal{F}$, and*

$$(4.2) \quad (\mathcal{P}(f_0, \lambda, f_1))^* = \mathcal{P}(f_0^*, \lambda, f_1^*).$$

Proof. See Appendix A or [4, Theorem 6.1]. $\square$

REMARK 4.4. Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in [0, 1]$. Set $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$ and $(f^*)_\lambda := \mathcal{P}(f_0^*, \lambda, f_1^*)$. Then Theorem 4.3 states that

$$(f_\lambda)^* = (f^*)_\lambda,$$

so we write these identical terms conveniently as f_λ^* .

We shall compute explicitly the proximal average of some indicator functions by specializing the following result.

EXAMPLE 4.5. Let C_0 and C_1 be two nonempty closed convex subsets of X , and let $\lambda \in]0, 1[$. Set $f_0 := \iota_{C_0}$, $f_1 := \iota_{C_1}$, and $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then for every $x \in X$,

$$f_\lambda(x) = (1 - \lambda)\lambda \inf \left\{ \frac{1}{2} \|c_0 - c_1\|^2 \mid (1 - \lambda)c_0 + \lambda c_1 = x \text{ and each } c_i \text{ belongs to } C_i \right\},$$

where $\inf \emptyset = +\infty$. Moreover, $\text{dom } f_\lambda = (1 - \lambda)C_0 + \lambda C_1$.

Proof. See Appendix B or [4, Theorem 6.1]. $\square$

EXAMPLE 4.6 (subspace and its orthogonal complement). Let Y be a subspace of X and let $\lambda \in]0, 1[$. Set $f_0 := \iota_Y$, $f_1 := \iota_{Y^\perp}$, and $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then

$$f_\lambda : X \rightarrow]-\infty, +\infty] : x \mapsto \frac{1}{2(1 - \lambda)\lambda} (\lambda^2 \|P_Y x\|^2 + (1 - \lambda)^2 \|P_{Y^\perp} x\|^2),$$

where P_Y and $P_{Y^\perp}$ denote the projectors onto Y and its orthogonal complement $Y^\perp$, respectively. Fig. 4.1 illustrates the case $Y = \mathbb{R} \times \{0\}$.

FIG. 4.1. The proximal average from the indicator of $\mathbb{R} \times \{0\}$ to the indicator of $\{0\} \times \mathbb{R}$ (see Example 4.6).

Proof. Fix $x \in X$. There is only one way to write x as a sum of two vectors from Y and $Y^\perp$, namely

$$x = P_Y x + P_{Y^\perp} x = (1 - \lambda)P_Y(x/(1 - \lambda)) + \lambda P_{Y^\perp}(x/\lambda).$$

In view of Example 4.5, we conclude that

$$\begin{aligned} f_\lambda(x) &= \frac{(1-\lambda)\lambda}{2} \|P_Y(x/(1-\lambda)) - P_{Y^\perp}(x/\lambda)\|^2 \\ &= \frac{1}{2(1-\lambda)\lambda} \|\lambda P_Y x - (1-\lambda)P_{Y^\perp} x\|^2 \\ &= \frac{1}{2(1-\lambda)\lambda} (\lambda^2 \|P_Y x\|^2 + (1-\lambda)^2 \|P_{Y^\perp} x\|^2). \end{aligned}$$

This completes the proof. $\square$

REMARK 4.7. Here are two special cases of the setting of Example 4.6.

(i) If $Y = \{0\}$, then $f_\lambda: x \mapsto \frac{1}{2}(\frac{1}{\lambda} - 1)\|x\|^2$.

(ii) $f_{\frac{1}{2}} = \frac{1}{2}\|\cdot\|^2$.

We now verify a considerable generalization of Remark 4.7.(ii), namely that *the proximal “midpoint” between a function and its Fenchel conjugate is always the energy*.

PROPOSITION 4.8. *Let $f \in \mathcal{F}$. Then $\mathcal{P}(f, \frac{1}{2}, f^*) = \frac{1}{2}\|\cdot\|^2$.*

Proof. Using Theorem 4.3, (3.1), and Proposition 4.2, we see that

$$(\mathcal{P}(f, \frac{1}{2}, f^*))^* = \mathcal{P}(f^*, \frac{1}{2}, f^{**}) = \mathcal{P}(f^*, 1 - \frac{1}{2}, f) = \mathcal{P}(f, \frac{1}{2}, f^*).$$

Thus $\mathcal{P}(f, \frac{1}{2}, f^*)$ is the same as its Fenchel conjugate. On the other hand, only $\frac{1}{2}\|\cdot\|^2$ possesses this self-conjugacy property (see (3.2)). $\square$

Another instructive case arises when we consider cones rather than subspaces. Recall that a subset K of X is a *convex cone* if K is convex and $(\forall \lambda > 0) \lambda K = K$. Let K be a nonempty closed convex cone. Then its *polar cone* is

$$K^\ominus := \{x^* \in X \mid (\forall k \in K) \langle k, x^* \rangle \leq 0\}.$$

Every subspace is a cone, but the converse is false in general (consider, e.g., $\mathbb{R}_+^d$). The polar cone of a subspace coincides with the orthogonal complement of the subspace; in general, these notions are different (e.g., in $\mathbb{R}$, we have $[0, +\infty[^\ominus =]-\infty, 0]$ while $[0, +\infty[^\perp = \{0\}$). We require the notion of the *projector* P_C of a nonempty closed convex set C in X . The operator P_C maps every $x \in X$ to its (unique) nearest point in C : $P_C x \in C$, $\|x - P_C x\| = \min_{c \in C} \|x - c\|$, and $(\forall c \in C \setminus \{P_C x\}) \|x - P_C x\| < \|x - c\|$.

