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Near Equality, Near Convexity, Sums of Maximally Monotone Operators, and Averages of Firmly Nonexpansive Mappings

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Dedicated to Jonathan Borwein on the occasion of his 60th birthday.

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Abstract We study nearly equal and nearly convex sets, ranges of maximally monotone operators, and ranges and fixed points of convex combinations of firmly nonexpansive mappings. The main result states that the range of an average of firmly nonexpansive mappings is nearly equal to the average of the ranges of the mappings. A striking application of this result yields that the average of asymptotically regular firmly nonexpansive mappings is also asymptotically regular. Throughout, examples are provided to illustrate the theory. We also obtain detailed information on the domain and range of the resolvent average.

Keywords asymptotic regularity · firmly nonexpansive mapping · nearly convex set · monotone operator · resolvent

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1 Overview

Throughout, we assume that

$$X \text{ is a real Euclidean space with inner product } \langle\cdot,\cdot\rangle \text{ and induced norm } \|\cdot\|, \tag{1.1}$$

that m is a strictly positive integer, and that

$$I = \{1, \ldots, m\}. \tag{1.2}$$

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Our aim is to study range properties of sums of maximally monotone operators as well as range and fixed point properties of firmly nonexpansive mappings. The required notions of near convexity and near equality are introduced in Section 2. Section 3 is concerned with maximally monotone operators, while firmly nonexpansive mappings are studied in Section 4. The notation we employ is standard and as in e.g., [4], [9], [31], and [38] to which we refer the reader for background material and further information.

2 Near Equality and Near Convexity

In this section, we introduce near equality for sets and show that this notion is useful in the study of nearly convex sets. These results are the key to study ranges of sums of maximal monotone operators in the sequel. Let C be a subset of X. We use $\operatorname{conv} C$ and $\operatorname{aff} C$ for the *convex hull* and *affine hull*, respectively; the *closure* of C is denoted by $\overline{C}$ and $\operatorname{ri} C$ is the *relative interior* of C (i.e., the interior with respect to the affine hull of C). See [31, Chapter 6] for more on this fundamental notion. The next result follows directly from the definition.

Lemma 2.1 *Let A and B be subsets of X such that $A \subseteq B$ and $\operatorname{aff} A = \operatorname{aff} B$. Then $\operatorname{ri} A \subseteq \operatorname{ri} B$.*

Fact 2.2 (Rockafellar) Let C and D be convex subsets of X, and let $\lambda \in \mathbb{R}$. Then the following hold.

(i) $\operatorname{ri} C$ and $\overline{C}$ are convex.
(ii) $C \neq \varnothing \Rightarrow \operatorname{ri} C \neq \varnothing$.
(iii) $\overline{\operatorname{ri} C} = \overline{C}$.
(iv) $\operatorname{ri} C = \operatorname{ri} \overline{C}$.
(v) $\operatorname{aff} \operatorname{ri} C = \operatorname{aff} C = \operatorname{aff} \overline{C}$.
(vi) $\operatorname{ri} C = \operatorname{ri} D \Leftrightarrow \overline{C} = \overline{D} \Leftrightarrow \operatorname{ri} C \subseteq D \subseteq \overline{C}$.
(vii) $\operatorname{ri} \lambda C = \lambda \operatorname{ri} C$.
(viii) $\operatorname{ri}(C + D) = \operatorname{ri} C + \operatorname{ri} D$.

Proof (i)&(ii): See [31, Theorem 6.2]. (iii)&(iv): See [31, Theorem 6.3]. (v): See [31, Theorem 6.2]. (vi): See [31, Corollary 6.3.1]. (vii): See [31, Corollary 6.6.1]. (viii): See [31, Corollary 6.6.2]. □

The key notion in this paper is defined next.

Definition 2.3 (near equality) Let A and B be subsets of X. We say that A and B are *nearly equal*, if

$$A \approx B \quad :\Leftrightarrow \quad \overline{A} = \overline{B} \quad \text{and} \quad \operatorname{ri} A = \operatorname{ri} B. \tag{2.1}$$

Observe that

$$A \approx B \;\Rightarrow\; \operatorname{int} A = \operatorname{int} B \tag{2.2}$$

since the relative interior coincides with the interior whenever the interior is nonempty.

Proposition 2.4 (equivalence relation) *The following hold for any subsets A, B, C of X.*

(i) $A \approx A$.
(ii) $A \approx B \Rightarrow B \approx A$.
(iii) $A \approx B$ *and* $B \approx C \Rightarrow A \approx C$.

Proposition 2.5 (squeeze theorem) *Let A, B, C be subsets of X such that $A \approx C$ and $A \subseteq B \subseteq C$. Then $A \approx B \approx C$.*

Proof By assumption, $\overline{A} = \overline{C}$ and $\operatorname{ri} A = \operatorname{ri} C$. Thus $\overline{A} = \overline{B} = \overline{C}$ and also $\operatorname{aff}(A) = \operatorname{aff}(\overline{A}) = \operatorname{aff}(\overline{C}) = \operatorname{aff}(C)$. Hence $\operatorname{aff} A = \operatorname{aff} B = \operatorname{aff} C$ and so, by Lemma 2.1, $\operatorname{ri} A \subseteq \operatorname{ri} B \subseteq \operatorname{ri} C$. Since $\operatorname{ri} A = \operatorname{ri} C$, we deduce that $\operatorname{ri} A = \operatorname{ri} B = \operatorname{ri} C$. Therefore, $A \approx B \approx C$. □

The equivalence relation "$\approx$" is best suited for studying nearly convex sets (the definition of which we recall next) as we do have that, e.g., $\mathbb{Q} \approx \mathbb{R} \smallsetminus \mathbb{Q}$!

Definition 2.6 (near convexity) (See Rockafellar and Wets's [34, Theorem 12.41].) Let A be a subset of X. Then A is *nearly convex* if there exists a convex subset C of X such that $C \subseteq A \subseteq \overline{C}$.

