AN ANSWER TO S. SIMONS' QUESTION ON THE MAXIMAL MONOTONICITY OF THE SUM OF A MAXIMAL MONOTONE LINEAR OPERATOR AND A NORMAL CONE OPERATOR

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Abstract

The question whether or not the sum of two maximal monotone operators is maximal monotone under Rockafellar's constraint qualification — that is, whether or not "the sum theorem" is true — is the most famous open problem in Monotone Operator Theory. In his 2008 monograph *"From Hahn-Banach to Monotonicity"*, Stephen Simons asked whether or not the sum theorem holds for the special case of a maximal monotone linear operator and a normal cone operator of a closed convex set provided that the interior of the set makes a nonempty intersection with the domain of the linear operator.

In this note, we provide an affirmative answer to Simons' question. In fact, we show that the sum theorem is true for a maximal monotone *linear relation* and a normal cone operator. The proof relies on Rockafellar's formula for the Fenchel conjugate of the sum as well as some results featuring the Fitzpatrick function.

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1 Introduction

Throughout this paper, we assume that X is a Banach space with norm $\|\cdot\|$, that X^* is its continuous dual space with norm $\|\cdot\|_*$, and that $\langle\cdot,\cdot\rangle$ denotes the usual pairing between these spaces. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \{(x,x^*) \in X \times X^* \mid x^* \in Ax\}$ be the *graph* of A. Then A is said to be *monotone* if

$$(1) \qquad \big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^*\rangle \geq 0,$$

and *maximal monotone* if no proper enlargement (in the sense of graph inclusion) of A is monotone. Monotone operators have proven to be a key class of objects in modern Optimization and Analysis; see, e.g., the books [6, 10, 15, 16, 14, 19] and the references therein. (We also adopt standard notation used in these books: $\operatorname{dom} A = \{x \in X \mid Ax \neq \varnothing\}$ is the *domain* of A. Given a subset C of X, $\operatorname{int} C$ is the *interior*, $\overline{C}$ is the *closure*, $\operatorname{bdry} C$ is the *boundary*, and $\operatorname{span} C$ is the *span* (the set of all finite linear combinations) of C. The *indicator function* ι_C of C takes the value 0 on C, and $+\infty$ on $X \smallsetminus C$. Given $f\colon X \to \left]-\infty,+\infty\right]$, $\operatorname{dom} f = f^{-1}(\mathbb{R})$ and $f^*\colon X^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}(\langle x,x^*\rangle - f(x))$ is the *Fenchel conjugate* of f. Furthermore, B_X is the *closed unit ball* $\{x \in X \mid \|x\| \leq 1\}$ of X, and $\mathbb{N} = \{0,1,2,3,\ldots\}$.)

Now assume that A is maximal monotone, and let $B\colon X \rightrightarrows X^*$ be maximal monotone as well. While the *sum operator* $A+B\colon X \rightrightarrows X^*\colon x \mapsto Ax + Bx = \{a^*+b^* \mid a^* \in Ax \text{ and } b^* \in Bx\}$ is clearly monotone, it may fail to be maximal monotone. When X is reflexive, the classical *constraint qualification* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \varnothing$ guarantees maximal monotonicity of $A+B$, this is a famous result due to Rockafellar [13, Theorem 1]. Various extensions of this *sum theorem* have been found, but the general version in nonreflexive Banach spaces remains elusive — this has led to the famous *sum problem*; see Simons' recent monograph [16] for the state-of-the-art.

The notorious difficulty of the sum problem makes it tempting to consider various special cases. In this paper, we shall focus on the case when A is a *linear relation* and B is the *normal cone operator* N_C of some nonempty closed convex subset C of X. (Recall that A is a linear relation if $\operatorname{gra} A$ is a linear subspace of $X \times X^*$, and that for every $x \in X$, the normal cone operator at x is defined by $N_C(x) = \{x^* \in X^* \mid \sup\langle C-x, x^*\rangle \leq 0\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$. Consult [7] for further information on linear relations.) If $A\colon X \rightrightarrows X^*$ is *at most single-valued* (i.e., for every $x \in X$, either $Ax = \varnothing$ or Ax is a singleton), then we follow the common slight abuse of notation of identifying A with a classical operator $\operatorname{dom} A \to X^*$. We thus include the classical case when $A\colon X \to X^*$ is a continuous linear monotone (thus *positive*) operator. Continuous and discontinuous linear operators — and lately even linear relations — have received some attention in Monotone Operator Theory [1, 2, 4, 5, 11, 17, 18] because they provide additional classes of examples apart from the well known and well understood *subdifferential operators* in the sense of Convex Analysis.