A second glance at the proof of Example 4.6 shows that the proof was easy because the decomposition of a vector into a sum of two vectors, one taken from the subspace and the other from the orthogonal complement, is unique. Decompositions with respect to cones need not be unique: e.g., if $K = [0, +\infty[$ in $\mathbb{R}$, then $K^\ominus = -K$ yet $2 = 2 + 0 = 3 + (-1)$ can be written in more than one way as the sum of two elements in K and $K^\ominus$, respectively. However, it turns out that Example 4.6 holds true for cones.

EXAMPLE 4.9 (cone and polar cone). *Let K be a nonempty closed convex cone in X and let $\lambda \in]0, 1[$. Set $f_0 := \iota_K$, $f_1 := \iota_{K^\ominus}$, and $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then*

$$f_\lambda: X \rightarrow]-\infty, +\infty] : x \mapsto \frac{1}{2(1-\lambda)\lambda} (\lambda^2 \|P_K x\|^2 + (1-\lambda)^2 \|P_{K^\ominus} x\|^2).$$

Proof. See Appendix C. $\square$

REMARK 4.10. Consider Example 4.9. Even though the given functions f_0 and f_1 are indicator functions, the proximal average f_λ is *not* an indicator function.

We conclude this section with observations that are required in later sections. We start with a result that is complementary to the “Moreover” part of Example 4.5.

THEOREM 4.11. *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in]0, 1[$. Set $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then the closure and the interior of $\text{dom } f_\lambda$ satisfy*

$$\overline{\text{dom } f_\lambda} = \overline{(1 - \lambda) \text{dom } f_0 + \lambda \text{dom } f_1}$$

and

$$\text{int } \text{dom } f_\lambda = \text{int } ((1 - \lambda) \text{dom } f_0 + \lambda \text{dom } f_1).$$

Proof. See Appendix D. $\square$

COROLLARY 4.12. *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in]0, 1[$. Suppose that f_0 or f_1 has full domain. Then $\mathcal{P}(f_0, \lambda, f_1)$ has full domain as well.*

Proof. Assume that f_0 has full domain. Then $\text{int } \text{dom } f_\lambda = \text{int } ((1 - \lambda) \text{dom } f_0 + \lambda \text{dom } f_1) = \text{int } ((1 - \lambda)X + \lambda \text{dom } f_1) = X$. $\square$

REMARK 4.13. Theorem 4.11 makes it clear why we observed in Fig. 1.2 that $\mathcal{P}(f_0, \lambda, f_1)$ has full domain for $\lambda \in]0, 1[$. The results in this section admit sharpenings as well as extensions to a general (possibly infinite-dimensional) Hilbert space setting; the required results from convex analysis can be found in [23].

PROPOSITION 4.14 (proximal average vs arithmetic average). *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in]0, 1[$. Suppose that $0 \leq f_0$, $0 \leq f_1$, and set $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then*

$$0 \leq f_\lambda \leq (1 - \lambda)f_0 + \lambda f_1.$$

Proof. Set $q := \frac{1}{2} \|\cdot\|^2$. Clearly, $q \leq q + f_0$ and $q \leq q + f_1$. Conjugate these inequalities, scale the results (by $1 - \lambda$ and λ , respectively), and then add to obtain $q \geq (1 - \lambda)(q + f_0)^* + \lambda(q + f_1)^*$. Conjugate one more time to obtain $q \leq ((1 - \lambda)(q + f_0)^* + \lambda(q + f_1)^*)^* = f_\lambda + q$. It follows that

$$\begin{aligned} q &\leq f_\lambda + q = ((1 - \lambda)(q + f_0)^* + \lambda(q + f_1)^*)^* \\ &\leq (1 - \lambda)(q + f_0)^{**} + \lambda(q + f_1)^{**} \\ &= q + (1 - \lambda)f_0 + \lambda f_1. \end{aligned}$$

The result follows by subtracting q . $\square$

REMARK 4.15. Suppose that f_{lower} , f_0 , f_1 , and f_{upper} belong to $\mathcal{F}$ and that $f_{\text{lower}} \leq \min\{f_0, f_1\} \leq \max\{f_0, f_1\} \leq f_{\text{upper}}$. We leave it to the reader to show that

$$(\forall \lambda \in]0, 1[) \quad f_{\text{lower}} \leq f_\lambda \leq f_{\text{upper}}.$$

5. Continuity and Homotopy.

The examples in the preceding sections show that the proximal average is not pointwise continuous, i.e., for $f_0 \in \mathcal{F}$, $f_1 \in \mathcal{F}$, and $x \in X$, the mapping $[0, 1] \rightarrow]-\infty, +\infty] : \lambda \mapsto \mathcal{P}(f_0, \lambda, f_1)(x)$ is in general *not* continuous. Nonetheless, the above examples and figures seem to suggest that *some* form of continuity is present. This is indeed the case when we use the *epi-topology*, a basic tool of modern variational analysis. Especially in our finite-dimensional setting, the corresponding theory is well

developed, beautiful, and ready to use. We will draw on results from the seminal book by Rockafellar and Wets [21].

DEFINITION 5.1 (epi-convergence and epi-topology). *Let f and $(f_n)_{n \in \mathbb{N}}$ be functions from X to $]-\infty, +\infty]$. Then $(f_n)_{n \in \mathbb{N}}$ epi-converges to f , in symbols, $f_n \xrightarrow{e} f$, if the following hold for every $x \in X$.*

- (i) *For every sequence $(x_n)_{n \in \mathbb{N}}$ in X converging to x , one has $f(x) \leq \underline{\lim} f_n(x_n)$.*
- (ii) *There exists a sequence $(y_n)_{n \in \mathbb{N}}$ in X converging to x such that $\lim f_n(y_n) \leq f(x)$.*

The epi-topology is the topology induced by epi-convergence.