Lemma 2.7 *Let A be a nearly convex subset of X, say $C \subseteq A \subseteq \overline{C}$, where C is a convex subset of X. Then*

$$A \approx \overline{A} \approx \operatorname{ri} A \approx \operatorname{conv} A \approx \operatorname{ri} \operatorname{conv} A \approx C. \tag{2.3}$$

In particular, the following hold.

(i) $\overline{A}$ *and* $\operatorname{ri} A$ *are convex.*
(ii) *If* $A \neq \varnothing$, *then* $\operatorname{ri} A \neq \varnothing$.

Proof We have

$$C \subseteq A \subseteq \operatorname{conv} A \subseteq \overline{C} \quad \text{and} \quad C \subseteq A \subseteq \overline{A} \subseteq \overline{C}. \tag{2.4}$$

Since $C \approx \overline{C}$ by Fact 2.2(iv), it follows from Proposition 2.5 that

$$A \approx \overline{A} \approx \operatorname{conv} A \approx C. \tag{2.5}$$

This implies

$$\operatorname{ri}(\operatorname{ri} A) = \operatorname{ri}(\operatorname{ri} C) = \operatorname{ri} C = \operatorname{ri} A \tag{2.6}$$

and

$$\overline{\operatorname{ri} A} = \overline{\operatorname{ri} C} = \overline{C} = \overline{A} \tag{2.7}$$

by Fact 2.2(iii). Therefore, $\operatorname{ri} A \approx A$. Applying this to $\operatorname{conv} A$, which is nearly convex, it also follows that $\operatorname{ri} \operatorname{conv} A \approx \operatorname{conv} A$. Finally, (i) holds because $A \approx C$ while (ii) follows from $\operatorname{ri} A = \operatorname{ri} C$ and Fact 2.2(ii). □

Remark 2.8 The assumption of near convexity in Lemma 2.7 is necessary: consider $A = \mathbb{Q}$ when $X = \mathbb{R}$.

Lemma 2.9 (characterization of near convexity) *Let $A \subseteq X$. Then the following are equivalent.*

(i) *A is nearly convex.*
(ii) $A \approx \operatorname{conv} A$.
(iii) *A is nearly equal to a convex set.*
(iv) *A is nearly equal to a nearly convex set.*
(v) $\operatorname{ri}\operatorname{conv} A \subseteq A$.

Proof "(i)⇒(ii)": Apply Lemma 2.7. "(ii)⇒(v)": Indeed, $\operatorname{ri}\operatorname{conv} A = \operatorname{ri} A \subseteq A$. "(v)⇒(i)": Set $C = \operatorname{ri}\operatorname{conv} A$. By Fact 2.2(iii), $C \subseteq A \subseteq \overline{\operatorname{conv}}\, A = \overline{\operatorname{ri}\operatorname{conv} A} = \overline{C}$. "(ii)⇒(iii)": Clear. "(iii)⇒(i)": Suppose that $A \approx C$, where C is convex. Then, using Fact 2.2(iii), $\operatorname{ri} C = \operatorname{ri} A \subseteq A \subseteq \overline{A} = \overline{C} = \overline{\operatorname{ri} C}$. Hence A is nearly convex. "(iii)⇒(iv)": Clear. "(iv)⇒(iii)": (The following simple proof was suggested by a referee.) Suppose $A \approx B$, where B is nearly convex. Then, applying the already verified implications "(i)⇒(ii)" and "(ii)⇒(iii)" to the set B, we see that $B \approx C$ for some convex set C. Using Proposition 2.4(iii), we conclude that $A \approx C$. □

Remark 2.10 The condition appearing in Lemma 2.9(v) was also used by Minty [25] and named "almost-convex".

Remark 2.11 Brézis and Haraux [14] define, for two subsets A and B of X,

$$A \simeq B \quad :\Leftrightarrow \quad \overline{A} = \overline{B} \quad \text{and} \quad \operatorname{int} A = \operatorname{int} B. \tag{2.8}$$

(i) In view of (2.2), it is clear that $A \approx B \Rightarrow A \simeq B$.
(ii) On the other hand, $A \simeq B \not\Rightarrow A \approx B$: indeed, consider $X = \mathbb{R}^2$, $A = \mathbb{Q} \times \{0\}$, and $B = \mathbb{R} \times \{0\}$.
(iii) The implications (iii)⇒(i) and (ii)⇒(i) in Lemma 2.9 fail for $\simeq$: indeed, consider $X = \mathbb{R}^2$, $A = (\mathbb{R} \smallsetminus \{0\}) \times \{0\}$ and $C = \operatorname{conv} A = \mathbb{R} \times \{0\}$. Then C is convex and $A \simeq C$. However, A is not nearly convex because $\operatorname{ri} A \neq \operatorname{ri} \overline{A}$.

Proposition 2.12 *Let A and B be nearly convex subsets of X. Then the following are equivalent.*

(i) $A \approx B$.
(ii) $\overline{A} = \overline{B}$.
(iii) $\operatorname{ri} A = \operatorname{ri} B$.
(iv) $\overline{\operatorname{conv} A} = \overline{\operatorname{conv} B}$.
(v) $\operatorname{ri}\operatorname{conv} A = \operatorname{ri}\operatorname{conv} B$.

