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Autoconjugate representers for linear monotone operators

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Abstract Monotone operators are of central importance in modern optimization and nonlinear analysis. Their study has been revolutionized lately, due to the systematic use of the Fitzpatrick function. Pioneered by Penot and Svaiter, a topic of recent interest has been the representation of maximal monotone operators by so-called autoconjugate functions. Two explicit constructions were proposed, the first by Penot and Zălinescu in 2005, and another by Bauschke and Wang in 2007. The former requires a mild constraint qualification while the latter is based on the proximal average.

We show that these two autoconjugate representers must coincide for continuous linear monotone operators on reflexive spaces. The continuity and the linearity assumption are both essential as examples of discontinuous linear operators and of subdifferential operators illustrate. Furthermore, we also construct an infinite family of autoconjugate representers for the identity operator on the real line.

Keywords Autoconjugate representer · Convex function · Fenchel conjugate · Fitzpatrick function · Linear monotone operator · Maximal monotone operator · Subdifferential operator

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1 Introduction

Throughout this paper, we assume that X is a real reflexive Banach space, with continuous dual space X^*, and pairing $\langle\cdot,\cdot\rangle$. The norm of X is denoted by $\|\cdot\|$, and the norm in the dual space X^* by $\|\cdot\|_*$.

Let $A\colon X \rightrightarrows X^*$ be a set-valued operator, with *graph*

$$\operatorname{gra} A = \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}, \tag{1.1}$$

with *inverse operator* $A^{-1}\colon X^* \rightrightarrows X$ given by

$$\operatorname{gra} A^{-1} = \big\{(x^*,x) \in X^* \times X \mid x^* \in Ax\big\}, \tag{1.2}$$

with *domain* $\operatorname{dom} A = \big\{x \in X \mid Ax \neq \varnothing\big\}$, and with *range* $\operatorname{ran} A = A(X)$. Recall that A is *monotone* if

$$\big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^*\rangle \geq 0. \tag{1.3}$$

A monotone operator A is *maximal monotone* if no proper enlargement (in the sense of graph inclusion) of A is monotone. Monotone operators are ubiquitous in Optimization and Analysis (see, e.g., [21,34,39,40,44,49]) since they contain the key classes of subdifferential operators and of positive linear operators. In [24], Fitzpatrick introduced the following tool in the study of monotone operators.

Definition 1.1 Let $A\colon X \rightrightarrows X^*$. The *Fitzpatrick function* of A is

$$F_A\colon (x,x^*) \mapsto \sup_{(y,y^*)\in \operatorname{gra} A} \big(\langle x,y^*\rangle + \langle y,x^*\rangle - \langle y,y^*\rangle\big). \tag{1.4}$$

Monotone Operator Theory has been revolutionized through the systematic use of the Fitzpatrick function; new results have been obtained and previously known result have been reproved in a simpler fashion — see, e.g., [1–3,6,8–11, 13–17,19,20,22,23,25,26,29–33,37,41–43,45–48,50] and also [18,36]. Before listing some of the key properties of the Fitzpatrick function, we introduce a convenient notation utilized by Penot [32]: If $F\colon X \times X^* \to \left]-\infty,+\infty\right]$, set

$$F^{\intercal}\colon X^* \times X\colon \to \left]-\infty,+\infty\right] : (x^*,x) \mapsto F(x,x^*), \tag{1.5}$$

and similarly for a function defined on $X^* \times X$. We now define an associated set-valued operator $S\colon X \rightrightarrows X^*$ by requiring that for $(x,x^*) \in X \times X^*$,

$$x^* \in S(F)x \quad \Leftrightarrow \quad F(x,x^*) = \langle x,x^*\rangle; \tag{1.6}$$

we also say that F is a *representer* for $S(F)$.

Fact 1.2 (Fitzpatrick) (See [24].) Let $A\colon X \rightrightarrows X^*$ be maximal monotone. Then the following hold.

(i) F_A is proper, lower semicontinuous, and convex.
(ii) $F_{A^{-1}} = F_A^{\intercal}$.
(iii) $F_A \geq \langle\cdot,\cdot\rangle$.
(iv) $A = S(F_A)$.

Item (iv) of Fact 1.2 states the key property that the Fitzpatrick function F_A is a representer for the maximal monotone operator A. It turns out that there are even more structured representers for A available. Let $F\colon X\times X^* \to \left]-\infty,+\infty\right]$. Then it is easy to verify that

$$F^{*\intercal} = F^{\intercal *}, \tag{1.7}$$

where for every $(x^*,x) \in X^* \times X$,

$$F^*(x^*,x) = \sup_{(y,y^*)\in X\times X^*} \big(\langle x,y^*\rangle + \langle y,x^*\rangle - F(y,y^*)\big). \tag{1.8}$$

If

$$F^* = F^{\intercal}, \tag{1.9}$$

then F is said to be *autoconjugate*. Autoconjugate representers are readily available for two important classes of maximal monotone operators.

Example 1.3 (subdifferential operator) Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper, lower semicontinuous, and convex. Then the separable sum of f and the Fenchel conjugate f^*, i.e.,

$$f \oplus f^*\colon X\times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto f(x) + f^*(x^*), \tag{1.10}$$

is an autoconjugate representer for the subdifferential operator ∂f.

Example 1.4 (antisymmetric operator) Let $A\colon X \to X^*$ be continuous, linear, and *antisymmetric*, i.e., $A^* = -A$. Then the indicator function of the graph of A, i.e.,

$$\iota_{\operatorname{gra} A}\colon X\times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \begin{cases} 0, & \text{if } x^* = Ax; \\ +\infty, & \text{otherwise} \end{cases} \tag{1.11}$$

is an autoconjugate representer for A.

Using (1.4), (1.5), and (1.7), we obtain the following useful reformulation of the Fitzpatrick function:

$$F_A = \big(\iota_{\operatorname{gra} A} + \langle\cdot,\cdot\rangle\big)^{*\intercal} = \big(\iota_{\operatorname{gra} A} + \langle\cdot,\cdot\rangle\big)^{\intercal *}. \tag{1.12}$$

We now list some very pleasing and well known properties of autoconjugate functions.

Fact 1.5 (Burachik-Penot-Simons-Svaiter-Zălinescu)
(See [22,31–33,46].) Let $F\colon X\times X^* \to \left]-\infty,+\infty\right]$ be autoconjugate. Then the following hold.

(i) F is proper, lower semicontinuous, and convex.
(ii) $F \geq \langle\cdot,\cdot\rangle$.
(iii) $S(F)$ is maximal monotone.
(iv) If $\widetilde{F}\colon X\times X^* \to \left]-\infty,+\infty\right]$ is autoconjugate and $\widetilde{F} \leq F$, then $\widetilde{F} = F$.

Unfortunately, the Fitzpatrick function F_A is usually *not* an autoconjugate representer for A. In view of Fact 1.5(iii), it was tempting to ask whether every general maximal monotone operator possesses an autoconjugate representer. Nonconstructive existence proofs were presented by Svaiter [47] and by Penot [31,32] in 2003 (see also Ghoussoub's preprint [26]). The first actual construction of an autoconjugate representer for a maximal monotone operator satisfying a mild constraint qualification was provided by Penot and Zălinescu [33] in 2005.