The notion of epi-convergence is especially useful in optimization. It has its roots in the theory of convergence of epigraphs (see (2.3)), viewed as subsets of $X \times \mathbb{R}$; the reader is referred to [21, Chapter 7] for further information. The following example (see [21, Section 7.B]) illustrates that epi-convergence is *independent* of the classical notion of pointwise convergence.

EXAMPLE 5.2. *For every $n \in \mathbb{N}$, define*

$$f_n: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \begin{cases} 0, & \text{if } x = 1/(n+1); \\ 1, & \text{otherwise.} \end{cases}$$

Then the pointwise-limit of $(f_n)_{n \in \mathbb{N}}$ is the function that is identically equal to 1; however, the epi-limit of $(f_n)_{n \in \mathbb{N}}$ is the function

$$f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The statement concerning the pointwise-limit is clear. Now fix $x \in \mathbb{R}$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ converging to x .

Case 1. $x = 0$. Then $f(x) = f(0) = 0 \leq \underline{\lim} f_n(x_n)$. Set $(\forall n \in \mathbb{N}) y_n := 1/(n+1)$. Then $y_n \rightarrow x$ and $(\forall n \in \mathbb{N}) f_n(y_n) = 0$. Hence $\lim f_n(y_n) = \lim 0 = 0 = f(0) = f(x)$. This verifies items (i) and (ii) of Definition 5.1 at $x = 0$.

Case 2. $x \neq 0$. Then eventually $|x_n| \geq |x|/2 > 0$ and, in turn, eventually $f_n(x_n) = 1$. It follows that $f(x) = 1 = \lim f_n(x_n) = \underline{\lim} f_n(x_n)$. Set $(\forall n \in \mathbb{N}) y_n := x$. Then $y_n \rightarrow x$ and eventually $f_n(y_n) = f_n(x) = 1$. Thus $\lim f_n(y_n) = \lim 1 = 1 = f(x)$. This verifies items (i) and (ii) of Definition 5.1 at x .

Hence in either case, items (i) and (ii) of Definition 5.1 hold, i.e., $f_n \xrightarrow{e} f$. $\square$

The following fact implies the continuity — with respect to the epi-topology — of various operations.

FACT 5.3. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $(f_n)_{n \in \mathbb{N}}$ epi-converges to a function $f \in \mathcal{F}$, set $q := \frac{1}{2} \|\cdot\|^2$, let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $[0, 1]$ converging to μ , and let $(g_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $(g_n)_{n \in \mathbb{N}}$ epi-converges to a function $g \in \mathcal{F}$. Then the following hold.*

- (i) $f_n \pm q \xrightarrow{e} f \pm q$.
- (ii) $f_n^* \xrightarrow{e} f^*$.
- (iii) $\mu_n f_n \xrightarrow{e} \mu f$ provided that $\mu > 0$.
- (iv) $\mu_n f_n \xrightarrow{e} \mu f$ provided that $\mu = 0$ and $\text{dom } f = X$.
- (v) $f_n + g_n \xrightarrow{e} f + g$ provided that $0 \in \text{int}(\text{dom } f - \text{dom } g)$.

Proof. (i): This is a consequence of [21, Exercise 7.8(a)].

(ii): See [21, Theorem 11.34].

(iii): See [21, Exercise 7.8(d)].

(iv): Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X converging to x . Then $-\infty < f(x) \leq \underline{\lim} f_n(x_n)$ and hence $(0 \cdot f)(x) = 0 \leq \underline{\lim} (\mu_n f_n)(x_n)$. On the other hand, there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in X such that $y_n \rightarrow y$ and $\overline{\lim} f_n(y_n) \leq f(x) < +\infty$. Hence $\overline{\lim} (\mu_n f_n)(y_n) \leq 0 = (0 \cdot f)(x)$. Altogether, $(\mu_n f_n)_{n \in \mathbb{N}}$ epi-converges to $0 \cdot f$.

(v): See [21, Exercise 7.47(b)]. $\square$

Based on Fact 5.3, we are now able to establish the continuity of the proximal average operator.

THEOREM 5.4 (continuity). *Suppose that $\mathcal{F}$ is equipped with the epi-topology. Then the proximal average operator $\mathcal{P}: \mathcal{F} \times [0, 1] \times \mathcal{F} \rightarrow \mathcal{F}$ is continuous.*

Proof. Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be sequences in $\mathcal{F}$, let f and g be in $\mathcal{F}$, let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $[0, 1]$, and let $\lambda \in [0, 1]$. Assume that

$$f_n \xrightarrow{e} f, \quad g_n \xrightarrow{e} g, \quad \text{and} \quad \lambda_n \rightarrow \lambda.$$

Set $q := \frac{1}{2} \|\cdot\|^2$. Fact 5.3.(i) implies that $f_n + q \xrightarrow{e} f + q$ and $g_n + q \xrightarrow{e} g + q$. Now Fact 5.3.(ii) yields $(f_n + q)^* \xrightarrow{e} (f + q)^*$ and $(g_n + q)^* \xrightarrow{e} (g + q)^*$. Since $\text{dom}(f + q)^* = X$ and $\text{dom}(g + q)^* = X$, Fact 5.3.(iii)&(iv) show that $(1 - \lambda_n)(f_n + q)^* \xrightarrow{e} (1 - \lambda)(f + q)^*$ and $\lambda_n(g_n + q)^* \xrightarrow{e} \lambda(g + q)^*$. Because $(1 - \lambda)(f + q)^*$ and $\lambda(g + q)^*$ each has full domain, we deduce from Fact 5.3.(v) that $(1 - \lambda_n)(f_n + q)^* + \lambda_n(g_n + q)^* \xrightarrow{e} (1 - \lambda)(f + q)^* + \lambda(g + q)^*$. By Fact 5.3.(ii), $((1 - \lambda_n)(f_n + q)^* + \lambda_n(g_n + q)^*)^* \xrightarrow{e} ((1 - \lambda)(f + q)^* + \lambda(g + q)^*)^*$. Finally, Fact 5.3.(i) implies that

$$((1 - \lambda_n)(f_n + q)^* + \lambda_n(g_n + q)^*)^* - q \xrightarrow{e} ((1 - \lambda)(f + q)^* + \lambda(g + q)^*)^* - q,$$

i.e., $\mathcal{P}(f_n, \lambda_n, g_n) \xrightarrow{e} \mathcal{P}(f, \lambda, g)$. $\square$