Proof "(i)⇒(ii)": This is clear from the definition of $\approx$. "(ii)⇒(iii)": $\operatorname{ri}\overline{A} = \operatorname{ri} A$ and $\operatorname{ri}\overline{B} = \operatorname{ri} B$ by Lemma 2.7. "(iii)⇒(iv)": $\overline{\operatorname{ri} A} = \overline{\operatorname{conv} A}$ and $\overline{\operatorname{ri} B} = \overline{\operatorname{conv} B}$ by Lemma 2.7. "(iv)⇒(v)": $\operatorname{ri}\overline{\operatorname{conv} A} = \operatorname{ri}\operatorname{conv} A$ and $\operatorname{ri}\overline{\operatorname{conv} B} = \operatorname{ri}\operatorname{conv} B$. "(v)⇒(i)": Lemma 2.7 gives that $\operatorname{ri}\operatorname{conv} A = \operatorname{ri} A$ and $\operatorname{ri}\operatorname{conv} B = \operatorname{ri} B$ so that $\operatorname{ri} A = \operatorname{ri} B$, $\overline{\operatorname{ri}\operatorname{conv} A} = \overline{\operatorname{conv} A} = \overline{A}$ and $\overline{\operatorname{ri}\operatorname{conv} B} = \overline{\operatorname{conv} B} = \overline{B}$ so that $\overline{A} = \overline{B}$. Hence (i) holds. □

In order to study addition of nearly convex sets, we require the following result.

Lemma 2.13 *Let $(A_i)_{i\in I}$ be a family of nearly convex subsets of X, and let $(\lambda_i)_{i\in I}$ be a family of real numbers. Then $\sum_{i\in I}\lambda_i A_i$ is nearly convex, and $\operatorname{ri}(\sum_{i\in I}\lambda_i A_i) = \sum_{i\in I}\lambda_i \operatorname{ri} A_i$.*

Proof For every $i \in I$, there exists a convex subset C_i of X such that $C_i \subseteq A_i \subseteq \overline{C_i}$. We have

$$\sum_{i\in I}\lambda_i C_i \subseteq \sum_{i\in I}\lambda_i A_i \subseteq \sum_{i\in I}\lambda_i \overline{C_i} \subseteq \overline{\sum_{i\in I}\lambda_i C_i}, \tag{2.9}$$

which yields the near convexity of $\sum_{i\in I}\lambda_i A_i$ and $\sum_{i\in I}\lambda_i A_i \approx \sum_{i\in I}\lambda_i C_i$ by Lemma 2.7. Moreover, by Fact 2.2(vii)&(viii) and Lemma 2.7,

$$\operatorname{ri}\Big(\sum_{i\in I}\lambda_i A_i\Big) = \operatorname{ri}\Big(\sum_{i\in I}\lambda_i C_i\Big) = \sum_{i\in I}\operatorname{ri}\big(\lambda_i C_i\big) = \sum_{i\in I}\lambda_i \operatorname{ri} C_i = \sum_{i\in I}\lambda_i \operatorname{ri} A_i. \tag{2.10}$$

This completes the proof. □

Theorem 2.14 *Let $(A_i)_{i\in I}$ be a family of nearly convex subsets of X, and let $(B_i)_{i\in I}$ be a family of subsets of X such that $A_i \approx B_i$, for every $i \in I$. Then $\sum_{i\in I} A_i$ is nearly convex and $\sum_{i\in I} A_i \approx \sum_{i\in I} B_i$.*

Proof Lemma 2.9 implies that B_i is nearly convex, for every $i \in I$. By Lemma 2.13, we have that $\sum_{i\in I} A_i$ is nearly convex and

$$\operatorname{ri}\sum_{i\in I} A_i = \sum_{i\in I}\operatorname{ri} A_i = \sum_{i\in I}\operatorname{ri} B_i = \operatorname{ri}\sum_{i\in I} B_i. \tag{2.11}$$

Furthermore,

$$\overline{\sum_{i\in I} A_i} = \overline{\sum_{i\in I} \overline{A_i}} = \overline{\sum_{i\in I} \overline{B_i}} = \overline{\sum_{i\in I} B_i} \tag{2.12}$$

and the result follows. □

Remark 2.15 Theorem 2.14 fails without the near convexity assumption: indeed, when $X = \mathbb{R}$ and $m = 2$, consider $A_1 = A_2 = \mathbb{Q}$ and $B_1 = B_2 = \mathbb{R} \smallsetminus \mathbb{Q}$. Then $A_i \approx B_i$, for every $i \in I$, yet $A_1 + A_2 = \mathbb{Q} \not\approx \mathbb{R} = B_1 + B_2$.

Theorem 2.16 *Let $(A_i)_{i\in I}$ be a family of nearly convex subsets of X, and let $(\lambda_i)_{i\in I}$ be a family of real numbers. For every $i \in I$, take*

$$B_i \in \big\{A_i, \overline{A_i}, \operatorname{conv} A_i, \operatorname{ri} A_i, \operatorname{ri}\operatorname{conv} A_i\big\}. \tag{2.13}$$

Then

$$\sum_{i\in I}\lambda_i A_i \approx \sum_{i\in I}\lambda_i B_i. \tag{2.14}$$

Proof By Lemma 2.7, $A_i \approx B_i$ for every $i \in I$. Now apply Theorem 2.14. □

Corollary 2.17 *Let $(A_i)_{i\in I}$ be a family of nearly convex subsets of X, and let $(\lambda_i)_{i\in I}$ be a family of real numbers. Suppose that $j \in I$ is such that $\lambda_j \neq 0$. Then*

$$\big(\operatorname{int} \lambda_j A_j\big) + \sum_{i\in I\smallsetminus\{j\}} \lambda_i \overline{A_i} \subseteq \operatorname{int} \sum_{i\in I} \lambda_i A_i; \tag{2.15}$$

consequently, if $0 \in (\operatorname{int} A_j) \cap \bigcap_{i\in I\smallsetminus\{j\}} \overline{A_i}$, *then* $0 \in \operatorname{int} \sum_{i\in I} \lambda_i A_i$.

