A NOTE ON THE PAPER BY ECKSTEIN AND SVAITER ON "GENERAL PROJECTIVE SPLITTING METHODS FOR SUMS OF MAXIMAL MONOTONE OPERATORS"

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May 27, 2009

Abstract

In their recent *SIAM J. Control Optim.* paper from 2009, J. Eckstein and B.F. Svaiter proposed a very general and flexible splitting framework for finding a zero of the sum of finitely many maximal monotone operators. In this short note, we provide a technical result that allows for the removal of Eckstein and Svaiter's assumption that the sum of the operators be maximal monotone or that the underlying Hilbert space be finite-dimensional.

2000 Mathematics Subject Classification:
Primary 47H05, 47H09; Secondary 47J25, 49M27, 52A41, 65J15, 90C25.

Keywords: Firmly nonexpansive mapping, maximal monotone operator, nonexpansive mapping, proximal algorithm, splitting algorithm.

Throughout, we assume that $\mathcal{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We shall assume basic notation and results from Fixed Point Theory and from Monotone Operator Theory; see, e.g., [1, 8, 9, 11, 12, 13, 14]. The *graph* of a maximal monotone operator $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is denoted by gra A, and its *resolvent* $(A+\mathrm{Id})^{-1}$ by J_A. Weak convergence is indicated by $\rightharpoonup$.

Lemma 1 *Let C be a closed linear subspace of $\mathcal{H}$ and let $F\colon \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive. Then $P_C F + (\mathrm{Id} - P_C)(\mathrm{Id} - F)$ is firmly nonexpansive.*

Proof. Since P_C and F are firmly nonexpansive, we have that $2P_C - \mathrm{Id}$ and $2F - \mathrm{Id}$ are both nonexpansive. Set $T = P_C F + (\mathrm{Id} - P_C)(\mathrm{Id} - F)$. Then $2T - \mathrm{Id} = (2P_C - \mathrm{Id})(2F - \mathrm{Id})$ is nonexpansive, and hence T is firmly nonexpansive. ■

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Theorem 2 *Let* $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ *be maximal monotone, and let* C *be a closed linear subspace of* $\mathcal{H}$*. Let* $(x_n, u_n)_{n\in\mathbb{N}}$ *be a sequence in* $\operatorname{gra} A$ *such that* $(x_n, u_n) \rightharpoonup (x, u) \in \mathcal{H} \times \mathcal{H}$*. Suppose that* $x_n - P_C x_n \to 0$ *and that* $P_C u_n \to 0$*, where* P_C *denotes the projector onto* C*. Then* $(x, u) \in (\operatorname{gra} A) \cap (C \times C^{\perp})$ *and* $\langle x_n, u_n \rangle \to \langle x, u \rangle = 0$*.*

Proof. Since P_C is a bounded linear operator, it is weakly continuous ([3, Theorem VI.1.1]). Thus $x \leftharpoonup x_n = (x_n - P_C x_n) + P_C x_n \rightharpoonup 0 + P_C x$ and hence $x = P_C x \in C$. Similarly, $0 \leftarrow P_C u_n \rightharpoonup P_C u$; hence $P_C u = 0$ and so $u \in C^{\perp}$. Altogether,

$$(x, u) \in C \times C^{\perp}. \tag{1}$$

Since $\operatorname{Id} - J_A$ is firmly nonexpansive, we see from Lemma 1 that

$$T = P_C(\operatorname{Id} - J_A) + (\operatorname{Id} - P_C) J_A = P_C + (\operatorname{Id} - 2P_C) J_A \tag{2}$$

is also firmly nonexpansive. Now $(\forall n \in \mathbb{N})$ $u_n \in A x_n$, i.e.,