Fact 1.6 (Penot-Zălinescu) (See [33].) Let $A\colon X \rightrightarrows X^*$ be maximal monotone. Suppose that the affine hull of $\operatorname{dom} A$ is closed. Then

$$\begin{aligned}\mathcal{A}_A\colon X \times X^* &\to \left]-\infty,+\infty\right] \\ (x,x^*) &\mapsto \inf_{y^*\in X^*} \left(\tfrac{1}{2}F_A(x,x^*+y^*) + \tfrac{1}{2}F_A^{*\intercal}(x,x^*-y^*)\right) \qquad (1.13)\end{aligned}$$

is an autoconjugate representer for A.

Another autoconjugate representer was very recently proposed in [9]. While this proximal-averaged based construction is more involved [4,5,7], it has the advantage of not imposing a constraint qualification.

Fact 1.7 (See [9].) Let $A\colon X \rightrightarrows X^*$ be maximal monotone. Then

$$\begin{aligned}\mathcal{B}_A\colon X \times X^* &\to \left]-\infty,+\infty\right] \\ (x,x^*) &\mapsto \inf_{(y,y^*)\in X\times X^*} \left(\tfrac{1}{2}F_A(x+y,x^*+y^*) + \tfrac{1}{2}F_A^{*\intercal}(x-y,x^*-y^*)\right. \\ &\qquad\qquad \left. + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|y^*\|_*^2\right) \qquad (1.14)\end{aligned}$$

is an autoconjugate representer for A.

It is natural to ask "How do the autoconjugate representers $\mathcal{A}_A$ and $\mathcal{B}_A$ compare?" We provide two answers to this question: First, we show that if $A\colon X \to X^*$ is continuous, linear, and monotone, then $\mathcal{A}_A$ and $\mathcal{B}_A$ coincide; furthermore, we provide a formula for this autoconjugate representer which agrees with a third autoconjugate representer $\mathcal{C}_A$ that is contained in the work by Ghoussoub (Theorem 3.1). Secondly, for *nonlinear* monotone subdifferential operators, the two autoconjugate representers may be different (Theorem 5.1).

The first answer raises the question on whether autoconjugate representers for continuous linear monotone operators are unique. We answer this question in the negative by providing a family of autoconjugate representers for the identity operator Id (Theorem 4.2). However, we show that the autoconjugate representers $\mathcal{A}_A$ and $\mathcal{B}_A$ in this setting are characterized by a pleasing symmetry property (Theorem 4.4).

We conclude by discussing *discontinuous* linear monotone operators. It turns out that $\mathcal{A}_A$ may fail to be autoconjugate (Example 6.5), which underlines not only the continuity assumption in Theorem 3.1 but also the importance of the constraint qualification in Fact 1.6.

The remainder of this paper is organized as follows. Section 2 contains some results on quadratic functions and another autoconjugate representer that will be used in later sections. In Section 3, we show that $\mathcal{A}_A$ and $\mathcal{B}_A$ coincide and provide a simple formula for it (see Theorem 3.1). Uniqueness of autoconjugate representations is discussed in Section 4, and a characterization in the symmetric case is also presented. In stark contrast, and as shown in Section 5, $\mathcal{A}_A$ and $\mathcal{B}_A$ may be different for (nonlinear) subdifferential operators. The final Section 6 reveals similar difference for discontinuous linear operators.

Notation utilized is standard as in Convex Analysis and Monotone Operator Theory; see, e.g., [38,39,49]. Thus, for a proper convex function $f\colon X \to \left]-\infty,+\infty\right]$, we write $f^*\colon x^* \mapsto \sup_{x\in X}(\langle x,x^*\rangle - f(x))$, $\partial f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y-x,x^*\rangle + f(x) \leq f(y)\big\}$, ∇f, and $\operatorname{dom} f = \big\{x \in X \mid f(x) < +\infty\big\}$, for the *Fenchel conjugate*, *subdifferential operator*, *gradient operator*, and *domain* of f, respectively. The strictly positive integers are $\mathbb{N} = \{1,2,\ldots\}$.

2 Auxiliary results

The following result is a consequence of results and proof techniques introduced by Penot, Simons, and Zălinescu [33,46]. It also extends [26, Lemma 2.2].

Proposition 2.1 *Let F_1 and F_2 be autoconjugate functions on $X \times X^*$ representing maximal monotone operators A_1 and A_2, respectively. Suppose that*

$$\bigcup_{\lambda>0} \lambda\big(P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\big) \quad \text{is a closed subspace of } X, \tag{2.1}$$

where $P_X\colon X \times X^* \to X\colon (x,x^*) \mapsto x$, and set

$$F\colon X \times X^* \to \left]-\infty,+\infty\right]\colon (x,x^*) \mapsto \inf_{y^*\in X^*} \big(F_1(x,y^*) + F_2(x,x^*-y^*)\big). \tag{2.2}$$

Then F is an autoconjugate representer for $A_1 + A_2$, and the infimum in (2.2) is attained.

Proof. Let $(x,x^*) \in X \times X^*$. Using Simons and Zălinescu's [46, Theorem 4.2] and the assumption that each F_i is autoconjugate, we obtain

$$\begin{aligned} F^*(x^*,x) &= \min_{x_1^*+x_2^*=x^*} \big(F_1^*(x_1^*,x) + F_2^*(x_2^*,x)\big) \\ &= \min_{x_1^*+x_2^*=x^*} \big(F_1(x,x_1^*) + F_2(x,x_2^*)\big) \\ &= F(x,x^*). \end{aligned} \tag{2.3}$$

Thus, F is autoconjugate and the infimum in (2.2) is attained.

It remains to show that $S(F) = S(F_1) + S(F_2)$. Since autoconjugates are greater than or equal to $\langle\cdot,\cdot\rangle$ (see Fact 1.5(ii)), the above implies the equivalences

$$
\begin{aligned}
& x^* \in S(F)x \\
& \Leftrightarrow F(x,x^*) = \langle x, x^*\rangle \\
& \Leftrightarrow (\exists\, y^* \in X^*)\ \ F_1(x,y^*) + F_2(x,x^*-y^*) = \langle x,y^*\rangle + \langle x, x^*-y^*\rangle \\
& \Leftrightarrow (\exists\, y^* \in X^*)\ \ F_1(x,y^*) = \langle x,y^*\rangle \ \text{ and } \ F_2(x,x^*-y^*) = \langle x, x^*-y^*\rangle \\
& \Leftrightarrow (\exists\, y^* \in X^*)\ \ y^* \in S(F_1)(x) \ \text{ and } \ x^*-y^* \in S(F_2)(x) \\
& \Leftrightarrow (\exists\, y^* \in X^*)\ \ y^* \in A_1x \ \text{ and } \ x^*-y^* \in A_2x \\
& \Leftrightarrow x^* \in (A_1+A_2)x.
\end{aligned} \tag{2.4}
$$

Therefore, $S(F) = A_1 + A_2$, i.e., F is a representer for $A_1 + A_2$. ■

Suppose that

$$A\colon X \to X^* \quad \text{is linear and continuous.} \tag{2.5}$$

Then A is *symmetric* (resp. *antisymmetric*) if $A^* = A$ (resp. $A^* = -A$). We denote the *symmetric part* and the *antisymmetric part* of A by

$$A_+ = \tfrac{1}{2}A + \tfrac{1}{2}A^* \quad \text{and} \quad A_\circ = \tfrac{1}{2}A - \tfrac{1}{2}A^*, \tag{2.6}$$

respectively. Throughout, we shall work with the quadratic function

$$q_A\colon X \to \mathbb{R}\colon x \mapsto \tfrac{1}{2}\langle x, Ax\rangle, \tag{2.7}$$

and we will use the well known facts (see, e.g., [35]) that $q_A = q_{A_+}$, that

$$\nabla q_A = A_+, \tag{2.8}$$

and that A is monotone $\Leftrightarrow$ q_A is convex ($\Leftrightarrow$ A is maximal monotone by [39, Example 12.7] or [51, Proposition 32.7]).