Proof By Theorem 2.16, $\operatorname{ri}(\lambda_j A_j + \sum_{i\in I\smallsetminus\{j\}} \lambda_i \overline{A_i}) = \operatorname{ri} \sum_{i\in I} \lambda_i A_i$. Since

$$\big(\operatorname{int} \lambda_j A_j\big) + \sum_{i\in I\smallsetminus\{j\}} \lambda_i \overline{A_i} \subseteq \operatorname{ri} \Big(\lambda_j A_j + \sum_{i\in I\smallsetminus\{j\}} \lambda_i \overline{A_i}\Big), \tag{2.16}$$

and $\big(\operatorname{int} \lambda_j A_j\big) + \sum_{i\in I\smallsetminus\{j\}} \lambda_i \overline{A_i}$ is an open set, (2.15) follows. In turn, the "consequently" follows from (2.15). □

We develop a complementary cancellation result whose proof relies on Rådström's cancellation.

Fact 2.18 (See [29].) Let A be a nonempty subset of X, let E be a nonempty bounded subset of X, and let B be a nonempty closed convex subset of X such that $A + E \subseteq B + E$. Then $A \subseteq B$.

Theorem 2.19 *Let A and B be nonempty nearly convex subsets of X, and let E be a nonempty compact subset of X such that $A + E \approx B + E$. Then $A \approx B$.*

Proof We have $A + E \subseteq \overline{A + E} = \overline{B + E} = \overline{B} + E$. Fact 2.18 implies $A \subseteq \overline{B}$; hence, $\overline{A} \subseteq \overline{B}$. Analogously, $\overline{B} \subseteq \overline{A}$ and thus $\overline{A} = \overline{B}$. Now apply Proposition 2.12. □

Finally, we give a result concerning the interior of nearly convex sets.

Proposition 2.20 *Let A be a nearly convex subset of X. Then*

$$\operatorname{int} A = \operatorname{int} \operatorname{conv} A = \operatorname{int} \overline{A}. \tag{2.17}$$

Proof By Lemma 2.7, $A \approx B$, where $B \in \big\{\overline{A}, \operatorname{conv} A\big\}$. Now recall (2.2). □

3 Maximally Monotone Operators

Let $A\colon X \rightrightarrows X$, i.e., A is a set-valued operator on X in the sense that $(\forall x \in X)$ $Ax \subseteq X$. The graph of A is denoted by $\operatorname{gr} A$. Then A is *monotone* (on X) if

$$(\forall (x,x^*) \in \operatorname{gr} A)(\forall (y,y^*) \in \operatorname{gr} A) \quad \langle x - y, x^* - y^*\rangle \geq 0, \tag{3.1}$$

and A is *maximally monotone* if A admits no proper monotone extension. Classical examples of monotone operators are subdifferential operators of functions that are convex, lower semicontinuous, and proper; linear operators with a positive symmetric part. See, e.g., [4], [9], [13], [15], [34], [35], [36], [38], [40], and [41] for applications and further information. As usual, the domain and range of A are denoted by $\operatorname{dom} A = \{x \in X : \ Ax \neq \varnothing\}$ and $\operatorname{ran} A = \bigcup_{x\in X} Ax$ respectively; $\operatorname{dom} f = \big\{x \in X \mid f(x) < +\infty\big\}$ stands for the domain of a function $f : X \to \left]-\infty, +\infty\right]$.

Fact 3.1 (Rockafellar) (See [32] or [34, Theorem 12.44].) Let A and B be maximally monotone on X. Suppose that $\operatorname{ri}\operatorname{dom} A \cap \operatorname{ri}\operatorname{dom} B \neq \varnothing$. Then $A+B$ is maximally monotone.

Fact 3.2 (Minty) (See [25] or [34, Theorem 12.41].) Let $A\colon X \rightrightarrows X$ be maximally monotone. Then $\operatorname{dom} A$ and $\operatorname{ran} A$ are nearly convex.

Remark 3.3 Fact 3.2 is optimal in the sense that the domain or the range of a maximally monotone operator may fail to be convex—even for a subdifferential operator—see, e.g., [34, page 555].

Sometimes quite precise information is available on the range of the sum of two maximally monotone operators. To formulate the corresponding statements, we need to review a few notions.

Definition 3.4 (Fitzpatrick function) (See [20], and also [16] or [24].) Let $A\colon X \rightrightarrows X$. Then the *Fitzpatrick function* associated with A is

$$F_A\colon X \times X \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gr} A} \big(\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big). \tag{3.2}$$

Example 3.5 (energy) (See, e.g., [6, Example 3.10].) Let $\operatorname{Id} : X \to X\colon x \mapsto x$ be the *identity operator*. Then $F_{\operatorname{Id}}\colon X \times X \to \mathbb{R}\colon (x, x^*) \mapsto \frac{1}{4}\|x + x^*\|^2$.

Definition 3.6 (Brézis-Haraux) (See [14].) Let $A\colon X \to X$ be monotone. Then A is *rectangular* (which is also known as star-monotone or 3^* monotone), if

$$\operatorname{dom} A \times \operatorname{ran} A \subseteq \operatorname{dom} F_A. \tag{3.3}$$

Remark 3.7 If $A\colon X \rightrightarrows X$ is maximally monotone and rectangular, then one obtains the "rectangle" $\overline{\operatorname{dom} F_A} = \overline{\operatorname{dom} A} \times \overline{\operatorname{ran} A}$, which prompted Simons [37] to call such an operator rectangular.

Fact 3.8 Let A and B be monotone on X, let $C\colon X \to X$ be linear and monotone, let $\alpha > 0$, and let $f\colon X \to \left]-\infty, +\infty\right]$ be convex, lower semicontinuous, and proper. Then the following hold.

(i) A is rectangular $\Leftrightarrow$ A^{-1} is rectangular.
(ii) A is rectangular $\Leftrightarrow$ αA is rectangular.
(iii) ∂f is maximally monotone and rectangular.
(iv) C is rectangular $\Leftrightarrow$ C^* is rectangular $\Leftrightarrow$ $(\exists \gamma > 0)(\forall x \in X)\ \langle x, Cx\rangle \geq \gamma\|Cx\|^2$.