$$(\forall n \in \mathbb{N}) \quad x_n = J_A(x_n + u_n). \tag{3}$$

Furthermore,

$$x_n + u_n \rightharpoonup x + u, \tag{4}$$

and (2) and (3) imply that $T(x_n + u_n) = P_C(x_n + u_n) + (\operatorname{Id} - 2P_C) J_A(x_n + u_n) = P_C x_n + P_C u_n + (\operatorname{Id} - 2P_C) x_n = x_n - P_C x_n + P_C u_n \to 0$, i.e., that

$$T(x_n + u_n) \to 0. \tag{5}$$

Since $\operatorname{Id} - T$ is (firmly) nonexpansive, the demiclosedness principle (see [8, 9]), applied to the sequence $(x_n + u_n)_{n\in\mathbb{N}}$ and the operator $\operatorname{Id} - T$, and (4) and (5) imply that $(\operatorname{Id} - (\operatorname{Id} - T))(x + u) = 0$, i.e., that $T(x + u) = 0$. Using (2), this means that

$$J_A(x + u) = 2P_C J_A(x + u) - P_C(x + u) \in C. \tag{6}$$

Applying P_C to both sides of (6), we deduce that $J_A(x + u) = P_C J_A(x + u)$; consequently, (6) simplifies to

$$J_A(x + u) = P_C x + P_C u. \tag{7}$$

However, (1) yields $P_C x = x$ and $P_C u = 0$, hence (7) becomes $J_A(x + u) = x$; equivalently, $u \in Ax$ or

$$(x, u) \in \operatorname{gra} A. \tag{8}$$

Combining (1) and (8), we see that $(x, u) \in (\operatorname{gra} A) \cap (C \times C^{\perp})$, as claimed. Finally, $\langle x_n, u_n \rangle = \langle P_C x_n, P_C u_n \rangle + \langle P_{C^{\perp}} x_n, P_{C^{\perp}} u_n \rangle \to \langle P_C x, 0 \rangle + \langle 0, P_{C^{\perp}} u \rangle = 0 = \langle P_C x, P_{C^{\perp}} u \rangle = \langle x, u \rangle$. ■

Corollary 3 *Let $A_1, \ldots, A_m$ be maximal monotone operators $\mathcal{H}$, and let $z_1, \ldots, z_m$ and $w_1, \ldots, w_m$ be vectors in $\mathcal{H}$. Suppose that for each i, $(x_{i,n}, y_{i,n})_{n\in\mathbb{N}}$ is a sequence in* gra A_i *such that for all i and j,*

$$(x_{i,n}, y_{i,n}) \rightharpoonup (z_i, w_i) \tag{9}$$

$$\sum_{i=1}^{m} y_{i,n} \to 0 \tag{10}$$

$$x_{i,n} - x_{j,n} \to 0. \tag{11}$$

Then $z_1 = \cdots = z_n$, $w_1 + \cdots + w_n = 0$, and each $w_i \in A_i z_i$.

Proof. We work in product Hilbert space $\boldsymbol{\mathcal{H}} = \mathcal{H}^m$, and we set

$$\mathbf{A} = A_1 \times \cdots \times A_m, \quad \text{and} \quad \mathbf{C} = \big\{(x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}} \mid x_1 = \cdots = x_m\big\}. \tag{12}$$

Note that $\mathbf{A}$ is maximal monotone on $\boldsymbol{\mathcal{H}}$, and that $\mathbf{C}$ is a closed linear subspace of $\boldsymbol{\mathcal{H}}$. Next, set $\mathbf{x} = (z_1, \ldots, z_m)$, $\mathbf{u} = (w_1, \ldots, w_m)$, and $(\forall n \in \mathbb{N})$ $\mathbf{x}_n = (x_{1,n}, \ldots, x_{m,n})$ and $\mathbf{u}_n = (y_{1,n}, \ldots, y_{m,n})$. By (9), $(\mathbf{x}_n, \mathbf{u_n})_{n\in\mathbb{N}}$ is a sequence in gra $\mathbf{A}$ such that $(\mathbf{x}_n, \mathbf{u}_n) \rightharpoonup (\mathbf{x}, \mathbf{u})$. Furthermore, (10) and (11) imply that $P_{\mathbf{C}}\mathbf{u}_n \to 0$ and that $\mathbf{x}_n - P_{\mathbf{C}}\mathbf{x}_n \to 0$, respectively. Therefore, by Theorem 2, $(\mathbf{x}, \mathbf{u}) \in (\text{gra}\,\mathbf{A}) \cap (\mathbf{C} \times \mathbf{C}^{\perp})$, which is precisely the announced conclusion. ■