COROLLARY 5.5 (homotopy). *Let $f_0 \in \mathcal{F}$ and let $f_1 \in \mathcal{F}$. Then*

$$\mathcal{P}(f_0, \cdot, f_1): [0, 1] \rightarrow \mathcal{F}$$

provides a homotopy between f_0 and f_1 . Consequently, all functions in $\mathcal{F}$ are homotopic.

REMARK 5.6. In infinite-dimensional spaces, the theory is more complicated and there are a number of topologies one might consider (most of which coincide in finite dimensions). We believe the most useful topology for carrying the results of this section over is the *Attouch-Wets topology*, for which addition and Fenchel conjugation are continuous operations. We refer the interested reader to the books [1, 6, 15] as well as to the articles [2, 19].

6. Inheritance.

Recall that $f \in \mathcal{F}$ is *strictly convex*, if

$$\left. \begin{array}{l} x \in \text{dom } f \\ y \in \text{dom } f \\ \lambda \in]0, 1[\\ x \neq y \end{array} \right\} \Rightarrow f((1 - \lambda)x + \lambda y) < (1 - \lambda)f(x) + \lambda f(y).$$

Strict convexity is important in optimization because it guarantees that there is at most one minimizer of any objective function with this property. Fenchel conjugacy shows that strict convexity and differentiability are dual to each other in the following immensely useful and beautiful sense.

FACT 6.1. *Let $f \in \mathcal{F}$ such that both f and f^* have full domain. Then*

$$f \text{ is differentiable} \Leftrightarrow f^* \text{ is strictly convex.}$$

We now show that the proximal average inherits good properties such as differentiability or strict convexity from the given functions.

THEOREM 6.2 (inheritance). *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in]0, 1[$. Set $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Suppose that f_0 or f_1 has full domain, and that f_0^* or f_1^* has full domain. Then the following hold.*

- (i) *Both f_λ and f_λ^* have full domain.*
- (ii) *If f_0 or f_1 is differentiable everywhere, then so is f_λ .*
- (iii) *If f_0 or f_1 is strictly convex and its Fenchel conjugate has full domain, then f_λ is strictly convex.*

Proof. (i): f_λ has full domain by Corollary 4.12. A dual argument (recall Remark 4.4) shows that f_λ^* has full domain as well.

(ii): Assume that f_0 is differentiable with full domain. Then so is $f_0 + q$ and hence (by Fact 6.1) $(f_0 + q)^* = f_0^* \square q$ is strictly convex with full domain. Thus $(1 - \lambda)(f_0 + q)^* + \lambda(f_1 + q)^*$ is strictly convex with full domain. Item (i) and Fact 6.1 imply that $f_\lambda + q$ is differentiable with full domain. Therefore, f_λ is differentiable with full domain.

(iii): Assume that f_0 is strictly convex and that $\text{dom } f_0^* = X$. Then f_0^* is differentiable everywhere and hence so is f_λ^* (by (ii) applied to f_0^* and f_1^*). We conclude that f_λ is strictly convex. $\square$

COROLLARY 6.3. *Let $f_0 \in \mathcal{F}$, let $f_1 \in \mathcal{F}$, and let $\lambda \in]0, 1[$. Suppose that f_0, f_1 , and their Fenchel conjugates have full domain, that f_0 is strictly convex, and that f_1 is differentiable. Set $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then both f_λ and f_λ^* are differentiable and strictly convex.*

REMARK 6.4. Assume that the hypothesis of Corollary 6.3 holds and that f_0 is *not* differentiable. The conclusion of Corollary 6.3 guarantees that the proximal average f_λ is differentiable. In contrast, the arithmetic average $(1 - \lambda)f_0 + \lambda f_1$ is *not* differentiable. This illustrates another advantage of the proximal average over the arithmetic average. Note that both averages produce a strictly convex function.

Recall that a *norm* $\|\cdot\|$ (on X) is a function $X \rightarrow \mathbb{R}$ satisfying the following four properties: $(\forall x \in X) \|\cdot\| \geq 0$; $(\forall x \in X) \|\cdot\| = 0 \Leftrightarrow x = 0$; $(\forall x \in X)(\forall r \in \mathbb{R}) \|rx\| = |r|\|\cdot\|$; and $(\forall x \in X)(\forall y \in X) \|x + y\| \leq \|x\| + \|y\|$. Every norm $\|\cdot\|$ can be written as $\|\cdot\| = \sqrt{2f}$, where the unique function $f \in \mathcal{F}$ is nonnegative, it vanishes only at 0, and it is *quadratic-homogeneous*, i.e., $(\forall x \in X)(\forall r \in \mathbb{R}) f(rx) = r^2 f(x)$; we shall say that f is the *function associated with* $\|\cdot\|$, and that $\|\cdot\|$ is the *norm associated with* f . Let $\|\cdot\|$ be a norm with associated function f . Then f^* is associated with the corresponding *dual norm* $\|\cdot\|^\circ: X \rightarrow \mathbb{R}: x^* \mapsto \sup \{ \langle x, x^* \rangle \mid \|x\| \leq 1 \}$. (See [20, Chapter 15].) The norm $\|\cdot\|$ is *strictly convex*, if f is; and $\|\cdot\|$ is *smooth*, if f is differentiable everywhere. Note that Fact 6.1 implies that $\|\cdot\|$ is strictly convex $\Leftrightarrow \|\cdot\|^\circ$ is smooth, and that $\|\cdot\|$ is smooth $\Leftrightarrow \|\cdot\|^\circ$ is strictly convex.