The next result provides a formula for q_A^* that will be useful later.

Proposition 2.2 *Let $A\colon X \to X^*$ be continuous, linear, symmetric, and monotone. Then*

$$\big(\forall (x,x^*) \in X \times X^*\big) \quad q_A^*(x^* + Ax) = q_A(x) + \langle x, x^*\rangle + q_A^*(x^*) \tag{2.9}$$

and

$$q_A^* \circ A = q_A. \tag{2.10}$$

Proof. Let $(x,x^*) \in X \times X^*$. Then

$$
\begin{aligned}
q_A^*(x^*+Ax) &= \sup_y \ \big(\langle y, x^*+Ax\rangle - q_A(y)\big) \\
&= \sup_y \ \big(\langle y, x^*\rangle - q_A(y) + \langle y, Ax\rangle\big) \\
&= q_A(x) + \sup_y \ \big(\langle y, x^*\rangle - q_A(y) + \langle y, Ax\rangle - q_A(x)\big) \\
&= q_A(x) + \sup_y \ \big(\langle y, x^*\rangle - q_A(y-x)\big) \\
&= q_A(x) + \langle x, x^*\rangle + \sup_y \ \big(\langle y-x, x^*\rangle - q_A(y-x)\big) \\
&= q_A(x) + \langle x, x^*\rangle + q_A^*(x^*),
\end{aligned} \tag{2.11}
$$

which verifies (2.9). To see (2.10), set $x^* = 0$ in (2.9). ■

Corollary 2.3 *Let $A\colon X \to X^*$ be continuous, linear, and monotone. Then*

$$\big(\forall (x,x^*) \in X \times X^*\big) \quad q_A^*(x^* + A_+x) = q_A(x) + \langle x, x^*\rangle + q_A^*(x^*). \tag{2.12}$$

Proposition 2.4 *Let $A\colon X \to X^*$ be continuous, linear, and monotone, and let $(x,x^*) \in X \times X^*$. Then*

$$F_A(x,x^*) = 2q_A^*(\tfrac{1}{2}x^* + \tfrac{1}{2}A^*x) = \tfrac{1}{2}q_A^*(x^* + A^*x) \tag{2.13}$$

and

$$F_A^*(x^*,x) = \iota_{\operatorname{gra} A}(x,x^*) + \langle x, Ax\rangle. \tag{2.14}$$

Proof. As in the proof of [3, Theorem 2.3(i)], we have

$$\begin{aligned} F_A(x,x^*) &= \sup_{y\in X} \big(\langle x, Ay\rangle + \langle y, x^*\rangle - \langle y, Ay\rangle\big) \\ &= 2\sup_{y\in X} \big(\langle y, \tfrac{1}{2}x^* + \tfrac{1}{2}A^*x\rangle - q_A(y)\big) \\ &= 2q_A^*(\tfrac{1}{2}x^* + \tfrac{1}{2}A^*x) \\ &= \tfrac{1}{2}q_A^*(x^* + A^*x). \end{aligned} \tag{2.15}$$

This verifies (2.13). Using (1.12), we see that $F_A^*(x^*,x) = (\iota_{\operatorname{gra} A} + \langle\cdot,\cdot\rangle)^{\intercal **}(x^*,x) = (\iota_{\operatorname{gra} A} + \langle\cdot,\cdot\rangle)^{\intercal}(x^*,x) = \iota_{\operatorname{gra} A}(x,x^*) + \langle x, Ax\rangle$. Hence (2.14) holds as well. A referee suggested the following more direct proof of (2.14). Using (2.13) and the fact that $(\forall y)\ \langle x, A^*y\rangle = \langle y, Ax\rangle$, we have

$$\begin{aligned} F_A^*(x^*,x) &= \sup_{(y,y^*)} \big(\langle y, x^*\rangle + \langle x, y^*\rangle - \tfrac{1}{2}q_A^*(y^* + A^*y)\big) \\ &= \sup_{(y,y^*)} \big(\langle y, x^* - Ax\rangle + \langle x, y^* + A^*y\rangle - \tfrac{1}{2}q_A^*(y^* + A^*y)\big) \\ &= \sup_{y} \langle y, x^* - Ax\rangle + \sup_{z^*} \big(\langle x, z^*\rangle - \tfrac{1}{2}q_A^*(z^*)\big) \\ &= \iota_{\{0\}}(x^* - Ax) + \big(\tfrac{1}{2}q_A^*\big)^*(x) \\ &= \iota_{\operatorname{gra} A}(x,x^*) + \tfrac{1}{2}q_A(2x) \\ &= \iota_{\operatorname{gra} A}(x,x^*) + 2q_A(x) \\ &= \iota_{\operatorname{gra} A}(x,x^*) + \langle x, Ax\rangle, \end{aligned} \tag{2.16}$$

as required. ■

Proposition 2.5 *Let $F_1\colon X \times X^* \to \left]-\infty,+\infty\right]$ be autoconjugate, and let $A_2\colon X \to X^*$ be continuous, linear, and antisymmetric. Then the function*

$$(x,x^*) \mapsto F_1(x, x^* - A_2x) \tag{2.17}$$

is an autoconjugate representer for $S(F_1) + A_2$.

Proof. Set $F_2 = \iota_{\operatorname{gra} A_2}$. By Example 1.4, F_2 is an autoconjugate representer for A_2. Let F be as in Proposition 2.1. Then for every $(x,x^*) \in X \times X^*$, we have

$$\begin{aligned} F(x,x^*) &= \inf_{z^*\in X^*} \big(F_1(x,x^*-z^*) + F_2(x,z^*)\big) \\ &= \inf_{z^*\in X^*} \big(F_1(x,x^*-z^*) + \iota_{\operatorname{gra} A_2}(x,z^*)\big) \\ &= F_1(x,x^*-A_2x). \end{aligned} \tag{2.18}$$

Thus, Proposition 2.1 yields that F represents $S(F_1) + A_2$. ■

Example 2.6 (Ghoussoub) (See also [26, Section 1].) Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper, lower semicontinuous, and convex, and let A be continuous, linear, and antisymmetric. Then the function

$$(x,x^*) \mapsto f(x) + f^*(x^* - Ax) \tag{2.19}$$

is an autoconjugate representer for $\partial f + A$.