(v) $(\operatorname{dom} A \cap \operatorname{dom} B) \times X \subseteq \operatorname{dom} F_B \Rightarrow A + B$ is rectangular.

Proof (i)&(ii): This follows readily from the definitions. (iii): The fact that ∂f is rectangular was pointed out in [14, Example 1 on page 166]. For maximal monotonicity of ∂f, see [27] (or [31, Corollary 31.5.2] or [34, Theorem 12.17]). (iv): [14, Proposition 2 and Remarque 2 on page 169]. (v): [4, Proposition 24.17]. □

Example 3.9 (See also [14, Example 3 on page 167] or [1, Example 6.5.2(iii)].) Let $A \colon X \rightrightarrows X$ be maximally monotone. Then $A + \operatorname{Id}$ and $(A + \operatorname{Id})^{-1}$ are maximally monotone and rectangular.

Proof Combining Fact 3.8(v) and Example 3.5, we see that $A+\operatorname{Id}$ is rectangular. Furthermore, $A+\operatorname{Id}$ is maximally monotone by Fact 3.1. Using Fact 3.8(i), we see that $(\operatorname{Id}+A)^{-1}$ is maximally monotone and rectangular. □

Proposition 3.10 (See [6, Proposition 4.2].) *Let A and B be monotone on X, and let $(x, x^*) \in X \times X$. Then $F_{A+B}(x, x^*) \leq \big(F_A(x, \cdot) \,\Box\, F_B(x, \cdot)\big)(x^*)$.*

Lemma 3.11 *Let A and B be rectangular on X. Then $A + B$ is rectangular.*

Proof Clearly, $\operatorname{dom}(A + B) = (\operatorname{dom} A) \cap (\operatorname{dom} B)$, and $\operatorname{ran}(A + B) \subseteq \operatorname{ran} A + \operatorname{ran} B$. Take $x \in \operatorname{dom}(A+B)$ and $y^* \in \operatorname{ran}(A+B)$. Then there exist $a^* \in \operatorname{ran} A$ and $b^* \in \operatorname{ran} B$ such that $a^* + b^* = y^*$. Furthermore, $(x, a^*) \in (\operatorname{dom} A) \times (\operatorname{ran} A) \subseteq \operatorname{dom} F_A$ and $(x, b^*) \in (\operatorname{dom} B) \times (\operatorname{ran} B) \subseteq \operatorname{dom} F_B$. Using Proposition 3.10 and the assumption that A and B are rectangular, we obtain

$$F_{A+B}(x, y^*) \leq F_A(x, a^*) + F_B(x, b^*) < +\infty. \tag{3.4}$$

Therefore, $\operatorname{dom}(A + B) \times \operatorname{ran}(A + B) \subseteq \operatorname{dom} F_{A+B}$ and we deduce that $A + B$ is rectangular. □

We are now ready to state the range result, which can be traced back to the seminal paper by Brézis and Haraux [14] (see also [35] or [37], and [30] for a Banach space version). The useful finite-dimensional formulation we record here was brought to light by Auslender and Teboulle [1].

Fact 3.12 (Brézis-Haraux) (See [1, Theorem 6.5.1(b) and Theorem 6.5.2].) Let A and B be monotone on X such that $A + B$ is maximally monotone. Suppose that one of the following holds.

(i) A and B are rectangular.
(ii) $\operatorname{dom} A \subseteq \operatorname{dom} B$ and B is rectangular.

Then $\overline{\operatorname{ran}(A + B)} = \overline{\operatorname{ran} A + \operatorname{ran} B}$, $\operatorname{int}(\operatorname{ran}(A + B)) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$, and $\operatorname{ri}\operatorname{conv}(\operatorname{ran} A + \operatorname{ran} B) \subseteq \operatorname{ran}(A + B)$.

Item (i) of the following result also follows from Chu's [17, Theorem 3.1].

Theorem 3.13 *Let A and B be monotone on X such that $A+B$ is maximally monotone. Suppose that one of the following holds.*

(i) *A and B are rectangular.*
(ii) $\operatorname{dom} A \subseteq \operatorname{dom} B$ *and B is rectangular.*

Then $\operatorname{ran}(A+B)$ *is nearly convex, and* $\operatorname{ran}(A+B) \approx \operatorname{ran} A + \operatorname{ran} B$.

Proof The near convexity of $\operatorname{ran}(A+B)$ follows from Fact 3.2. Using Fact 3.12 and Fact 2.2(iii),

$$\operatorname{ri}\operatorname{conv}(\operatorname{ran} A + \operatorname{ran} B) \subseteq \operatorname{ran}(A+B) \subseteq \operatorname{ran} A + \operatorname{ran} B \subseteq \overline{\operatorname{conv}}\,(\operatorname{ran} A + \operatorname{ran} B) \tag{3.5a}$$

$$= \overline{\operatorname{ri}\operatorname{conv}(\operatorname{ran} A + \operatorname{ran} B)}. \tag{3.5b}$$

Proposition 2.5 and Lemma 2.7 imply that $\operatorname{ran}(A+B) \approx \operatorname{ran} A + \operatorname{ran} B \approx \operatorname{ri}\operatorname{conv}(\operatorname{ran} A + \operatorname{ran} B)$. □

Remark 3.14 Considering $A+0$, where A is the rotator by $\pi/2$ on $\mathbb{R}^2$ which is not rectangular, we see that $A+B$ need not be rectangular under assumption (ii) in Theorem 3.13.

If we let $S_i = \operatorname{ran} A_i$ and $\lambda_i = 1$ for every $i \in I$ in Theorem 3.15, then we obtain a result that is related to Pennanen's [28, Corollary 6].