We now show that the proximal average can be employed to create a new norm that inherits good properties from two given norms.

THEOREM 6.5 (renorming). *Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms on X , and denote their associated functions by f_0 and f_1 . Let $\lambda \in]0, 1[$ and set $f_\lambda := \mathcal{P}(f_0, \lambda, f_1)$. Then f_λ is associated with some norm $\|\cdot\|_\lambda$, f_λ^* is associated with $\|\cdot\|_\lambda^\circ$, and*

$$(6.1) \quad 0 \leq \left(\frac{1}{1 - \lambda} f_0 \right) \square \left(\frac{1}{\lambda} f_1 \right) \leq f_\lambda \leq (1 - \lambda)f_0 + \lambda f_1.$$

If $\|\cdot\|_0$ is strictly convex and $\|\cdot\|_1$ is smooth, then $\|\cdot\|_\lambda$ and $\|\cdot\|_\lambda^\circ$ are strictly convex and smooth.

Proof. Since $0 \leq f_0$ and $0 \leq f_1$, the first inequality in (6.1) follows immediately. The third inequality in (6.1) was already recorded in Proposition 4.14. Applying this proposition to f_0^* and f_1^* , which are the functions associated with the dual norms $\|\cdot\|_0^\circ$ and $\|\cdot\|_1^\circ$, we see that $f_\lambda^* \leq (1-\lambda)f_0^* + \lambda f_1^*$. Conjugate this inequality to obtain

$$f_\lambda \geq \left(((1-\lambda)f_0^*)^* \right) \square \left((\lambda f_1^*)^* \right) = \left((1-\lambda)f_0(\cdot/(1-\lambda)) \right) \square \left(\lambda f_1(\cdot/\lambda) \right).$$

The second inequality in (6.1) thus follows from the quadratic homogeneity of f_0 and f_1 . Let us show that f_λ is associated with some norm $\|\cdot\|_\lambda$. It is clear that f_λ belongs to $\mathcal{F}$, and (6.1) yields $0 \leq f_\lambda$. Suppose that $x \in X$ satisfies $f_\lambda(x) = 0$. Since f_0 and f_1 are both coercive (i.e., each $\lim_{\|x\| \rightarrow +\infty} f_i(x) = +\infty$), the infimal convolution of (6.1) is exact at x (i.e., the infimum in the definition of the infimal convolution at the point x is actually attained, i.e., the infimum is a minimum). Hence there exist vectors x_0 and x_1 in X such that $x = x_0 + x_1$ and $0 \leq (1/(1-\lambda))f_0(x_0) + (1/\lambda)f_1(x_1) \leq f_\lambda(x) = 0$; Thus $f_0(x_0) = f_1(x_1) = 0$, which implies $x_0 = 0 = x_1$ and further $x = 0$. It follows that f_λ vanishes only at 0. The definition of f_λ now leads readily to the quadratic homogeneity of f_λ . Therefore, f_λ is associated with some norm $\|\cdot\|_\lambda$ and f_λ^* is associated with $\|\cdot\|_\lambda^\circ$. Finally, we assume that f_0 is strictly convex and f_1 is differentiable everywhere. In view of Corollary 6.3, f_λ and f_λ^* are strictly convex and differentiable; equivalently, $\|\cdot\|_\lambda$ and $\|\cdot\|_\lambda^\circ$ are strictly convex and smooth. $\square$

Define two norms at any point $(x_1, x_2) \in \mathbb{R}^2$ via

$$(6.2) \quad \|(x_1, x_2)\|_0 = \begin{cases} \frac{1}{2}|x_1|, & \text{if } |x_1| > 2|x_2|; \\ \frac{1}{2}|x_2|, & \text{if } |x_2| > 2|x_1|; \\ |x_1| + |x_2| - \sqrt{2|x_1||x_2|}, & \text{otherwise,} \end{cases}$$

and

$$(6.3) \quad \|(x_1, x_2)\|_1 = \frac{|x_1| + \sqrt{4|x_1|^2 + 3|x_2|^2}}{3}.$$

FIG. 6.1. An illustration of Theorem 6.5 when $\|\cdot\|_0$ and $\|\cdot\|_1$ are defined by (6.2) and (6.3), respectively. The unit spheres of the norms $\|\cdot\|_\lambda$, where $\lambda \in [0, 1]$, are visualized.

Note that $\|\cdot\|_0$ is smooth but not strictly convex, and that $\|\cdot\|_1$ is strictly convex but not smooth. The unit spheres of these norms — as well as those of all norms $\|\cdot\|_\lambda$ obtained by Theorem 6.5 — are illustrated in Fig. 6.1. As predicted by the theory, the intermediate norms $\|\cdot\|_\lambda$ are both smooth and strictly convex.

7. Exercises.

In this section, we present a collection of examples some of which were previously considered by the first-named author and A. Jarvis [14]. We do leave the details of these examples to the interested reader as exercises and we encourage him/her to utilize available software such as [5, 7, 12, 16, 17], which was very beneficial for the creation of the examples presented in this paper.

In all these examples, we shall specify two functions f_0 and f_1 in $\mathcal{F}$, and we shall assume that

$$\lambda \in]0, 1[\quad \text{and} \quad f_\lambda := \mathcal{P}(f_0, \lambda, f_1).$$

The first example shows that the proximal average coincides with the arithmetic average if one function is simply a (vertical) translate of the other.