Proof. By Example 1.3, $f \oplus f^*$ is an autoconjugate representer for ∂f. The result thus follows from Proposition 2.5. ■

Corollary 2.7 *Let $A\colon X \to X^*$ be continuous, linear, and monotone. Then*

$$\begin{aligned} \mathcal{C}_A\colon X \times X^* &\to \left]-\infty,+\infty\right] \\ (x,x^*) &\mapsto q_A(x) + q_A^*(x^* - A_\circ x) \end{aligned} \tag{2.20}$$

is an autoconjugate representer for A. In particular, if A is symmetric, then

$$\mathcal{C}_A = q_A \oplus q_A^*. \tag{2.21}$$

Proof. This follows from (2.8) and Example 2.6 (when applied to the function $f = q_{A_+}$ and to the antisymmetric operator $A_\circ$). ■

We now show that the Ghoussoub representers are closed under the partial infimal convolution operation of Proposition 2.1.

Proposition 2.8 *Let A and B be continuous, linear, and monotone on X. Then the function*

$$F\colon X \times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \min_{y^*\in X^*} \big(\mathcal{C}_A(x,x^*-y^*) + \mathcal{C}_B(x,y^*)\big) \tag{2.22}$$

coincides with the autoconjugate representer $\mathcal{C}_{A+B}$ for $A+B$.

Proof. In view of Proposition 2.1, we only need to show that $F = \mathcal{C}_{A+B}$. Let $(x,x^*) \in X \times X^*$. Using (2.22) and Corollary 2.7, we obtain

$$\begin{aligned} F(x,x^*) &= \min_{y^*\in X^*} \big(q_A(x) + q_A^*(x^*-y^*-A_\circ x) + q_B(x) + q_B^*(y^*-B_\circ x)\big) \\ &= q_A(x) + q_B(x) + (q_A^* \Box q_B^*)(x^* - A_\circ x - B_\circ x) \\ &= q_{A+B}(x) + \big(q_A + q_B\big)^*(x^* - A_\circ x - B_\circ x) \qquad (2.23) \\ &= q_{A+B}(x) + q_{A+B}^*\big(x^* - (A+B)_\circ x\big) \\ &= \mathcal{C}_{A+B}(x,x^*). \qquad (2.24) \end{aligned}$$

Here "□" denotes *infimal convolution* and (2.23) holds because both q_A and q_B are convex and continuous on X. The proof is complete. ■

3 Coincidence

We are now ready for one of our main results.

Theorem 3.1 (coincidence) *Let $A\colon X \to X^*$ be continuous, linear, and monotone. Then all three autoconjugate representers $\mathcal{A}_A$, $\mathcal{B}_A$, $\mathcal{C}_A$ for A coincide with the function*

$$(x, x^*) \mapsto \langle x, x^*\rangle + q_A^*(x^* - Ax). \tag{3.1}$$

Proof. The proof proceeds by proving a succession of claims. Let $(x, x^*) \in X \times X^*$.

Claim 1: $\mathcal{A}_A = \mathcal{C}_A$.

Using (1.13), (2.14), (2.13), and (2.20), we obtain

$$\begin{aligned}
\mathcal{A}_A(x, x^*) &= \inf_{y^*\in X^*} \big(\tfrac{1}{2}F_A(x, x^* + y^*) + \tfrac{1}{2}F_A^*(x^* - y^*, x)\big)\\
&= \inf_{y^*\in X^*} \big(\tfrac{1}{2}F_A(x, x^* + y^*) + \iota_{\operatorname{gra} A}(x, x^* - y^*) + q_A(x)\big)\\
&= \tfrac{1}{2}F_A(x, 2x^* - Ax) + q_A(x)\\
&= q_A^*\big(x^* - \tfrac{1}{2}Ax + \tfrac{1}{2}A^*x\big) + q_A(x)\\
&= q_A^*\big(x^* - A_\circ x\big) + q_A(x) \qquad (3.2)\\
&= \mathcal{C}_A(x, x^*). \qquad (3.3)
\end{aligned}$$

This verifies Claim 1.

Claim 2: $\mathcal{A}_A$ coincides with the function of (3.1).

In view of (3.2) and Corollary 2.3, we see that

$$\begin{aligned}
\mathcal{A}_A(x, x^*) &= q_A^*\big(x^* - A_\circ x\big) + q_A(x)\\
&= q_A^*(x^* - Ax + A_+x) + q_A(x)\\
&= 2q_A(x) + \langle x, x^* - Ax\rangle + q_A^*(x^* - Ax)\\
&= \langle x, x^*\rangle + q_A^*(x^* - Ax), \qquad (3.4)
\end{aligned}$$

which establishes Claim 2.

Claim 3: $\mathcal{A}_A = \mathcal{B}_A$.

Using (1.14), (2.14), (2.13), Corollary 2.3, and Claim 2, we have

$$\begin{aligned}
\mathcal{B}_A(x, x^*) = &\inf_{(y,y^*)\in X\times X^*} \big(\tfrac{1}{2}F_A(x + y, x^* + y^*) + \tfrac{1}{2}F_A^*(x^* - y^*, x - y)\\
&\qquad + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|y^*\|_*^2\big)\\
= &\inf_{(y,y^*)\in X\times X^*} \big(\tfrac{1}{2}F_A(x + y, x^* + y^*) + \iota_{\operatorname{gra} A}(x - y, x^* - y^*)\\
&\qquad + \tfrac{1}{2}\langle x - y, A(x - y)\rangle + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|y^*\|_*^2\big)
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{y\in X} \big(\tfrac{1}{2}F_A(x+y, 2x^* - A(x-y)) + q_A(x-y) \\
&\qquad\quad + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|x^* - A(x-y)\|_*^2\big) \\
&= \inf_{y\in X} \big(q_A^*\big(x^* - \tfrac{1}{2}A(x-y) + \tfrac{1}{2}A^*(x+y)\big) + q_A(x-y) \\
&\qquad\quad + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|x^* - A(x-y)\|_*^2\big) \\
&= \inf_{y\in X} \big(q_A^*\big(x^* - Ax + A_+(x+y)\big) + q_A(x-y) \\
&\qquad\quad + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|x^* - A(x-y)\|_*^2\big) \\
&= \inf_{y\in X} \big(q_A^*(x^* - Ax) + \langle x+y, x^* - Ax\rangle + q_A(x+y) + q_A(x-y) \\
&\qquad + \tfrac{1}{2}\|y\|^2 + \tfrac{1}{2}\|x^* - A(x-y)\|_*^2\big) \\
&= \inf_{y\in X} \big(q_A^*(x^* - Ax) + \langle x+y, x^* - Ax\rangle + 2q_A(x) + 2q_A(y) \\
&\qquad + \tfrac{1}{2}\|-y\|^2 + \tfrac{1}{2}\|x^* - A(x-y)\|_*^2\big) \\
&\geq q_A^*(x^* - Ax) + \langle x, x^*\rangle \\
&\qquad\quad + \inf_{y\in X} \big(\langle y, x^* - Ax\rangle + 2q_A(y) + \langle -y, x^* - A(x-y)\rangle\big) \\
&= q_A^*(x^* - Ax) + \langle x, x^*\rangle + \inf_{y\in X} \big(2q_A(y) + \langle -y, Ay\rangle\big) \\
&= q_A^*(x^* - Ax) + \langle x, x^*\rangle \\
&= \mathcal{A}_A(x, x^*).
\end{aligned} \tag{3.5}
$$

Hence $\mathcal{B}_A \geq \mathcal{A}_A$. On the other hand, both $\mathcal{A}_A$ and $\mathcal{B}_A$ are autoconjugate (see Fact 1.6 and Fact 1.7). Altogether, Fact 1.5(iv) implies Claim 3.