Theorem 3.15 *Let $(A_i)_{i\in I}$ be a family of maximally monotone rectangular operators on X with $\bigcap_{i\in I} \operatorname{ri}\operatorname{dom} A_i \neq \varnothing$, let $(S_i)_{i\in I}$ be a family of subsets of X such that*

$$(\forall i \in I) \quad S_i \in \big\{ \operatorname{ran} A_i, \overline{\operatorname{ran} A_i}, \operatorname{ri}(\operatorname{ran} A_i)\big\}, \tag{3.6}$$

and let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers. Then $\sum_{i\in I} \lambda_i A_i$ is maximally monotone, rectangular, and $\operatorname{ran} \sum_{i\in I} \lambda_i A_i \approx \sum_{i\in I} \lambda_i S_i$ *is nearly convex.*

Proof To see that $\sum_{i\in I} \lambda_i A_i$ is maximally monotone and rectangular, use Fact 3.1, Lemma 3.11, and induction. With Theorem 2.14, Fact 3.1 and Lemma 3.11 in mind, Theorem 3.13(i) and the principle of mathematical induction yields $\operatorname{ran} \sum_{i\in I} \lambda_i A_i \approx \sum_{i\in I} \lambda_i \operatorname{ran} A_i$ and the near convexity. Finally, as $\operatorname{ran} A_i$ is nearly convex for every $i \in I$ by Fact 3.2, $\operatorname{ran} \sum_{i\in I} \lambda_i A_i \approx \sum_{i\in I} \lambda_i S_i$ follows from Theorem 2.16. □

The main result of this section is the following.

Theorem 3.16 *Let $(A_i)_{i\in I}$ be a family of maximally monotone rectangular operators on X such that $\bigcap_{i\in I} \operatorname{ri}\operatorname{dom} A_i \neq \varnothing$, let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers, and let $j \in I$. Set*

$$A = \sum_{i\in I} \lambda_i A_i. \tag{3.7}$$

Then the following hold.

(i) *If* $\sum_{i\in I} \lambda_i \operatorname{ran} A_i = X$, *then* $\operatorname{ran} A = X$.
(ii) *If A_j is surjective, then A is surjective.*

(iii) *If* $0 \in \bigcap_{i\in I} \overline{\operatorname{ran} A_i}$*, then* $0 \in \overline{\operatorname{ran} A}$*.*
(iv) *If* $0 \in (\operatorname{int}\operatorname{ran} A_j) \cap \bigcap_{i\in I\smallsetminus\{j\}} \overline{\operatorname{ran} A_i}$*, then* $0 \in \operatorname{int}\operatorname{ran} A$*.*

Proof Theorem 3.15 implies that $\operatorname{ran}\sum_{i\in I}\lambda_i A_i \approx \sum_{i\in I}\lambda_i \operatorname{ran} A_i$ is nearly convex. Hence

$$\operatorname{ri}\operatorname{ran} A = \operatorname{ri}\operatorname{ran}\sum_{i\in I}\lambda_i A_i = \operatorname{ri}\Big(\sum_{i\in I}\lambda_i \operatorname{ran} A_i\Big) = \sum_{i\in I}\lambda_i \operatorname{ri}\operatorname{ran} A_i \tag{3.8}$$

and

$$\overline{\operatorname{ran} A} = \overline{\operatorname{ran}\sum_{i\in I}\lambda_i A_i} = \overline{\sum_{i\in I}\lambda_i \operatorname{ran} A_i}\,. \tag{3.9}$$

(i): Indeed, using (3.8), $X = \operatorname{ri} X = \operatorname{ri}\sum_{i\in I}\lambda_i \operatorname{ran} A_i = \operatorname{ri}\operatorname{ran} A \subseteq \operatorname{ran} A \subseteq X$. (ii): Clear from (i). (iii): It follows from (3.9) that $0 \in \sum_{i\in I}\lambda_i\overline{\operatorname{ran} A_i} \subseteq \overline{\sum_{i\in I}\lambda_i\operatorname{ran} A_i} = \overline{\operatorname{ran} A}$. (iv): By Fact 3.2, $\operatorname{ran} A_i$ is nearly convex for every $i \in I$. Thus, $0 \in \operatorname{int}\sum_{i\in I}\lambda_i \operatorname{ran} A_i$ by Corollary 2.17. On the other hand, (3.8) implies that $\operatorname{int}\sum_{i\in I}\lambda_i \operatorname{ran} A_i \subseteq \operatorname{ri}\sum_{i\in I}\lambda_i \operatorname{ran} A_i = \operatorname{ri}\operatorname{ran} A$. Altogether, $0 \in \operatorname{ri}\operatorname{ran} A = \operatorname{int}\operatorname{ran} A$ because $\operatorname{int}\operatorname{ran} A \neq \varnothing$. □

4 Firmly Nonexpansive Mappings

To find zeros of maximally monotone operators, one often utilizes firmly nonexpansive mappings [4,18,19,33]. In this section, we apply the result of Section 3 to firmly nonexpansive mappings. Let $T\colon X \to X$. Recall that T is *firmly nonexpansive* (see also Zarantonello's seminal work [39] for further results) if

$$(\forall x \in X)(\forall y \in X) \quad \langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2. \tag{4.1}$$

The following characterizations are well known.

Fact 4.1 (See, e.g., [4], [21], or [22].) Let $T\colon X \to X$. Then the following are equivalent.

(i) T is firmly nonexpansive.
(ii) $(\forall x \in X)(\forall y \in X)\ \|Tx-Ty\|^2 + \|(\operatorname{Id}-T)x - (\operatorname{Id}-T)y\|^2 \leq \|x-y\|^2$.
(iii) $(\forall x \in X)(\forall y \in X)\ 0 \leq \langle Tx-Ty, (\operatorname{Id}-T)x - (\operatorname{Id}-T)y\rangle$.
(iv) $\operatorname{Id}-T$ is firmly nonexpansive.
(v) $2T-\operatorname{Id}$ is *nonexpansive*, i.e., Lipschitz continuous with constant 1.