On page 199 in his monograph [16] from 2008, Stephen Simons asked the question whether or not $A + N_C$ is maximal monotone when $A\colon \operatorname{dom} A \to X^*$ is linear and maximal monotone and Rockafellar's constraint qualification $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$ holds. In this manuscript, we provide an affirmative answer to Simons' question. In fact, maximality of $A+N_C$ is guaranteed even when A is a maximal monotone linear relation, i.e., A is simultaneously a maximal monotone operator and a linear relation.

The paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. The main result (Theorem 3.1) is proved in Section 3.

We conclude this section with some topological comments. If $x \in X$ and $x^* \in X^*$, then $\langle x, x^*\rangle$ is the evaluation of the functional x^* at the point x. We identify X with its canonical image in the bidual space X^{**}. Furthermore, $X \times X^*$ and $(X \times X^*)^* = X^* \times X^{**}$ are likewise paired via $\langle (x,x^*),(y^*,y^{**})\rangle = \langle x,y^*\rangle + \langle x^*,y^{**}\rangle$, where $(x,x^*) \in X \times X^*$ and $(y^*,y^{**}) \in X^* \times X^{**}$.

2 Auxiliary Results

Fact 2.1 (Rockafellar) (See [12, Theorem 3(a)], [16, Corollary 10.3], or [19, Theorem 2.8.7(iii)].) *Let f and g be proper convex functions from X to $]-\infty,+\infty]$. Assume that there exists a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0. Then for every $z^* \in X^*$, there exists $y^* \in X^*$ such that*

$$(f+g)^*(z^*) = f^*(y^*) + g^*(z^*-y^*). \tag{2}$$

Fact 2.2 (Fitzpatrick) (See [8, Corollary 3.9].) *Let $A\colon X \rightrightarrows X^*$ be maximal monotone, and set*

$$F_A\colon X \times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\big), \tag{3}$$

which is the Fitzpatrick function *associated with A. Then for every $(x,x^*) \in X \times X^*$, the inequality $\langle x,x^*\rangle \le F_A(x,x^*)$ is true, and equality holds if and only if $(x,x^*) \in \operatorname{gra} A$.*

Fact 2.3 (Simons) (See [16, Corollary 28.2].) *Let $A\colon X \rightrightarrows X^*$ be maximal monotone. Then*

$$\overline{\operatorname{span}(P_X \operatorname{dom} F_A)} = \overline{\operatorname{span} \operatorname{dom} A}, \tag{4}$$

where $P_X : X \times X^* \to X : (x,x^*) \mapsto x$.

Fact 2.4 (Simons) (See [16, Lemma 19.7 and Section 22].) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that $\operatorname{gra} A \neq \varnothing$. Then the function*

$$g\colon X \times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \langle x,x^*\rangle + \iota_{\operatorname{gra} A}(x,x^*) \tag{5}$$

is proper and convex.

Proof. We thank the referee for suggesting this simple proof, which we reproduce here in our current setting for the reader's convenience. It is clear that g is proper because $\operatorname{gra} A \neq \varnothing$. To see that g is convex, let (a,a^*) and (b,b^*) be in $\operatorname{gra} A$, and let $\lambda \in \left]0,1\right[$. Set $\mu = 1-\lambda \in \left]0,1\right[$ and observe that $\lambda(a,a^*)+\mu(b,b^*)=(\lambda a+\mu b,\lambda a^*+\mu b^*)\in \operatorname{gra} A$ by convexity of $\operatorname{gra} A$. Since A is monotone, it follows that

$$\begin{aligned}(6)\qquad \lambda g(a,a^*)+\mu g(b,b^*)-g\big(\lambda(a,a^*)+\mu(b,b^*)\big) &= \lambda\langle a,a^*\rangle+\mu\langle b,b^*\rangle-\langle \lambda a+\mu b,\lambda a^*+\mu b^*\rangle\\ &=\lambda\mu\langle a-b,a^*-b^*\rangle\\ &\geq 0.\end{aligned}$$