Finally, observe that Claims 1–3 yield the result. ■

Example 3.2 Suppose that X is the Euclidean plane $\mathbb{R}^2$, let $\theta \in \left[0, \frac{\pi}{2}\right[$, and set

$$
A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad A_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}
$$

Then for every $(x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$
\begin{aligned}
\mathcal{A}_A(x, x^*) = \mathcal{B}_A(x, x^*) &= \mathcal{C}_A(x, x^*) \\
&= \frac{1}{2\cos\theta}\|x^* - Ax\|^2 + \langle x, x^*\rangle \\
&= \frac{1}{2\cos\theta}\|x^* - (\sin\theta)A_{\pi/2}x\|^2 + \frac{\cos\theta}{2}\|x\|^2.
\end{aligned} \tag{3.7}
$$

Proof. This is a consequence of Theorem 3.1 because $A_+ = (\cos\theta)\operatorname{Id}$, $q_A = (\cos\theta)\frac{1}{2}\|\cdot\|^2$, and $A_\circ = (\sin\theta)A_{\pi/2}$. ■

4 Observations on nonuniqueness

Theorem 3.1 might nurture the conjecture that for continuous linear monotone operators, all autoconjugate representers coincide. This conjecture is false — we shall provide a whole family of distinct autoconjugate representers for the identity on $\mathbb{R}$. Our constructions rest on the following result.

Proposition 4.1 *Let $g\colon \mathbb{R} \to \left]-\infty,+\infty\right]$ be such that*

$$(\forall x \in \mathbb{R}) \quad g^*(-x) = g(x) \geq 0. \tag{4.1}$$

Then

$$g(0) = 0. \tag{4.2}$$

Moreover, each of the following functions satisfies (4.1)*:*

(i) *the indicator function* $\iota_{[0,+\infty[}\colon x \mapsto \begin{cases} 0, & \text{if } x \geq 0; \\ +\infty, & \text{if } x < 0; \end{cases}$

(ii) *the halved energy function* $\frac{1}{2}|\cdot|^2$*;*

(iii) *for $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the function*

$$x \mapsto \begin{cases} \frac{1}{p}x^p, & \text{if } x \geq 0; \\ \frac{1}{q}(-x)^q, & \text{if } x < 0. \end{cases}$$

Proof. On the one hand, $g(0) \geq 0$. On the other hand, $g(0) = g^*(-0) = g^*(0) = \sup_{y\in\mathbb{R}} -g(y) = -\inf_{y\in\mathbb{R}} g(y) \leq 0$. Altogether, $g(0) = 0$ and so (4.2) holds. It is straightforward to verify that each of the given functions satisfies (4.1). ■

Theorem 4.2 *Let $g\colon \mathbb{R} \to \left]-\infty,+\infty\right]$ be such that for every $x \in \mathbb{R}$, $g^*(-x) = g(x) \geq 0$, and set $q\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \frac{1}{2}|x|^2$. Then*

$$F\colon \mathbb{R}^2 \to \left]-\infty,+\infty\right] : (x,y) \mapsto q\Big(\frac{x+y}{\sqrt{2}}\Big) + g\Big(\frac{x-y}{\sqrt{2}}\Big). \tag{4.3}$$

is an autoconjugate representer for $\mathrm{Id}\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x$.

Proof. Let $(x,y) \in \mathbb{R}^2$. Using the fact that $q^* = q$ and the assumption on g, we see that

$$\begin{aligned}
F^*(y,x) &= \sup_{(u,v)\in\mathbb{R}^2} \Big(uy + vx - q\big(\tfrac{u+v}{\sqrt{2}}\big) - g\big(\tfrac{u-v}{\sqrt{2}}\big)\Big) \\
&= \sup_{(u,v)\in\mathbb{R}^2} \Big(\tfrac{u+v}{2}(x+y) - \tfrac{u-v}{2}(x-y) - q\big(\tfrac{u+v}{\sqrt{2}}\big) - g\big(\tfrac{u-v}{\sqrt{2}}\big)\Big) \\
&= \sup_{(u,v)\in\mathbb{R}^2} \Big(\tfrac{u+v}{\sqrt{2}}\tfrac{x+y}{\sqrt{2}} - \tfrac{u-v}{\sqrt{2}}\tfrac{x-y}{\sqrt{2}} - q\big(\tfrac{u+v}{\sqrt{2}}\big) - g\big(\tfrac{u-v}{\sqrt{2}}\big)\Big) \\
&= q^*\big(\tfrac{x+y}{\sqrt{2}}\big) + g^*\big(-\tfrac{x-y}{\sqrt{2}}\big) \\
&= q\big(\tfrac{x+y}{\sqrt{2}}\big) + g\big(\tfrac{x-y}{\sqrt{2}}\big) \\
&= F(x,y).
\end{aligned} \tag{4.4}$$

Hence F is autoconjugate. In view of (4.2), we have $(x,y) \in \operatorname{gra}\big(S(F)\big) \Leftrightarrow y \in S(F)x \Leftrightarrow F(x,y) = xy \Leftrightarrow q\big((x+y)/\sqrt{2}\big) + g\big((x-y)/\sqrt{2}\big) = xy \Leftrightarrow \frac{1}{4}(x+y)^2 + g\big((x-y)/\sqrt{2}\big) = xy \Leftrightarrow \frac{1}{4}(x-y)^2 + g\big((x-y)/\sqrt{2}\big) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow (x,y) \in \operatorname{gra}(\mathrm{Id})$. ■

Remark 4.3 Consider Theorem 4.2. If we set $g = q = \frac{1}{2}|\cdot|^2$, then $F = q \oplus q = q_{\mathrm{Id}} \oplus q^*_{\mathrm{Id}} = \mathcal{C}_{\mathrm{Id}}$ by Corollary 2.7. Thus, this pleasingly symmetric choice of g gives rise to $\mathcal{A}_{\mathrm{Id}} = \mathcal{B}_{\mathrm{Id}} = \mathcal{C}_{\mathrm{Id}}$. Proposition 4.1 provides other choices of g that lead to different autoconjugate representers for Id.

Having settled the nonuniqueness of autoconjugate representers, it is natural to ask "What makes the autoconjugate representers of Theorem 3.1 special?" The next result provides a complete answer for a large class of linear operators.

Theorem 4.4 *Let $A\colon X \to X^*$ be continuous, linear, monotone, symmetric, and such that* $\operatorname{ran} A$ *is closed. Furthermore, let $F\colon X \times X^* \to \left]-\infty, +\infty\right]$. Then*

$$F = \mathcal{C}_A \quad \Leftrightarrow \quad \begin{cases} F \textit{ is autoconjugate}, \\ F(0,0) = 0, \\ \big(\forall (x,y) \in X \times X\big) \;\; F(x, Ay) = F(y, Ax). \end{cases} \tag{4.5}$$

Proof. "$\Rightarrow$": By Corollary 2.7, F is autoconjugate and $F(0,0) = (q_A \oplus q_A^*)(0,0) = 0$. Let x and y be in X. Using (2.10), we have $F(x, Ay) = (q_A \oplus q_A^*)(x, Ay) = q_A(x) + q_A^*(Ay) = q_A(x) + q_A(y) = q_A(y) + q_A^*(Ax) = (q_A \oplus q_A^*)(y, Ax) = F(y, Ax)$.