EXAMPLE 7.1. *Let $f_0 \in \mathcal{F}$, let $\gamma \in \mathbb{R}$, and set $f_1 := f_0 + \gamma$. Then $f_\lambda = f_0 + \lambda\gamma = (1 - \lambda)f_0 + \lambda f_1$.*

On the other hand, if we consider the proximal average between 0 and a nonzero linear functional, then we will not obtain the arithmetic average.

EXAMPLE 7.2. *Let $a \in X \setminus \{0\}$, set $f_0 := 0$ and $f_1 := \langle \cdot, a \rangle$. Then $f_\lambda = \lambda \langle \cdot, a \rangle - (1 - \lambda)\lambda \frac{1}{2} \|a\|^2$.*

Next let us demonstrate that simple scaling may lead to relatively complicated proximal averages.

EXAMPLE 7.3. *Let $\alpha > 0$, set $f_0 := \frac{1}{2} \|\cdot\|^2$ and $f_1 := \alpha \frac{1}{2} \|\cdot\|^2$. Then*

$$f_\lambda = \frac{(1 + \lambda)\alpha + 1 - \lambda}{(1 - \lambda)\alpha + 1 + \lambda} \frac{1}{2} \|\cdot\|^2.$$

By discussing the case of true quadratics and linear functions separately, it is possible to obtain the following explicit formula for transforming between two quadratics. Note that the proximal average is a quadratic as well unless both functions are linear.

EXAMPLE 7.4 (two quadratics). *Let $\alpha_0 \geq 0$, let $\alpha_1 \geq 0$, let $b_0 \in X$, let $b_1 \in X$, let $\gamma_0 \in \mathbb{R}$, and let $\gamma_1 \in \mathbb{R}$. Define $f_0: x \mapsto \alpha_0 \|x\|^2 + \langle x, b_0 \rangle + \gamma_0$ and $f_1: x \mapsto \alpha_1 \|x\|^2 + \langle x, b_1 \rangle + \gamma_1$, and set*

- (i) $\delta_\lambda := 1 + 2(\lambda\alpha_0 + (1 - \lambda)\alpha_1)$,
- (ii) $\alpha_\lambda := \frac{2\alpha_0\alpha_1 + (1 - \lambda)\alpha_0 + \lambda\alpha_1}{\delta_\lambda}$,
- (iii) $b_\lambda := \frac{(1 - \lambda)(2\alpha_1 + 1)b_0 + \lambda(2\alpha_0 + 1)b_1}{\delta_\lambda}$, and
- (iv) $\gamma_\lambda := (1 - \lambda)\gamma_0 + \lambda\gamma_1 - \frac{(1 - \lambda)\lambda\|b_0 - b_1\|^2}{2\delta_\lambda}$.

Then

$$f_\lambda: x \mapsto \alpha_\lambda \|x\|^2 + \langle x, b_\lambda \rangle + \gamma_\lambda.$$

Fig. 1.1(b) illustrates the case $\alpha_0 = 0, b_0 = 1, \gamma_0 = 2$, and $\alpha_1 = 1, b_1 = 0, \gamma_1 = 0$.

The transformation from a quadratic to a singleton indicator follows along a quadratic path.

EXAMPLE 7.5. *Let $\alpha_0 \geq 0$, let $b_0 \in X$, let $b_1 \in X$, let $\gamma_0 \in \mathbb{R}$, and let $\gamma_1 \in \mathbb{R}$. Define $f_0: x \mapsto \alpha_0 \|x\|^2 + \langle x, b_0 \rangle + \gamma_0$ and $f_1 := \iota_{\{b_1\}} + \gamma_1$, and set*

- (i) $\alpha_\lambda := \frac{2\alpha_0 + \lambda}{2(1 - \lambda)}$,

$$\begin{aligned}
\text{(ii)} \quad b_\lambda &:= \frac{(1-\lambda)b_0 - \lambda(1+2\alpha_0)b_1}{1-\lambda}, \text{ and} \\
\text{(iii)} \quad \gamma_\lambda &:= \frac{2(1-\lambda)^2\gamma_0 + 2\lambda(1-\lambda)\gamma_1 + \lambda\langle b_1, (1+2\lambda\alpha_0)b_1 - 2(1-\lambda)b_0 \rangle}{2(1-\lambda)}.
\end{aligned}$$

Then

$$f_\lambda: x \mapsto \alpha_\lambda \|x\|^2 + \langle x, b_\lambda \rangle + \gamma_\lambda.$$

Corollary 6.3 implies that for each $\lambda \in]0, 1[$, the proximal average f_λ is both strictly convex and differentiable. Fig. 3.1(b) illustrates the case when $\alpha_0 = \frac{1}{2}$, $b_0 = 0$, $\gamma_0 = 0$, and $b_1 = 0$, $\gamma_1 = 1$. If $\alpha_0 = 0$, then f_0 is differentiable but not strictly convex, and f_1 is strictly convex but not differentiable.

We conclude this section with two singleton indicator functions.

EXAMPLE 7.6. Let $b_0 \in X$, let $b_1 \in X$, let $\gamma_0 \in \mathbb{R}$, and let $\gamma_1 \in \mathbb{R}$. Set $f_0 := \iota_{\{b_0\}} + \gamma_0$ and $f_1 := \iota_{\{b_1\}} + \gamma_1$. Then

$$f_\lambda = \iota_{\{(1-\lambda)b_0 + \lambda b_1\}} + (1-\lambda)\gamma_0 + \lambda\gamma_1 + \frac{(1-\lambda)\lambda}{2} \|b_0 - b_1\|^2.$$

Appendix A. Proof of Theorem 4.3.