Therefore, g is convex. ■

Lemma 2.5 *Let C be a nonempty closed convex subset of X such that $\operatorname{int} C\neq\varnothing$. Let $c_0\in\operatorname{int} C$ and suppose that $z\in X\smallsetminus C$. Then there exists $\lambda\in\,]0,1[$ such that $\lambda c_0+(1-\lambda)z\in\operatorname{bdry} C$.*

Proof. Let $\lambda=\inf\big\{t\in[0,1]\mid tc_0+(1-t)z\in C\big\}$. Since C is closed,

$$(7)\qquad \lambda=\min\big\{t\in[0,1]\mid tc_0+(1-t)z\in C\big\}.$$

Because $z\notin C$, $\lambda>0$. We now show that $\lambda c_0+(1-\lambda)z\in\operatorname{bdry} C$. Assume to the contrary that $\lambda c_0+(1-\lambda)z\in\operatorname{int} C$. Then there exists $\delta\in\,]0,\lambda[$ such that $\lambda c_0+(1-\lambda)z-\delta(c_0-z)\in C$. Hence $(\lambda-\delta)c_0+(1-\lambda+\delta)z\in C$, which contradicts (7). Therefore, $\lambda c_0+(1-\lambda)z\in\operatorname{bdry} C$. Since $c_0\notin\operatorname{bdry} C$, we also have $\lambda<1$. ■

The following useful result is a variant of [3, Theorem 2.14].

Lemma 2.6 *Let $A\colon X\rightrightarrows X^*$ be a set-valued operator, let C be a nonempty closed convex subset of X, and let $(z,z^*)\in X\times X^*$. Set*

$$(8)\qquad I_C\colon X\rightrightarrows X^*\colon x\mapsto\begin{cases}\{0\}, & \text{if } x\in C;\\ \varnothing, & \text{otherwise.}\end{cases}$$

Then (z,z^*) is monotonically related to $\operatorname{gra}(A+N_C)$ if and only if

$$(9)\qquad (z,z^*)\text{ is monotonically related to }\operatorname{gra}(A+I_C)\quad\text{and}\quad z\in\bigcap_{a\in(\operatorname{dom} A)\cap C}\big(a+T_C(a)\big),$$

where $(\forall a\in C)$ $T_C(a)=\big\{x\in X\mid \sup\langle x,N_C(a)\rangle\leq 0\big\}$.

Proof. "$\Rightarrow$": Since $\operatorname{gra} I_C\subseteq\operatorname{gra} N_C$, it follows that $\operatorname{gra}(A+I_C)\subseteq\operatorname{gra}(A+N_C)$; consequently, (z,z^*) is monotonically related to $\operatorname{gra}(A+I_C)$. Now assume that $a\in\operatorname{dom} A\cap C$, and let $a^*\in Aa$. Then $(a,a^*+N_C(a))\subseteq\operatorname{gra}(A+N_C)$ and hence $\inf\langle a-z,a^*+N_C(a)-z^*\rangle\geq 0$. This implies $+\infty>\langle a-z,a^*-z^*\rangle\geq\sup\langle z-a,N_C(a)\rangle$. Since $N_C(a)$ is a cone, it follows that $\sup\langle z-a,N_C(a)\rangle\leq 0$ and hence $z\in a+T_C(a)$. "$\Leftarrow$": Assume that $a\in\operatorname{dom} A\cap C$. Then $Aa=(A+I_C)a$, which yields $\sup\langle z-a,Aa-z^*\rangle\leq 0$, and also $z-a\in T_C(a)$, i.e., $\sup\langle z-a,N_C(a)\rangle\leq 0$. Adding the last two inequalities, we obtain $\sup\langle z-a,Aa+N_C(a)-z^*\rangle\leq 0$, i.e., $\inf\langle a-z,(A+N_C)(a)-z^*\rangle\geq 0$, as required. ■