"$\Leftarrow$": Let $(x, x^*) \in X \times X^*$. We proceed by verifying the next two claims.

Claim 1: $x^* \notin \operatorname{ran} A \Rightarrow F(x, x^*) = +\infty$.
Assume that $x^* \notin \operatorname{ran} A$. The Separation Theorem yields $z \in X$ such that

$$\langle z, x^* \rangle > 0 \tag{4.6}$$

and $\max\langle z, \operatorname{ran} A\rangle = 0$. Since A is symmetric, we deduce that $Az = 0$. This implies $(\forall \rho \in \mathbb{R})$ $F(\rho z, 0) = F(\rho z, A0) = F\big(0, A(\rho z)\big) = F(0,0) = 0$. Thus

$$\begin{aligned} (\forall \rho \in \mathbb{R}) \quad F(x, x^*) &= F(x, x^*) + F(\rho z, 0) = F(x, x^*) + F^*(0, \rho z) \\ &\geq \langle x, 0 \rangle + \langle \rho z, x^* \rangle = \rho \langle z, x^* \rangle. \end{aligned} \tag{4.7}$$

In view of (4.6), we see that Claim 1 follows by letting $\rho \to +\infty$ in (4.7).

Claim 2: $x^* \in \operatorname{ran} A \Rightarrow F(x, x^*) \geq \mathcal{C}_A(x, x^*)$.
Assume that $x^* \in \operatorname{ran} A$, say $x^* = Ay$. Then $2F(x, x^*) = 2F(x, Ay) = F(x, Ay) + F(y, Ax) = F(x, Ay) + F^*(Ax, y) \geq \langle x, Ax\rangle + \langle y, Ay\rangle$ and hence, using (2.10),

$$F(x, x^*) \geq q_A(x) + q_A(y) = q_A(x) + q_A^*(Ay) = (q_A \oplus q_A^*)(x, x^*). \tag{4.8}$$

This and (2.21) yield Claim 2.

Note that Claim 1 and Claim 2 yield $F \geq \mathcal{C}_A$. Therefore, Fact 1.5(iv) implies that $F = \mathcal{C}_A$. ∎

5 Autoconjugate representers for $\partial(-\ln)$

Theorem 3.1 showed that three ostensibly different autoconjugate representers are in fact identical for continuous linear monotone operators. It is tempting to consider a subdifferential operator ∂f, and to compare $\mathcal{A}_{\partial f}$, $\mathcal{B}_{\partial f}$, and $f \oplus f^*$. It turns out that these autoconjugate representers for ∂f may all be different. To aid in the construction of this example, it will be convenient to work in this section with the negative natural logarithm function

$$f\colon \mathbb{R} \to \left]-\infty, +\infty\right] : x \mapsto \begin{cases} -\ln(x), & \text{if } x > 0; \\ +\infty, & \text{if } x \le 0, \end{cases} \tag{5.1}$$

and with the set

$$C = \big\{(x, x^*) \in \mathbb{R} \times \mathbb{R} \mid x^* \le -\tfrac{1}{x} < 0\big\}. \tag{5.2}$$

It is well known that

$$(\forall x \in \mathbb{R}) \quad f^*(x) = -1 + f(-x) \tag{5.3}$$

and straightforward to verify that

$$\tfrac{1}{\sqrt{2}}C = \big\{(x, x^*) \in \mathbb{R} \times \mathbb{R} \mid x^* \le -\tfrac{1}{2x} < 0\big\}, \tag{5.4}$$

$$\tfrac{1}{2}C = \big\{(x, x^*) \in \mathbb{R} \times \mathbb{R} \mid x^* \le -\tfrac{1}{4x} < 0\big\}, \tag{5.5}$$

and

$$\tfrac{1}{\sqrt{2}}C \subsetneqq \tfrac{1}{2}C \subsetneqq \left]0, +\infty\right[\times \left]-\infty, 0\right[. \tag{5.6}$$

Furthermore, [8, Example 3.4] yields

$$\big(\forall (x, x^*) \in \mathbb{R} \times \mathbb{R}\big) \quad F_{\partial f}(x, x^*) = \begin{cases} 1 - 2\sqrt{-xx^*}, & \text{if } x \ge 0 \text{ and } x^* \le 0; \\ +\infty, & \text{otherwise,} \end{cases} \tag{5.7}$$

and

$$F_{\partial f}^{*\intercal} = -1 + \iota_C. \tag{5.8}$$

Theorem 5.1 *The functions $\mathcal{A}_{\partial f}$, $\mathcal{B}_{\partial f}$, and $f \oplus f^*$ have domains $\frac{1}{\sqrt{2}}C$, $\frac{1}{2}C$, and $\left]0, +\infty\right[\times \left]-\infty, 0\right[$, respectively. Consequently, they are three different autoconjugate representers for ∂f.*

Proof. Using (1.13), (5.7), (5.8) and (5.4), we see that

$$\begin{aligned}
&\operatorname{dom} \mathcal{A}_{\partial f} \\
&= \big\{(x, \tfrac{1}{2}x_1^* + \tfrac{1}{2}x_2^*) \in \mathbb{R} \times \mathbb{R} \mid (x, x_1^*) \in \operatorname{dom} F_{\partial f} \text{ and } (x, x_2^*) \in \operatorname{dom} F_{\partial f}^{*\intercal}\big\} \\
&= \big\{(x, \tfrac{1}{2}x_1^* + \tfrac{1}{2}x_2^*) \in \mathbb{R} \times \mathbb{R} \mid x \ge 0,\ x_1^* \le 0, \text{ and } (x, x_2^*) \in C\big\} \\
&= \big\{(x, x^*) \in \mathbb{R} \times \mathbb{R} \mid x^* \le -\tfrac{1}{2x} < 0\big\} \\
&= \tfrac{1}{\sqrt{2}}C,
\end{aligned} \tag{5.9}$$

as claimed. Similarly, by (1.14), (5.7), and (5.8),

$$\begin{aligned}\operatorname{dom}\mathcal{B}_{\partial f} &= \tfrac{1}{2}\operatorname{dom}F_{\partial f} + \tfrac{1}{2}\operatorname{dom}F_{\partial f}^{*\intercal}\\ &= \tfrac{1}{2}\big(\left[0,+\infty\right[\times\left]-\infty,0\right]\big) + \tfrac{1}{2}C\\ &= \tfrac{1}{2}C. \end{aligned}\tag{5.10}$$

Furthermore, by (5.1) and (5.3),

$$\operatorname{dom}(f\oplus f^*) = (\operatorname{dom} f)\times(\operatorname{dom} f^*) = \left]0,+\infty\right[\times\left]-\infty,0\right[.\tag{5.11}$$

We thus have verified the statements concerning the domains. Fact 1.6, Fact 1.7, and Example 1.3 imply that all three functions are autoconjugate representers for ∂f. In view of (5.6), these functions are all different since their domains are also all different. ■

Remark 5.2 Using (5.7) and (5.8), one may verify that

$$\big(\forall (x,x^*)\in\mathbb{R}\times\mathbb{R}\big)\quad \mathcal{A}_{\partial f}(x,x^*) = \begin{cases} -\sqrt{-1-2xx^*}, & \text{if } (x,x^*)\in\frac{1}{\sqrt{2}}C;\\ +\infty, & \text{otherwise.}\end{cases}\tag{5.12}$$

However, we do not have an explicit formula for $\mathcal{B}_{\partial f}$.