The result is clear if $\lambda \in \{0, 1\}$ so we assume that $\lambda \in]0, 1[$. Set $\lambda_0 := (1-\lambda)$ and $\lambda_1 := \lambda$, and set $q := \frac{1}{2} \|\cdot\|^2$ so that $q = q^*$ (see (3.2)) and $\text{dom } q = X$. For $i \in \{0, 1\}$, set $g_i := \lambda_i(f_i + q)^* = \lambda_i(f_i^* \square q)$. Now each f_i^* belongs to $\mathcal{F}$, hence so does each g_i and each g_i has full domain. Thus $g_0 + g_1 \in \mathcal{F}$ and $\text{dom}(g_0 + g_1) = X$. It follows that $(g_0 + g_1)^*$ is proper. Now fix $x^* \in X^*$. Then convex calculus yields

$$\begin{aligned}
(\mathcal{P}(f_0, \lambda, f_1))^*(x^*) &= ((g_0 + g_1)^* - q)^*(x^*) \\
&= \sup_{z^*} (g_0 + g_1)^{**}(x^* + z^*) - q(z^*) \\
&= \sup_{z^*} (g_0 + g_1)(x^* + z^*) - q(z^*) \\
&= \sup_{y^*} (g_0 + g_1)(y^*) - q(y^* - x^*) \\
&= \sup_{y^*} \lambda_0(f_0^* \square q)(y^*) + \lambda_1(f_1^* \square q)(y^*) - (q(y^*) + q(x^*) - \langle x^*, y^* \rangle) \\
&= \sup_{y^*} \left(\lambda_0(q(y^*) - (f_0 \square q)(y^*)) + \lambda_1(q(y^*) - (f_1 \square q)(y^*)) \right. \\
&\quad \left. - (q(y^*) + q(x^*) - \langle y^*, x^* \rangle) \right) \\
&= -q(x^*) + \sup_{y^*} \langle x^*, y^* \rangle - (\lambda_0(f_0 \square q) + \lambda_1(f_1 \square q))(y^*) \\
&= (\lambda_0(f_0 \square q) + \lambda_1(f_1 \square q))^*(x^*) - q(x^*) \\
&= (\lambda_0(f_0^* + q)^* + \lambda_1(f_1^* + q)^*)^*(x^*) - q(x^*) \\
&= ((1-\lambda)(f_0^* + q)^* + \lambda(f_1^* + q)^*)^*(x^*) - q(x^*) \\
&= (\mathcal{P}(f_0^*, \lambda, f_1^*))^*(x^*).
\end{aligned}$$

This verifies (4.2). Hence

$$(\mathcal{P}(f_0, \lambda, f_1))^{**} = (\mathcal{P}(f_0^*, \lambda, f_1^*))^* = \mathcal{P}(f_0^{**}, \lambda, f_1^{**}) = \mathcal{P}(f_0, \lambda, f_1)$$

and therefore $\mathcal{P}(f_0, \lambda, f_1) \in \mathcal{F}$. The proof of Theorem 4.3 is complete.

Appendix B. Proof of Example 4.5.

Fix $x \in X$. Using the notation of Appendix A, we obtain

$$\begin{aligned}
f(x) &= (g_0 + g_1)^*(x) - q(x) \\
&= (g_0^* \square g_1^*)(x) - q(x) \\
&= \inf_{x_0+x_1=x} g_0^*(x_0) + g_1^*(x_1) - q(x) \\
&= \inf_{x_0+x_1=x} \lambda_0(\iota_{C_0} + q)(\lambda_0^{-1}x_0) + \lambda_1(\iota_{C_1} + q)(\lambda_1^{-1}x_1) - q(x) \\
&= \inf_{x_0+x_1=x \text{ and each } \lambda_i^{-1}x_i \in C_i} \lambda_0 q(\lambda_0^{-1}x_0) + \lambda_1 q(\lambda_1^{-1}x_1) - q(x_0 + x_1) \\
&= \inf_{\lambda_0 c_0 + \lambda_1 c_1 = x \text{ and each } c_i \in C_i} \lambda_0 q(c_0) + \lambda_1 q(c_1) - q(\lambda_0 c_0 + \lambda_1 c_1) \\
&= \inf_{\lambda_0 c_0 + \lambda_1 c_1 = x \text{ and each } c_i \in C_i} \lambda_0 \frac{1}{2} \|c_0\|^2 + \lambda_1 \frac{1}{2} \|c_1\|^2 - \frac{1}{2} \|\lambda_0 c_0 + \lambda_1 c_1\|^2 \\
&= \inf_{\lambda_0 c_0 + \lambda_1 c_1 = x \text{ and each } c_i \in C_i} \frac{1}{2} \lambda_0 \lambda_1 \|c_0 - c_1\|^2 \\
&= \inf_{(1-\lambda)c_0 + \lambda c_1 = x \text{ and each } c_i \in C_i} \frac{1}{2} (1-\lambda) \lambda \|c_0 - c_1\|^2.
\end{aligned}$$

This verifies the formula for f_λ and the “Moreover” part is now clear. This concludes the proof of Example 4.5.

Appendix C. Proof of Example 4.9.

We begin by stating a classical result due to Moreau. This result can be interpreted as the Pythagorean decomposition for cones.

FACT C.1. *Let K be a nonempty closed convex cone in X . Then for every x in X , $x = P_K x + P_{K^\ominus} x$ and $\langle P_K x, P_{K^\ominus} x \rangle = 0$.*

We require the following consequence of Moreau’s result.

PROPOSITION C.2. *Let K be a nonempty closed convex cone in X and let $x \in X$. Then the unique solution to the optimization problem*

$$(C.1) \quad \text{minimize } \|k\| \quad \text{subject to } k \in K \text{ and } x - k \in K^\ominus$$

is the vector $P_K x$.