3 Main Result

Theorem 3.1 *Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation, let C be a nonempty closed convex subset of X, and suppose that $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. Then $A + N_C$ is maximal monotone.*

Proof. Let $(z, z^*) \in X \times X^*$ and suppose that

(10) $$(z, z^*) \text{ is monotonically related to } \operatorname{gra}(A + N_C).$$

It suffices to show that

(11) $$(z, z^*) \in \operatorname{gra}(A + N_C).$$

Set

(12) $$g\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^* \rangle + \iota_{\operatorname{gra} A}(x, x^*).$$

By Fact 2.4, g is convex. Hence,

(13) $$h = g + \iota_{C \times X^*}$$

is convex as well. Let

(14) $$c_0 \in \operatorname{dom} A \cap \operatorname{int} C,$$

and let $c_0^* \in Ac_0$. Then $(c_0, c_0^*) \in \operatorname{gra} A \cap (\operatorname{int} C \times X^*) = \operatorname{dom} g \cap \operatorname{int} \operatorname{dom} \iota_{C \times X^*}$, and $\iota_{C \times X^*}$ is continuous at (c_0, c_0^*). By Fact 2.1 (applied to g and $\iota_{C \times X^*}$), there exists $(y^*, y^{**}) \in X^* \times X^{**}$ such that

(15) $$\begin{aligned} h^*(z^*, z) &= g^*(y^*, y^{**}) + \iota^*_{C \times X^*}(z^* - y^*, z - y^{**}) \\ &= g^*(y^*, y^{**}) + \iota^*_C(z^* - y^*) + \iota_{\{0\}}(z - y^{**}). \end{aligned}$$

Let $(x, x^*) \in \operatorname{dom} h = \operatorname{gra} A \cap (C \times X^*)$. Then $x^* + 0 \in (A + N_C)x$ and hence $(x, x^*) \in \operatorname{gra}(A + N_C)$. In view of (10), $\langle x - z, x^* - z^* \rangle \geq 0$, from which $\langle (x, x^*), (z^*, z) \rangle - h(x, x^*) = \langle x, z^* \rangle + \langle z, x^* \rangle - \langle x, x^* \rangle \leq \langle z, z^* \rangle$. Consequently,

(16) $$h^*(z^*, z) \leq \langle z, z^* \rangle.$$

Combining (15) with (16), we obtain

(17) $$g^*(y^*, y^{**}) + \iota^*_C(z^* - y^*) + \iota_{\{0\}}(z - y^{**}) \leq \langle z, z^* \rangle.$$

Therefore, $y^{**} = z$ and $g^*(y^*, z) + \iota^*_C(z^* - y^*) \leq \langle z, z^* \rangle$. Since $g^*(y^*, z) = F_A(z, y^*)$, we deduce that $F_A(z, y^*) + \iota^*_C(z^* - y^*) \leq \langle z, z^* \rangle$; equivalently,

(18) $$(\forall c \in C) \quad F_A(z, y^*) - \langle z, y^* \rangle + \langle c - z, z^* - y^* \rangle \leq 0.$$

We now claim that

$$z \in C. \tag{19}$$

Assume to the contrary that (19) fails, i.e., that $z \notin C$. By (18), $(z, y^*) \in \operatorname{dom} F_A$. Using Fact 2.3 and the fact that $\operatorname{dom} A$ is a linear subspace of X, we see that $z \in P_X(\operatorname{dom} F_A) \subseteq \overline{\operatorname{span} P_X(\operatorname{dom} F_A)} = \overline{\operatorname{span} \operatorname{dom} A} = \overline{\operatorname{dom} A}$. Hence there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in $(\operatorname{dom} A) \smallsetminus C$ such that $z_n \to z$. By Lemma 2.5, $(\forall n \in \mathbb{N})$ $(\exists \lambda_n \in \,]0,1[\,)$ $\lambda_n z_n + (1-\lambda_n)c_0 \in \operatorname{bdry} C$. Thus,

$$(\forall n \in \mathbb{N}) \quad \lambda_n z_n + (1-\lambda_n)c_0 \in \operatorname{dom} A \cap \operatorname{bdry} C. \tag{20}$$

