A note on the paper by Eckstein and Svaiter on “General projective splitting methods for sums of maximal monotone operators”

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Abstract

In their recent SIAM J. Control Optim. paper from 2009, J. Eckstein and B.F. Svaiter proposed a very general and flexible splitting framework for finding a zero of the sum of finitely many maximal monotone operators. In this short note, we provide a technical result that allows for the removal of Eckstein and Svaiter’s assumption that the sum of the operators be maximal monotone or that the underlying Hilbert space be finite-dimensional.

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Throughout, we assume that $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We shall assume basic notation and results from Fixed Point Theory and from Monotone Operator Theory; see, e.g., [1, 8, 9, 11, 12, 13, 14]. The graph of a maximal monotone operator $A: \mathcal{H} \rightrightarrows \mathcal{H}$ is denoted by $\text{gra} A$, and its resolvent $(A + \text{Id})^{-1}$ by $J_A$. Weak convergence is indicated by $\rightharpoonup$.

Lemma 1 Let $C$ be a closed linear subspace of $\mathcal{H}$ and let $F: \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive. Then $P_C F + (\text{Id} - P_C)(\text{Id} - F)$ is firmly nonexpansive.

Proof. Since $P_C$ and $F$ are firmly nonexpansive, we have that $2P_C - \text{Id}$ and $2F - \text{Id}$ are both nonexpansive. Set $T = P_C F + (\text{Id} - P_C)(\text{Id} - F)$. Then $2T - \text{Id} = (2P_C - \text{Id})(2F - \text{Id})$ is nonexpansive, and hence $T$ is firmly nonexpansive. ■

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Theorem 2 Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximal monotone, and let $C$ be a closed linear subspace of $\mathcal{H}$. Let $(x_n, u_n)_{n \in \mathbb{N}}$ be a sequence in $\text{gra } A$ such that $(x_n, u_n) \rightharpoonup (x, u) \in \mathcal{H} \times \mathcal{H}$. Suppose that $x_n - P_Cx_n \to 0$ and that $P_Cu_n \to 0$, where $P_C$ denotes the projector onto $C$. Then $(x, u) \in (\text{gra } A) \cap (C \times C^\perp)$ and $\langle x_n, u_n \rangle \to \langle x, u \rangle = 0$.

Proof. Since $P_C$ is a bounded linear operator, it is weakly continuous ([3, Theorem VI.1.1]). Thus $x \leftarrow x_n = (x_n - P_Cx_n) + P_Cx_n \to 0 + P_Cx$ and hence $x = P_Cx \in C$. Similarly, $0 \leftarrow P_Cu_n \rightharpoonup P_Cu$; hence $P_Cu = 0$ and so $u \in C^\perp$. Altogether, 

\begin{equation}
(1) \quad (x, u) \in C \times C^\perp.
\end{equation}

Since $\text{Id} - J_A$ is firmly nonexpansive, we see from Lemma 1 that 

\begin{equation}
(2) \quad T = P_C(\text{Id} - J_A) + (\text{Id} - P_C)J_A = P_C + (\text{Id} - 2P_C)J_A
\end{equation}

is also firmly nonexpansive. Now $(\forall n \in \mathbb{N})$ $u_n \in Ax_n$, i.e., 

\begin{equation}
(3) \quad (\forall n \in \mathbb{N}) \quad x_n = J_A(x_n + u_n).
\end{equation}

Furthermore, 

\begin{equation}
(4) \quad x_n + u_n \rightharpoonup x + u,
\end{equation}

and (2) and (3) imply that $T(x_n + u_n) = P_C(x_n + u_n) + (\text{Id} - 2P_C)J_A(x_n + u_n) = P_Cx_n + P_Cu_n + (\text{Id} - 2P_C)x_n = x_n - P_Cx_n + P_Cu_n \to 0$, i.e., that 

\begin{equation}
(5) \quad T(x_n + u_n) \to 0.
\end{equation}

Since $\text{Id} - T$ is (firmly) nonexpansive, the demiclosedness principle (see [8, 9]), applied to the sequence $(x_n + u_n)_{n \in \mathbb{N}}$ and the operator $\text{Id} - T$, and (4) and (5) imply that $(\text{Id} - (\text{Id} - T))(x + u) = 0$, i.e., that $T(x + u) = 0$. Using (2), this means that 

\begin{equation}
(6) \quad J_A(x + u) = 2P_CJ_A(x + u) - P_C(x + u) \in C.
\end{equation}

