

Fixed Points of Averages of Resolvents: Geometry and Algorithms

Heinz H. Bauschke*, Xianfu Wang[†] and Calvin J.S. Wylie[‡]

September 27, 2011 revision 2 (original: February 7; revision 1: August 29)

Abstract

To provide generalized solutions if a given problem admits no actual solution is an important task in mathematics and the natural sciences. It has a rich history dating back to the early 19th century when Carl Friedrich Gauss developed the method of least squares of a system of linear equations — its solutions can be viewed as fixed points of averaged projections onto hyperplanes. A powerful generalization of this problem is to find fixed points of averaged resolvents (i.e., firmly nonexpansive mappings).

This paper concerns the relationship between the set of fixed points of averaged resolvents and certain fixed point sets of compositions of resolvents. It partially extends recent work for two mappings on a question of C. Byrne. The analysis suggests a reformulation in a product space. Algorithmic consequences are also presented.

2010 Mathematics Subject Classification: Primary 47H05, 47H09; Secondary 47J25, 65K05, 65K10, 90C25.

Keywords: averaged mapping, firmly nonexpansive mapping, fixed point, Hilbert space, least squares solutions, maximal monotone operator, nonexpansive mapping, normal equation, projection, resolvent, resolvent average.

*Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: heinz.bauschke@ubc.ca.

[†]Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: shawn.wang@ubc.ca.

[‡]Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: calvin.wylie@gmail.com.

1 Introduction

Throughout this paper,

$$(1) \quad X \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle$$

and induced norm $\| \cdot \|$. We impose that $X \neq \{0\}$. To motivate the results of this paper, let us assume that C_1, \dots, C_m are finitely many nonempty closed convex subsets of X , with projections (nearest point mappings) P_1, \dots, P_m . Many problems in mathematics and the physical sciences can be recast as the **convex feasibility problem** of finding a point in the intersection $C_1 \cap \dots \cap C_m$. However, in applications it may well be that this intersection is empty. In this case, a powerful and very useful generalization of the intersection is the set of fixed points of the operator

$$(2) \quad X \rightarrow X: x \mapsto \frac{P_1x + \dots + P_mx}{m}.$$

(See, e.g., [17] for applications.) Indeed, these fixed points are precisely the minimizers of the convex function

$$(3) \quad X \rightarrow \mathbb{R}: x \mapsto \sum_{i=1}^m \|x - P_ix\|^2$$

and—when each C_i is a suitably described hyperplane—there is a well known connection to the set of **least squares solutions** in the sense of linear algebra (see Appendix).

A problem open for a long time is to find precise relationships between the fixed points of the operator defined in (2) and the fixed points of the composition $P_m \circ \dots \circ P_2 \circ P_1$ when the intersection $C_1 \cap \dots \cap C_m$ is empty. (It is well known that both fixed points sets coincide with $C_1 \cap \dots \cap C_m$ provided this intersection is nonempty.) This problem was recently explicitly stated and nicely discussed in [14, Chapter 50] and [15, Open Question 2 on page 101 in Subsection 8.3.2]. For other related work¹, see [3], [5], [16], [19], and the references therein. When $m = 2$, the recent work [34] contains some precise relationships. For instance, the results in [34, Section 3] show that

$$(4) \quad \text{Fix}(P_2 \circ P_1) \rightarrow \text{Fix}\left(\frac{1}{2}P_1 + \frac{1}{2}P_2\right): x \mapsto \frac{1}{2}x + \frac{1}{2}P_1x$$

is a well defined bijection.

¹In passing, we mention that when C_1, C_2, C_3 are line segments forming a triangle in the Euclidean plane, then the minimizer of (3) is known as the **symmedian point** (also known as the Grebe-Lemoine point) of the given triangle; see [24, Theorem 349 on page 216].

Our goal in this paper is two-fold. First, we wish to find a suitable extension to describe these fixed point sets when $m \geq 3$. Second, we build on these insights to obtain algorithms for finding these fixed points.

The results provided are somewhat surprising. While we completely generalize some of the two-set work from [34], the generalized intersection is *not* formulated as the fixed point set of a simple composition, but rather as the fixed point set of a more complicated operator described in a product space. Nonetheless, the geometric insight obtained will turn out to be quite useful in the design of new algorithms that show better convergence properties when compared to straight iteration of the averaged projection operator. Furthermore, the results actually hold for very general firmly nonexpansive operators—equivalently, resolvents of maximally monotone operators—although the optimization-based interpretation as a set of minimizers analogous to (3) is then unavailable.