6 Discontinuous symmetric operators

In this final section, we investigate discontinuous symmetric operators. Specifically, we assume throughout this section that $A\colon X\rightrightarrows X^*$ is maximal monotone, at most single-valued, $\operatorname{dom} A$ is a linear subspace, and $A|_{\operatorname{dom} A}$ is linear and symmetric. Put differently, we assume that

$$A\colon \operatorname{dom} A\to X^* \quad \text{is linear, symmetric, and maximal montone.}\tag{6.1}$$

It is convenient to extend the definition of q_A in (2.7) to this more general setting via

$$q_A\colon X\to\mathbb{R}\colon x\mapsto \begin{cases}\frac{1}{2}\langle x,Ax\rangle, & \text{if } x\in\operatorname{dom}A;\\ +\infty, & \text{otherwise.}\end{cases}\tag{6.2}$$

A key tool is the function

$$f\colon X\to\left]-\infty,+\infty\right]\colon x\mapsto \sup_{y\in\operatorname{dom}A}\big(\langle x,Ay\rangle - \tfrac{1}{2}\langle y,Ay\rangle\big),\tag{6.3}$$

which was introduced by Phelps and Simons.

Fact 6.1 (Phelps-Simons) (See [35].) The following hold.

(i) f is proper, lower semicontinuous, and convex.
(ii) $A=\partial f$.
(iii) $\operatorname{dom}A\subseteq\operatorname{dom}f\subseteq\overline{\operatorname{dom}}\,A$ and $(\forall x\in\operatorname{dom}A)\ f(x)=\frac{1}{2}\langle x,Ax\rangle$.
(iv) A is continuous $\Leftrightarrow$ $\operatorname{dom}A=X$ $\Leftrightarrow$ $\operatorname{dom}f=X$.

Corollary 6.2 *The following hold.*

(i) $f + \iota_{\operatorname{dom} A} = q_A$.
(ii) $f = q_A \Leftrightarrow \operatorname{dom} f = \operatorname{dom} A$.
(iii) $q_A^{**} = f$.
(iv) *If A is one-to-one, then $f = q^*_{A^{-1}}$.*

Proof. (i): Clear from Fact 6.1(iii). (ii): Since $\operatorname{dom} q_A = \operatorname{dom} A$, this item is a consequence of (i). (iii): Using Fact 6.1(i)&(ii) and a result by J. Borwein (see [12, Theorem 1] or [49, Theorem 3.1.4(i)]), we see that $f = f^{**} = (f + \iota_{\operatorname{dom}\partial f})^{**} = (f + \iota_{\operatorname{dom} A})^{**} = q_A^{**}$.

The following alternative proof of (iii) was suggested by a referee. By (i), $q_A = f + \iota_{\operatorname{dom} A} \geq f$. Biconjugating this inequality and then invoking Fact 6.1(i), we see that

$$q_A^{**} \geq f. \tag{6.4}$$

Now fix $x^* \in X^*$. We claim that

$$\big(\forall x \in X\big)\big(\exists\, y \in \operatorname{dom}(\partial f)\big) \quad \langle x, x^*\rangle - f(x) \leq \langle y, x^*\rangle - f(y). \tag{6.5}$$

Let $x \in X$. If $x \in \operatorname{dom}(\partial f)$, then (6.5) holds with $y = x$. So assume that $x \notin \operatorname{dom}(\partial f)$. Then $(x, x^*) \notin \operatorname{gra}(\partial f)$. Rockafellar's classical result on the maximality of ∂f yields $(y, y^*) \in \operatorname{gra}(\partial f)$ such that $\langle x - y, x^* - y^*\rangle < 0$, i.e., $\langle x - y, x^*\rangle < \langle x - y, y^*\rangle$. Since $y^* \in \partial f(y)$, we also have $f(y) + \langle x - y, y^*\rangle \leq f(x)$. Altogether, $f(y) + \langle x - y, x^*\rangle < f(x)$. This verifies (6.5).

Since $\operatorname{dom}(\partial f) = \operatorname{dom} A$ (by Fact 6.1(ii)), we deduce from (6.5) and (i) that $f^*(x^*) = \sup_{x \in X}(\langle x, x^*\rangle - f(x)) \leq \sup_{y \in \operatorname{dom} A}(\langle y, x^*\rangle - f(y)) = (f + \iota_{\operatorname{dom} A})^*(x^*) = q_A^*(x^*)$. Thus $f^* \leq q_A^*$ and hence, using Fact 6.1(i) one more time,

$$f \geq q_A^{**}. \tag{6.6}$$

Therefore, (iii) follows by combining (6.4) and (6.6).

(iv): If A is one-to-one, then, for every $x \in X$,

$$\begin{aligned} f(x) &= \sup_{y^* \in \operatorname{dom} A^{-1}} \big(\langle x, y^*\rangle - \tfrac{1}{2}\langle A^{-1}y^*, y^*\rangle\big) \\ &= \sup_{y^* \in \operatorname{dom} q_{A^{-1}}} \big(\langle x, y^*\rangle - q_{A^{-1}}(y^*)\big) \\ &= q^*_{A^{-1}}(x). \end{aligned} \tag{6.7}$$

This completes the proof. ■

Proposition 6.3 *We have:* $\operatorname{dom} A = \operatorname{dom} f \Leftrightarrow$ *every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\operatorname{dom} A$ *such that* $(x_n)_{n\in\mathbb{N}}$ *and* $(\langle x_n, Ax_n\rangle)_{n\in\mathbb{N}}$ *are convergent must satisfy* $\lim x_n \in \operatorname{dom} A$.

Proof. "⇒": Assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\operatorname{dom} A$ such that $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ and $(\langle x_n, Ax_n\rangle)_{n\in\mathbb{N}}$ is also convergent. Using Fact 6.1(i) and Corollary 6.2(i), we have $x \in \operatorname{dom} f$ and thus $x \in \operatorname{dom} A$.

"⇐": In view of Fact 6.1(iii), it suffices to show that $\operatorname{dom} f \subseteq \operatorname{dom} A$. To this end, let $x \in \operatorname{dom} f$. By Corollary 6.2(iii), [49, Theorem 2.3.1(iv) and Theorem 2.3.4(i)], there exists a sequence $(x_n, \rho_n)_{n\in\mathbb{N}}$ in $X \times \mathbb{R}$ such that

$x_n \to x$, $\rho_n \to f(x)$, and $(\forall n \in \mathbb{N})\ q_A(x_n) \leq \rho_n$. Using Corollary 6.2(i) and Fact 6.1(i), we see that $f(x) = \lim \rho_n = \overline{\lim}\, \rho_n \geq \overline{\lim}\, q_A(x_n) \geq \underline{\lim}\, q_A(x_n) = \underline{\lim}\, f(x_n) \geq f(x)$. Hence $x_n \to x$ and $\frac{1}{2}\langle x_n, Ax_n\rangle = q_A(x_n) \to f(x)$. By assumption, $x \in \operatorname{dom} A$, as required. ■

Theorem 6.4 *Let $B\colon X^* \to X$ be continuous, linear, symmetric, monotone, and one-to-one. Suppose that $A = B^{-1}$. Then*

$$\mathcal{A}_A = q_A \oplus q_B = (q_B^* + \iota_{\operatorname{dom} A}) \oplus q_B. \tag{6.8}$$

and

$$\mathcal{B}_A = \mathcal{A}_A^{**} = q_B^* \oplus q_B \tag{6.9}$$

are both representers for A. Furthermore, $\mathcal{A}_A = \mathcal{B}_A \Leftrightarrow \operatorname{dom} q_B^* = \operatorname{dom} A$.