Minty [26] first observed—while Eckstein and Bertsekas [19] made this fully precise—a fundamental correspondence between maximally monotone operators and firmly nonexpansive mappings. It is based on the *resolvent* of A,

$$J_A := (\operatorname{Id}+A)^{-1}, \tag{4.2}$$

which satisfies the useful identity

$$J_A + J_{A^{-1}} = \operatorname{Id}, \tag{4.3}$$

and which allows for the beautiful *Minty parametrization*

$$\operatorname{gr} A = \big\{(J_A x, x - J_A x) \mid x \in X\big\} \tag{4.4}$$

of the graph of A.

Fact 4.2 (See [19] and [26].) Let $T\colon X \to X$ and let $A\colon X \rightrightarrows X$. Then the following hold.

(i) If T is firmly nonexpansive, then $B := T^{-1} - \mathrm{Id}$ is maximally monotone and $J_B = T$.
(ii) If A is maximally monotone, then J_A has full domain, and it is single-valued and firmly nonexpansive.

Corollary 4.3 *Let $T\colon X \to X$ be firmly nonexpansive. Then T is maximally monotone and rectangular, and* $\operatorname{ran} T$ *is nearly convex.*

Proof Combine Example 3.9, Fact 4.2(i), and Fact 3.2. □

It is also known that the class of firmly nonexpansive mappings is closed under taking convex combinations. For completeness, we include a short proof of this result.

Lemma 4.4 *Let $(T_i)_{i\in I}$ be a family of firmly nonexpansive mappings on X, and let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \lambda_i = 1$. Then $\sum_{i\in I} \lambda_i T_i$ is also firmly nonexpansive.*

Proof Set $T = \sum_{i\in I} \lambda_i T_i$. By Fact 4.1, $2T_i - \mathrm{Id}$ is nonexpansive for every $i \in I$, so $2T - \mathrm{Id} = \sum_{i\in I} \lambda_i (2T_i - \mathrm{Id})$ is also nonexpansive. Applying Fact 4.1 once more, we deduce that T is firmly nonexpansive. □

We are now ready for the first main result of this section.

Theorem 4.5 (averages of firmly nonexpansive mappings) *Let $(T_i)_{i\in I}$ be a family of firmly nonexpansive mappings on X, let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \lambda_i = 1$, and let $j \in I$. Set $T = \sum_{i\in I} \lambda_i T_i$. Then the following hold.*

(i) *T is firmly nonexpansive and* $\operatorname{ran} T \approx \sum_{i\in I} \lambda_i \operatorname{ran} T_i$ *is nearly convex.*
(ii) *If T_j is surjective, then T is surjective.*
(iii) *If* $0 \in \bigcap_{i\in I} \overline{\operatorname{ran} T_i}$*, then* $0 \in \overline{\operatorname{ran} T}$.
(iv) *If* $0 \in (\operatorname{int} \operatorname{ran} T_j) \cap \bigcap_{i\in I\smallsetminus\{j\}} \overline{\operatorname{ran} T_i}$*, then* $0 \in \operatorname{int} \operatorname{ran} T$.

Proof By Corollary 4.3, each T_i is maximally monotone, rectangular and $\operatorname{ran} T_i$ is nearly convex. (i): Lemma 2.7, Lemma 4.4, and Theorem 3.15. (ii): Theorem 3.16(ii). (iii): Theorem 3.16(iii). (iv): Theorem 3.16(iv). □

The following averaged-projection operator plays a role in methods for solving (potentially inconsistent) convex feasibility problems because its fixed point set consists of least-squares solutions; see, e.g., [3, Section 6], [8] and [18] for further information.

Example 4.6 Let $(C_i)_{i\in I}$ be a family of nonempty closed convex subsets of X with associated projection operators P_i, and let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I}\lambda_i = 1$. Then

$$\operatorname{ran}\sum_{i\in I}\lambda_i P_i \approx \sum_{i\in I}\lambda_i C_i. \tag{4.5}$$

Proof This follows from Theorem 4.5(i) since $(\forall i\in I)\ \operatorname{ran} P_i = C_i$. □

Remark 4.7 Let C_1 and C_2 be nonempty closed convex subsets of X with associated projection operators P_1 and P_2 respectively, and—instead of averaging as in Example 4.6—consider the composition $T = P_2\circ P_1$, which is still *nonexpansive.* It is obvious that $\operatorname{ran} T\subseteq \operatorname{ran} P_2 = C_2$, but $\operatorname{ran} T$ need not be even nearly convex: indeed, suppose that $X=\mathbb{R}^2$, let C_2 be the unit ball centered at 0 of radius 1, and let $C_1 = \mathbb{R}\times\{2\}$. Then $\operatorname{ran} T$ is the intersection of the open upper halfplane and the boundary of C_2, which is very far from being nearly convex. Thus the near convexity part of Corollary 4.3 has no counterpart for nonexpansive mappings.

Definition 4.8 Let $T\colon X\to X$ be firmly nonexpansive. The set of *fixed points* is denoted by

$$\operatorname{Fix} T = \big\{x\in X \mid x = Tx\big\}. \tag{4.6}$$

We say that T is *asymptotically regular* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_n - Tx_n\to 0$; equivalently, if $0\in\overline{\operatorname{ran}(\operatorname{Id}-T)}$.

Remark 4.9 If the sequence $(x_n)_{n\in\mathbb{N}}$ in Definition 4.8 has a cluster point, say $\bar{x}$, then continuity of T implies that $\bar{x}\in\operatorname{Fix} T$.

The next result is a consequence of fundamental work [2] by Baillon, Bruck and Reich.