Applying $P_C$ to both sides of (6), we deduce that $J_A(x + u) = P_CJ_A(x + u)$; consequently, (6) simplifies to 

\begin{equation}
(7) \quad J_A(x + u) = P_Cx + P_Cu.
\end{equation}

However, (1) yields $P_Cx = x$ and $P_Cu = 0$, hence (7) becomes $J_A(x + u) = x$; equivalently, $u \in Ax$ or 

\begin{equation}
(8) \quad (x, u) \in \text{gra } A.
\end{equation}

Combining (1) and (8), we see that $(x, u) \in (\text{gra } A) \cap (C \times C^\perp)$, as claimed. Finally, $\langle x_n, u_n \rangle = \langle P_Cx_n, P_Cu_n \rangle + \langle P_C^\perp x_n, P_C^\perp u_n \rangle \to \langle P_Cx, 0 \rangle + \langle 0, P_C^\perp u \rangle = 0 = \langle P_Cx, P_C^\perp u \rangle = \langle x, u \rangle$. \qed
Corollary 3 Let \( A_1, \ldots, A_m \) be maximal monotone operators \( \mathcal{H} \), and let \( z_1, \ldots, z_m \) and \( w_1, \ldots, w_m \) be vectors in \( \mathcal{H} \). Suppose that for each \( i \), \((x_{i,n}, y_{i,n})_{n \in \mathbb{N}} \) is a sequence in \( \text{gra} \ A_i \) such that for all \( i \) and \( j \),

\[
(9) \quad (x_{i,n}, y_{i,n}) \rightharpoonup (z_i, w_i)
\]

\[
(10) \quad \sum_{i=1}^{m} y_{i,n} \to 0
\]

\[
(11) \quad x_{i,n} - x_{j,n} \to 0.
\]

Then \( z_1 = \cdots = z_m \), \( w_1 + \cdots + w_m = 0 \), and each \( w_i \in A_i z_i \).

Proof. We work in product Hilbert space \( \mathcal{H} = \mathcal{H}^m \), and we set

\[
(12) \quad A = A_1 \times \cdots \times A_m, \quad \text{and} \quad C = \{(x_1, \ldots, x_m) \in \mathcal{H} \mid x_1 = \cdots = x_m \}.
\]

Note that \( A \) is maximal monotone on \( \mathcal{H} \), and that \( C \) is a closed linear subspace of \( \mathcal{H} \). Next, set \( x = (z_1, \ldots, z_m), u = (w_1, \ldots, w_m) \), and \((\forall n \in \mathbb{N}) x_n = (x_{1,n}, \ldots, x_{m,n}) \) and \( u_n = (y_{1,n}, \ldots, y_{m,n}) \).

By (9), \((x_n, u_n)_{n \in \mathbb{N}} \) is a sequence in \( \text{gra} \ A \) such that \((x_n, u_n) \rightharpoonup (x, u)\). Furthermore, (10) and (11) imply that \( P_C u_n \to 0 \) and that \( x_n - P_C x_n \to 0 \), respectively. Therefore, by Theorem 2, \((x, u) \in (\text{gra} \ A) \cap (C \times C^\perp)\), which is precisely the announced conclusion.

Remark 4 Corollary 3 is a considerable strengthening of [7, Proposition A.1], where it was additionally assumed that \( A_1 + \cdots + A_m \) is maximal monotone, and where part of the conclusion of Corollary 3, namely \( z_1 = \cdots = z_m \), was an additional assumption.

Remark 5 Because of the removal of the assumption that \( A_1 + \cdots + A_m \) be maximal monotone (see the previous remark), a second look at the proofs in Eckstein and Svaiter’s paper [7] reveals that — in our present notation — the assumption that

“either \( \mathcal{H} \) is finite-dimensional or \( A_1 + \cdots + A_m \) is maximal monotone”

is superfluous in both [7, Proposition 3.2 and Proposition 4.2]. This is important in the infinite-dimensional case, where the maximality of the sum can typically be only guaranteed when a constraint qualification is satisfied; consequently, Corollary 3 helps to widen the scope of the powerful algorithmic framework of Eckstein and Svaiter.

Remark 6 The author is grateful for the following comments.

(i) Dr. Patrick Combettes brought to our attention the recent paper [2] on the use of product space techniques in monotone operator splitting problems.

(ii) Dr. Jonathan Eckstein observed that Lemma 1 remains true if \( P_C \) is replaced by an arbitrary linear firmly nonexpansive operator and that the operator \( P_C F + (\text{Id} - P_C)(\text{Id} - F) = (2P_C - \text{Id})(2F - \text{Id}) \) lies at the heart of splitting methods including the Douglas-Rachford splitting method [4, 5, 6, 10].
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References