Proof. Since $q_A = q_B^* + \iota_{\operatorname{dom} A}$ by Corollary 6.2(i)&(iv), it suffices to verify the left equality in (6.8). Let $(x, x^*) \in X \times X^*$. Using (1.13), Fact 1.2(ii), and Proposition 2.4, we see that

$$\begin{aligned}\mathcal{A}_A(x,x^*) &= \inf_{y^*} \big(\tfrac{1}{2}F_A(x, x^*+y^*) + \tfrac{1}{2}F_A^*(x^*-y^*, x)\big)\\ &= \inf_{y^*} \big(\tfrac{1}{2}F_B(x^*+y^*, x) + \tfrac{1}{2}F_B^*(x, x^*-y^*)\big)\\ &= \inf_{y^*} \big(\tfrac{1}{2}F_B(x^*+y^*, x) + \tfrac{1}{2}\big(\iota_{\operatorname{gra} B}(x^*-y^*, x)\\ &\qquad + \langle x^*-y^*, B(x^*-y^*)\rangle\big)\big).\end{aligned} \tag{6.10}$$

If $x \notin \operatorname{ran} B = \operatorname{dom} A$, then (6.10) shows that $\mathcal{A}_A(x,x^*) = +\infty$, as required. So assume that $x \in \operatorname{ran} B = \operatorname{dom} A$. In view of (6.10), (2.13), and (2.10), we deduce that

$$\begin{aligned}\mathcal{A}_A(x,x^*) &= \tfrac{1}{2}F_B(2x^* - Ax, x) + \tfrac{1}{2}\langle x, Ax\rangle\\ &= q_B^*\big(\tfrac{1}{2}x + \tfrac{1}{2}B(2x^* - Ax)\big) + q_A(x)\\ &= q_B^*(Bx^*) + q_A(x)\\ &= q_B(x^*) + q_A(x).\end{aligned} \tag{6.11}$$

Hence (6.8) holds. Using (6.8), Corollary 6.2(iii)&(iv), (2.21), and Theorem 3.1, we see that $\mathcal{A}_A^{**} = (q_A \oplus q_B)^{**} = q_A^{**} \oplus q_B^{**} = q_B^* \oplus q_B = (q_B \oplus q_B^*)^* = \mathcal{B}_B^* = \mathcal{B}_B^{\intercal} = \mathcal{B}_{B^{-1}} = \mathcal{B}_A$, so that (6.9) holds. Furthermore, $\mathcal{A}_A = \mathcal{B}_A \Leftrightarrow q_A \oplus q_B = q_A^{**} \oplus q_B \Leftrightarrow q_A = q_A^{**} \Leftrightarrow \operatorname{dom} q_A^{**} = \operatorname{dom} A \Leftrightarrow \operatorname{dom} q_B^* = \operatorname{dom} A$ by Corollary 6.2. ■

Example 6.5 Suppose that X is the Hilbert space $\ell_2(\mathbb{N})$ of square-summable sequences; thus, $X^* = X$. Set

$$B\colon X \to X\colon (\xi_k)_{k\in\mathbb{N}} \mapsto (\tfrac{1}{k}\xi_k)_{k\in\mathbb{N}} \tag{6.12}$$

and suppose that $A = B^{-1}$. Then $\operatorname{ran} B = \operatorname{dom} A$ is dense in X, but it is not closed (since, e.g., $(\frac{1}{k})_{k\in\mathbb{N}} \in X \smallsetminus (\operatorname{ran} B)$). Now set

$$x = \big(\tfrac{1}{k^{4/3}}\big)_{k\in\mathbb{N}} \quad \text{and} \quad (\forall n \in \mathbb{N})\ x_n = \big(\tfrac{1}{1^{4/3}}, \tfrac{1}{2^{4/3}}, \ldots, \tfrac{1}{n^{4/3}}, 0, 0, \ldots\big). \tag{6.13}$$

On the one hand, $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{dom} A$ and $x_n \to x \in X \smallsetminus (\operatorname{dom} A)$. On the other hand, $\langle x_n, Ax_n\rangle = \sum_{k=1}^{n} \frac{1}{k^{4/3}} \frac{k}{k^{4/3}} = \sum_{k=1}^{n} \frac{1}{k^{5/3}} \to \zeta(5/3) \in \mathbb{R}$. Altogether, Proposition 6.3 implies that $\operatorname{dom} A \subsetneq \operatorname{dom} q_B^*$. Therefore, by Theorem 6.4, $\mathcal{A}_A$ is neither lower semicontinuous nor equal to $\mathcal{B}_A$. While $\mathcal{A}_A$ is still a representer for A, it cannot be autoconjugate.

Remark 6.6 Several comments are in order.

(i) Without the constraint qualification, Fact 1.6 fails (see Example 6.5, where $\operatorname{dom} A$ is a subspace that is not closed).
(ii) It is conceivable that $\mathcal{A}_A^{**}$ is always an autoconjugate representer for A — this would sharpen Fact 1.6 and it would be consistent with Theorem 6.4.
(iii) Suppose that B is as in Theorem 6.4, that $A = B^{-1}$, and that $\operatorname{dom} A = \operatorname{ran} B$ is a dense subspace of X with $\operatorname{dom} A \neq X$.
We do not know whether $(\operatorname{dom} f) \smallsetminus (\operatorname{dom} A) \neq \varnothing$ must hold (as it does in Example 6.5), i.e. (see Proposition 6.3), whether there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ such that $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in X \smallsetminus (\operatorname{dom} A)$, yet $(\langle x_n, Ax_n\rangle)_{n\in\mathbb{N}}$ converges to a real number.
In contrast, there does exist a point $x \in X \smallsetminus (\operatorname{dom} A)$ such that every sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ converging to x must have $\langle x_n, Ax_n\rangle \to +\infty$. (Indeed, since $\operatorname{dom} A \neq X$, it follows from Fact 6.1(iv) that $\operatorname{dom} f \neq X$. Take $x \in X \smallsetminus (\operatorname{dom} f)$ and assume that $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{dom} A$ and converges to x. Then $+\infty = f(x) \leq \varliminf f(x_n) = \varliminf \frac{1}{2}\langle x_n, Ax_n\rangle$ by Fact 6.1(i)&(iii)). Thus for every sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* such that $Bx_n^* \to x \notin \operatorname{ran} B$, it follows that $\|Bx_n^*\| \cdot \|x_n^*\|_* \geq \langle Bx_n^*, x_n^*\rangle = \langle Bx_n^*, A(Bx_n^*)\rangle \to +\infty$. Since $0 \in \operatorname{dom} f$ and so $x \neq 0$, we deduce $\|x_n^*\|_* \to +\infty$, which is a well known result from Functional Analysis (see [28, Corollary 17.G]).

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