Theorem 4.10 *Let $T\colon X\to X$ be firmly nonexpansive. Then T is asymptotically regular if and only if for every $x_0\in X$, the sequence defined by*

$$(\forall n\in\mathbb{N})\quad x_{n+1} = Tx_n \tag{4.7}$$

satisfies $x_n - x_{n+1}\to 0$. Moreover, if $\operatorname{Fix} T\neq\varnothing$, then $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point; otherwise, $\|x_n\|\to+\infty$.

Proof This follows from [2, Corollary 2.3, Theorem 1.2, and Corollary 2.2]. □

Here is the second main result of this section.

Theorem 4.11 (asymptotic regularity of the average) *Let $(T_i)_{i\in I}$ be a family of firmly nonexpansive mappings on X, and let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I}\lambda_i = 1$. Suppose that T_i is asymptotically regular, for every $i\in I$. Then $\sum_{i\in I}\lambda_i T_i$ is also asymptotically regular.*

Proof Set $T = \sum_{i\in I} \lambda_i T_i$. Then

$$\operatorname{Id} - T = \sum_{i\in I} \lambda_i (\operatorname{Id} - T_i).$$

Since each $\operatorname{Id} - T_i$ is firmly nonexpansive and $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_i)}$ by the asymptotic regularity of T_i, the conclusion follows from Theorem 4.5(iii). ☐

Remark 4.12 Consider Theorem 4.11. Even when $\operatorname{Fix} T_i \neq \varnothing$, for every $i \in I$, it is impossible to improve the conclusion to $\operatorname{Fix} \sum_{i\in I} \lambda_i T_i \neq \varnothing$. Indeed, suppose that $X = \mathbb{R}^2$, and set $C_1 = \mathbb{R} \times \{0\}$ and $C_2 = \operatorname{epi} \exp$. Set $T = \frac{1}{2} P_{C_1} + \frac{1}{2} P_{C_2}$. Then $\operatorname{Fix} T_1 = C_1$ and $\operatorname{Fix} T_2 = C_2$, yet $\operatorname{Fix} T = \varnothing$.

The proof of the following useful result is straightforward and hence omitted.

Lemma 4.13 *Let $A\colon X \rightrightarrows X$ be maximally monotone. Then J_A is asymptotically regular if and only if $0 \in \overline{\operatorname{ran} A}$.*

We conclude this paper with an application to the resolvent average of monotone operators. Let $(A_i)_{i\in I}$ be a family of maximally monotone operators on X. Compute and average the corresponding resolvents to obtain $T := \sum_{i\in I} \lambda_i J_{A_i}$. By Lemma 4.4, T is firmly nonexpansive; hence, again by Fact 4.2, $T = J_A$ for some maximally monotone operator A. The operator A is called the *resolvent average* of the family $(A_i)_{i\in I}$ with respect to the weights $(\lambda_i)_{i\in I}$; it was analyzed in detail for real symmetric positive semidefinite matrices in [7].

Corollary 4.14 (resolvent average) *Let $(A_i)_{i\in I}$ be a family of maximally monotone operators on X, let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \lambda_i = 1$, let $j \in I$, and set*

$$A = \Big(\sum_{i\in I} \lambda_i (\operatorname{Id} + A_i)^{-1} \Big)^{-1} - \operatorname{Id}. \tag{4.8}$$

Then the following hold.

(i) *A is maximally monotone.*
(ii) $\operatorname{dom} A \approx \sum_{i\in I} \lambda_i \operatorname{dom} A_i$.
(iii) $\operatorname{ran} A \approx \sum_{i\in I} \lambda_i \operatorname{ran} A_i$.
(iv) *If $0 \in \bigcap_{i\in I} \overline{\operatorname{ran} A_i}$, then $0 \in \overline{\operatorname{ran} A}$.*
(v) *If $0 \in \operatorname{int} \operatorname{ran} A_j \cap \bigcap_{i\in I\setminus\{j\}} \overline{\operatorname{ran} A_i}$, then $0 \in \operatorname{int} \operatorname{ran} A$.*
(vi) *If $\operatorname{dom} A_j = X$, then $\operatorname{dom} A = X$.*
(vii) *If $\operatorname{ran} A_j = X$, then $\operatorname{ran} A = X$.*

Proof Observe that

$$J_A = \sum_{i\in I} \lambda_i J_{A_i} \tag{4.9}$$

and

$$J_{A^{-1}} = \sum_{i\in I} \lambda_i J_{A_i^{-1}} \tag{4.10}$$

by using (4.3). Furthermore, using (4.4), we note that

$$\operatorname{ran} J_A = \operatorname{dom} A \quad \text{and} \quad \operatorname{ran} J_{A^{-1}} = \operatorname{ran} A. \tag{4.11}$$

(i): This follows from (4.9) and Fact 4.2. (ii): Apply Theorem 4.5(i) to $(J_{A_i})_{i\in I}$, and use (4.9) and (4.11). (iii): Apply Theorem 4.5(i) to $(\mathrm{Id} - J_{A_i})_{i\in I}$, and use (4.3) and (4.11). (iv): Combine Theorem 4.11 and Lemma 4.13, and use (4.9). (v): Apply Theorem 4.5(iv) to (4.10), and use (4.11). (vi) and (vii): These follow from (ii) and (iii), respectively. □

Remark 4.15 (proximal average) In Corollary 4.14, one may also start from a family $(f_i)_{i\in I}$ of functions on X that are convex, lower semicontinuous, and proper, and with corresponding subdifferential operators $(A_i)_{i\in I} = (\partial f_i)_{i\in I}$. This relates to the *proximal average*, p, of the family $(f_i)_{i\in I}$, where ∂p is the resolvent average of the family $(\partial f_i)_{i\in I}$. See [5] for further information and references. Corollary 4.14(vii) essentially states that p is *supercoercive* provided that some f_j is. Analogously, Corollary 4.14(v) shows that that *coercivity* of p follows from the coercivity of some function f_j. Similar comments apply to *sharp minima*; see [23, Lemma 3.1 and Theorem 4.3] for details.

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