The paper is organized as follows. In the remainder of this introductory section, we describe some central notions fundamental to our analysis. The main result of Section 2 is Theorem 2.1 where we provide a precise correspondence between the fixed point set of an averaged resolvent J_A and a certain set \mathbf{S} in a product space. In Section 3, it is shown that \mathbf{S} is in fact the fixed point set of an averaged mapping (see Corollary 3.8). This insight is brought to good use in Section 4, where we design a new algorithm for finding a point in \mathbf{S} (and hence in $\text{Fix } J_A$) and where we provide a rigorous convergence proof. Akin to the Gauss-Seidel variant of the Jacobi iteration in numerical linear algebra, we propose another new algorithm. An appendix connecting fixed points of averages of projections onto hyperplanes to classical least squares solutions concludes the paper. The notation we utilize is standard and as in [4], [8], [29], [30], [32], [33], or [35] to which we also refer for background.

Recall that a mapping

$$(5) \quad T: X \rightarrow X$$

is **firmly nonexpansive** (see [36] for the first systematic study) if

$$(6) \quad (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2,$$

where $\text{Id}: X \rightarrow X: x \mapsto x$ denotes the **identity operator**. The prime examples of firmly nonexpansive mappings are **projection operators** (also known as nearest point mappings) with respect to nonempty closed convex subsets of X . It is clear that if T is firmly nonexpansive, then it is **nonexpansive**, i.e., Lipschitz continuous with constant 1,

$$(7) \quad (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\| \leq \|x - y\|;$$

the converse, however, is false (consider $-\text{Id}$). The set of **fixed points** of T is

$$(8) \quad \text{Fix } T = \{x \in X \mid x = Tx\}.$$

The following characterization of firm nonexpansiveness is well known and will be used repeatedly.

Fact 1.1 (See, e.g., [4, 21, 22].) *Let $T: X \rightarrow X$. Then the following are equivalent:*

- (i) *T is firmly nonexpansive.*
- (ii) *$\text{Id} - T$ is firmly nonexpansive.*
- (iii) *$2T - \text{Id}$ is nonexpansive.*
- (iv) $(\forall x \in X)(\forall y \in X) \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle.$
- (v) $(\forall x \in X)(\forall y \in X) 0 \leq \langle Tx - Ty, (\text{Id} - T)x - (\text{Id} - T)y \rangle.$

Firmly nonexpansive mappings are also intimately tied with maximally monotone operators. Recall that a set-valued operator $A: X \rightrightarrows X$ (i.e., $(\forall x \in X) Ax \subseteq X$) with graph $\text{gr } A$ is **monotone** if

$$(9) \quad (\forall (x, u) \in \text{gr } A)(\forall (y, v) \in \text{gr } A) \quad \langle x - y, u - v \rangle \geq 0,$$

and that A is **maximally monotone** if it is monotone and every proper extension of A fails to be monotone. We write $\text{dom } A = \{x \in X \mid Ax \neq \emptyset\}$ and $\text{ran } A = A(X) = \bigcup_{x \in X} Ax$ for the **domain** and **range** of A , respectively. The **inverse** of A is defined via $\text{gr } A^{-1} = \{(u, x) \in X \times X \mid u \in Ax\}$. Monotone operators are ubiquitous in modern analysis and optimization; see, e.g., the books [4], [8], [9], [13], [32], [33], [35], [37], [38], and [39]. Two key examples of maximally monotone operators are continuous linear monotone operators and subdifferential operators (in the sense of convex analysis) of functions that are convex, lower semicontinuous, and proper.

Now let $A: X \rightrightarrows X$ be maximally monotone and denote the associated **resolvent** by

$$(10) \quad J_A = (\text{Id} + A)^{-1}.$$

In [27], Minty made the seminal observation that J_A is in fact a firmly nonexpansive operator from X to X and that, conversely, every firmly nonexpansive operator arises this way:

Fact 1.2 (Minty) (See, e.g., [27] or [20].) *Let $T: X \rightarrow X$ be firmly nonexpansive, and let $A: X \rightrightarrows X$ be maximally monotone. Then the following hold.*

- (i) *$B = T^{-1} - \text{Id}$ is maximally monotone (and $J_B = T$).*
- (ii) *J_A is firmly nonexpansive (and $A = J_A^{-1} - \text{Id}$).*

One of the motivations to study the correspondence between firmly nonexpansive mappings and maximally monotone operators is the very useful correspondence

$$(11) \quad A^{-1}(0) = \text{Fix } J_A,$$

where $A: X \rightrightarrows X$ is maximally monotone.