After passing to a subsequence and relabeling if necessary, we assume that $\lambda_n \to \lambda \in [0,1]$. Taking the limit in (20), we deduce that $\lambda z + (1-\lambda)c_0 \in \operatorname{bdry} C$. Since $c_0 \in \operatorname{int} C$ and $z \in X \smallsetminus C$, we have $0 < \lambda$ and $\lambda < 1$. Hence

$$\lambda_n \to \lambda \in \,]0,1[\,. \tag{21}$$

Since $\operatorname{int} C \neq \varnothing$, Mazur's Separation Theorem (see, e.g., [9, Theorem 2.2.19]) yields a sequence $(c_n^*)_{n\in\mathbb{N}}$ in X^* such that

$$(\forall n \in \mathbb{N}) \quad c_n^* \in N_C\big(\lambda_n z_n + (1-\lambda_n)c_0\big) \text{ and } \|c_n^*\|_* = 1. \tag{22}$$

Since $c_0 \in \operatorname{int} C$, there exists $\delta > 0$ such that $c_0 + \delta B_X \subseteq C$. It follows that

$$(\forall n \in \mathbb{N}) \quad \delta \leq \lambda_n \langle z_n - c_0, c_n^* \rangle. \tag{23}$$

Since the sequence $(c_n^*)_{n\in\mathbb{N}}$ is bounded, we pass to a weak* convergent *subnet* $(c_\gamma^*)_{\gamma\in\Gamma}$, say $c_\gamma^* \overset{\mathrm{w}^*}{\rightharpoonup} c^* \in X^*$. Passing to the limit in (23) along subnets, we see that $\delta \leq \lambda\langle z - c_0, c^*\rangle$; hence, using (21),

$$0 < \langle z - c_0, c^* \rangle. \tag{24}$$

On the other hand and borrowing the notation of Lemma 2.6, we deduce from (20), (10), and Lemma 2.6 that $(\forall n \in \mathbb{N})$ $z \in (\operatorname{Id} + T_C)(\lambda_n z_n + (1-\lambda_n)c_0)$, which in view of (22) yields

$$(\forall n \in \mathbb{N}) \quad \langle z - (\lambda_n z_n + (1-\lambda_n)c_0), c_n^* \rangle \leq 0. \tag{25}$$

Taking limits in (25) along subnets, we deduce $\langle z - (\lambda z + (1-\lambda)c_0), c^* \rangle \leq 0$. Dividing by $1-\lambda$ and recalling (21), we thus have

$$\langle z - c_0, c^* \rangle \leq 0. \tag{26}$$

Considered together, the inequalities (24) and (26) are absurd — we have thus verified (19).

Substituting (19) into (18), we deduce that

$$F_A(z, y^*) \leq \langle z, y^* \rangle. \tag{27}$$

By Fact 2.2,

$$(z, y^*) \in \operatorname{gra} A \tag{28}$$

and $F_A(z, y^*) = \langle z, y^* \rangle$. Thus, using (18) again, we see that $\sup_{c \in C} \langle c - z, z^* - y^* \rangle \leq 0$, i.e., that

$$(z, z^* - y^*) \in \operatorname{gra} N_C. \tag{29}$$

Adding (28) and (29), we obtain (11), and this completes the proof. ■

Corollary 3.2 *Let $A : X \rightrightarrows X^*$ be maximal monotone and at most single-valued, and let C be a nonempty closed convex subset of X. Suppose that $A|_{\operatorname{dom} A}$ is linear, and that $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. Then $A + N_C$ is maximal monotone.*

Remark 3.3 Corollary 3.2 provides an affirmative answer to a question Stephen Simons raised in his 2008 monograph [16, page 199] concerning [15, Theorem 41.6].

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