Proof. Set $K^\oplus = -K^\ominus$. By Fact C.1, $K \cap (x - K^\ominus)$ is nonempty and thus the optimization problem (C.1) is *feasible*, i.e., the minimum is not taken over the empty set. Hence (C.1) is equivalent to another feasible optimization problem, namely

$$(C.2) \quad \text{minimize } \|k\| \quad \text{subject to } k \in K \text{ and } k \in x + K^\oplus.$$

Since $K \cap (x + K^\oplus) \neq \emptyset$, the unique solution of (C.2) is $P_{K \cap (x + K^\oplus)} 0$. On the other hand, we have

$$P_{x+K^\oplus} 0 = x + P_{K^\oplus}(0 - x) = x + P_{-K^\ominus}(-x) = x - P_{K^\ominus} x = P_K x.$$

(See also [3, Remark 3.3].) Altogether, we conclude that $P_{K \cap (x + K^\oplus)} 0 = P_K x$. $\square$

Let us now verify Example 4.9. Fix $x \in X$. Utilizing Example 4.5, Fact C.1, and

Proposition C.2, we estimate that

$$\begin{aligned}
f_\lambda(x) &\leq \frac{(1-\lambda)\lambda}{2} \|P_K(\frac{1}{1-\lambda}x) - P_{K^\ominus}(\frac{1}{\lambda}x)\|^2 \\
&= \frac{(1-\lambda)\lambda}{2} \left(\frac{1}{(1-\lambda)^2} \|P_K x\|^2 + \frac{1}{\lambda^2} \|P_{K^\ominus} x\|^2 \right) \\
&= \frac{(1-\lambda)\lambda}{2} \left(\frac{1}{(1-\lambda)^2} \inf_{k \in K, x-k \in K^\ominus} \|k\|^2 + \frac{1}{\lambda^2} \inf_{l \in K^\ominus, x-l \in K} \|l\|^2 \right) \\
&\leq \frac{(1-\lambda)\lambda}{2} \inf_{k+l=x, k \in K, l \in K^\ominus} \left(\frac{1}{(1-\lambda)^2} \|k\|^2 + \frac{1}{\lambda^2} \|l\|^2 \right) \\
&= \frac{(1-\lambda)\lambda}{2} \inf_{(1-\lambda)u+\lambda v=x, u \in K, v \in K^\ominus} (\|u\|^2 + \|v\|^2) \\
&\leq \frac{(1-\lambda)\lambda}{2} \inf_{(1-\lambda)u+\lambda v=x, u \in K, v \in K^\ominus} (\|u\|^2 - 2\langle u, v \rangle + \|v\|^2) \\
&= \frac{(1-\lambda)\lambda}{2} \inf_{(1-\lambda)u+\lambda v=x, u \in K, v \in K^\ominus} \|u - v\|^2 \\
&= f_\lambda(x).
\end{aligned}$$

Therefore,

$$f_\lambda(x) = \frac{(1-\lambda)\lambda}{2} \left(\frac{1}{(1-\lambda)^2} \|P_K x\|^2 + \frac{1}{\lambda^2} \|P_{K^\ominus} x\|^2 \right)$$

and the proof of Example 4.9 is complete.

Appendix D. Proof of Theorem 4.11.

Let us first recall the required notions from convex analysis. Fix $f \in \mathcal{F}$. The *subdifferential operator* ∂f is the set-valued mapping

$$x \mapsto \{x^* \in X \mid (\forall h \in X) \ f(x) + \langle h, x^* \rangle \leq f(x+h)\},$$

with domain $\text{dom } \partial f := \{x \in X \mid \partial f(x) \neq \emptyset\}$. Set $q := \frac{1}{2} \|\cdot\|^2$. Note that $\text{ran } \partial(q+f)^* = \text{ran}((\partial(q+f))^{-1}) = \text{dom } \partial(q+f) = \text{dom } \partial f \subseteq \text{dom } f$ and thus $\overline{\text{ran } \partial(q+f)^*} = \overline{\text{dom } f}$; see, e.g., [20, Chapters 23 and 24]. The *proximal mapping* of a function $g \in \mathcal{F}$ is defined by $\partial(q+g)^*$; see, e.g., [18, 20]. Now abbreviate the proximal mapping of f_μ by P_μ , for $\mu \in \{0, \lambda, 1\}$. Then [4, Theorem 2.6] states that

$$P_\lambda = (1-\lambda)P_0 + \lambda P_1.$$

Set $A := (1-\lambda)P_0$ and $B := \lambda P_1$. Observe that A and B are subdifferential operators of functions in $\mathcal{F}$, namely of $(1-\lambda)(q+f_0)^*$ and $\lambda(q+f_1)^*$, respectively. Thus A and B are *maximal monotone* (see [20, Corollary 31.5.2] and also [10, 22] for further information), and both satisfy condition $(*)$ of Brézis and Haraux [11]. Since $A+B = P_\lambda$ is also maximal monotone, [11, Théorème 3] (see also [22, Section 19]) implies that $\text{ran}(A+B) = \text{ran } A + \text{ran } B$ and $\text{int}(\text{ran}(A+B)) = \text{int}(\text{ran } A + \text{ran } B)$. Hence $\text{dom } f_\lambda = \text{ran } \partial(q+f_\lambda)^* = \text{ran } P_\lambda = \text{ran}(A+B) = \text{ran } A + \text{ran } B = (1-\lambda)\text{ran } P_0 + \lambda\text{ran } P_1 = (1-\lambda)\text{dom } \partial f_0 + \lambda\text{dom } \partial f_1 = (1-\lambda)\text{dom } f_0 + \lambda\text{dom } f_1$. Consequently, $\text{int dom } f_\lambda = \text{int dom } f_\lambda = \text{int}((1-\lambda)\text{dom } f_0 + \lambda\text{dom } f_1) = \text{int}((1-\lambda)\text{dom } f_0 + \lambda\text{dom } f_1)$. This completes the proof of Theorem 4.11.

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