Remark 4 Corollary 3 is a considerable strengthening of [7, Proposition A.1], where it was *additionally assumed* that $A_1 + \cdots + A_m$ is maximal monotone, and where part of the *conclusion* of Corollary 3, namely $z_1 = \cdots = z_m$, was an additional *assumption*.

Remark 5 Because of the removal of the assumption that $A_1 + \cdots + A_m$ be maximal monotone (see the previous remark), a second look at the proofs in Eckstein and Svaiter's paper [7] reveals that — in our present notation — the assumption that

"either $\mathcal{H}$ is finite-dimensional or $A_1 + \cdots + A_m$ is maximal monotone"

is superfluous in both [7, Proposition 3.2 and Proposition 4.2]. This is important in the infinite-dimensional case, where the maximality of the sum can typically be only guaranteed when a constraint qualification is satisfied; consequently, Corollary 3 helps to widen the scope of the powerful algorithmic framework of Eckstein and Svaiter.

Remark 6 The author is grateful for the following comments.

(i) Dr. Patrick Combettes brought to our attention the recent paper [2] on the use of product space techniques in monotone operator splitting problems.

(ii) Dr. Jonathan Eckstein observed that Lemma 1 remains true if P_C is replaced by an arbitrary linear firmly nonexpansive operator and that the operator $P_C F + (\mathrm{Id} - P_C)(\mathrm{Id} - F) = (2P_C - \mathrm{Id})(2F - \mathrm{Id})$ lies at the heart of splitting methods including the Douglas-Rachford splitting method [4, 5, 6, 10].

Acknowledgment

The author was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program.

References

[1] R.S. Burachik and A.N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer-Verlag, 2008.

[2] P.L. Combettes, "Iterative construction of the resolvent of a sum of maximal monotone operators", *Journal of Convex Analysis*, vol. 16, 2009.

[3] J.B. Conway, *A Course in Functional Analysis*, 2nd edition, Springer-Verlag, 1990.

[4] J. Eckstein, *Splitting Methods for Monotone Operators with Applications to Parallel Optimization*, Ph.D. Thesis, Department of Civil Engineering, Massachusetts Institute of Technology, 1989.

[5] J. Eckstein and D.P. Bertsekas, "On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators", *Mathematical Programming*, vol. 55, pp. 293–318, 1992.

[6] J. Eckstein and M.C. Ferris, "Operator splitting methods for monotone affine variational inequalities, with a parallel application to optimal control", *INFORMS Journal on Computing*, vol. 10, pp. 218–235, 1998.

[7] J. Eckstein and B.F. Svaiter, "General projective splitting methods for sums of maximal monotone operators", *SIAM Journal on Control and Optimization*, vol. 48, pp. 787–811, 2009.

[8] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[9] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.

[10] J. Lawrence and J.E. Spingarn, "On fixed points of non-expansive piecewise isometric mappings", *Proceedings of the London Mathematical Society*, vol. 55, pp. 605–624, 1987.

[11] R.T. Rockafellar and R.J-B Wets, *Variational Analysis*, 2nd printing, Springer-Verlag, 2004.

[12] S. Simons, *Minimax and Monotonicity*, Springer-Verlag, 1998.

[13] S. Simons, *From Hahn-Banach to Monotonicity*, Springer-Verlag, 2008.

[14] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing, 2002.