From now on we assume that

$$(12) \quad A_1, \dots, A_m \text{ are maximally monotone operators on } X, \quad \text{where } m \in \{2, 3, \dots\},$$

that

$$(13) \quad \lambda_1, \dots, \lambda_m \text{ belong to }]0, 1[\text{ such that } \sum_{i \in I} \lambda_i = 1, \quad \text{where } I = \{1, 2, \dots, m\},$$

and we set

$$(14) \quad A = \left(\sum_{i \in I} \lambda_i J_{A_i} \right)^{-1} - \text{Id}.$$

Then the definition of the resolvent yields

$$(15) \quad J_A = \sum_{i \in I} \lambda_i J_{A_i};$$

thus, since it is easy to see that J_A is firmly nonexpansive, it follows from Fact 1.2 that A is maximally monotone. We refer to the operator A as the **resolvent average** of the maximally monotone operators A_1, \dots, A_m and we note that J_A is the **weighted average of the resolvents** J_{A_i} . The operator J_A is the announced generalization of the averaged projection operator considered in (2), and $\text{Fix } J_A$ is the generalization of the minimizers of the function in (3). For a related discussion, see also [18, Section 6.3].

This introductory section is now complete. In the next section, we shall derive an alternative description of $\text{Fix } J_A$.

2 The Fixed Point Set Viewed in a Product Space

It will be quite convenient to define numbers complementary to the convex coefficients fixed in (13); thus, we let

$$(16) \quad \mu_i = 1 - \lambda_i, \quad \text{for every } i \in I.$$

Several of the results will be formulated in the Hilbert product space

$$(17) \quad \mathbf{X} = X^m, \quad \text{with inner product } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle,$$

where $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ are generic vectors in \mathbf{X} . The set \mathbf{S} , defined by

$$(18) \quad \mathbf{S} = \left\{ \mathbf{x} = (x_i)_{i \in I} \in \mathbf{X} \mid (\forall i \in I) \ x_i = J_{\mu_i^{-1}A_i} \left(\sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} x_j \right) \right\},$$

turns out to be fundamental in describing $\text{Fix } J_A$.

Theorem 2.1 (correspondence between \mathbf{S} and $\text{Fix } J_A$) *The operator*

$$(19) \quad L: \mathbf{S} \rightarrow \text{Fix } J_A: \mathbf{x} = (x_i)_{i \in I} \mapsto \sum_{i \in I} \lambda_i x_i$$

is well defined, bijective, and Lipschitz continuous with constant 1. Furthermore, the inverse operator of L satisfies

$$(20) \quad L^{-1}: \text{Fix } J_A \rightarrow \mathbf{S}: x \mapsto (J_{A_i} x)_{i \in I}$$

and L^{-1} is Lipschitz continuous with constant \sqrt{m} .

Proof. We proceed along several steps.

Claim 1: $(\forall \mathbf{x} \in \mathbf{S}) \ L\mathbf{x} \in \text{Fix } J_A$ and $\mathbf{x} = (J_{A_i} L\mathbf{x})_{i \in I}$; consequently, L is well defined.

Let $\mathbf{x} = (x_i)_{i \in I} \in \mathbf{S}$ and set $\bar{x} = \sum_{i \in I} \lambda_i x_i = L\mathbf{x}$. Using the definition of the resolvent, we have, for every $i \in I$,

$$\begin{aligned}
(21a) \quad & \sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} x_j \in (\text{Id} + \mu_i^{-1} A_i) x_i \Leftrightarrow \sum_{j \in I \setminus \{i\}} \lambda_j x_j \in \mu_i x_i + A_i x_i \\
(21b) \quad & \Leftrightarrow \sum_{j \in I \setminus \{i\}} \lambda_j x_j \in (1 - \lambda_i) x_i + A_i x_i \\
(21c) \quad & \Leftrightarrow \bar{x} = \sum_{j \in I} \lambda_j x_j \in (\text{Id} + A_i) x_i \\
(21d) \quad & \Leftrightarrow x_i = J_{A_i} \bar{x} = J_{A_i} L\mathbf{x}.
\end{aligned}$$

Hence $\mathbf{x} = (J_{A_i} L\mathbf{x})_{i \in I}$, as claimed. Moreover, $(\forall i \in I) \lambda_i x_i = \lambda_i J_{A_i} \bar{x}$, which, after summing over $i \in I$ and recalling (15), yields $\bar{x} = \sum_{i \in I} \lambda_i x_i = \sum_{i \in I} \lambda_i J_{A_i} \bar{x} = J_A \bar{x}$. Thus $L\mathbf{x} = \bar{x} \in \text{Fix } J_A$ and Claim 1 is verified.

Claim 2: $(\forall x \in \text{Fix } J_A) (J_{A_i} x)_{i \in I} \in \mathbf{S}$.

Assume that $x \in \text{Fix } J_A$ and set $(\forall i \in I) y_i = J_{A_i} x$. Then, using (15), we see that

$$(22) \quad \sum_{i \in I} \lambda_i y_i = \sum_{i \in I} \lambda_i J_{A_i} x = J_A x = x.$$

Furthermore, for every $i \in I$, and using (22) in the derivation of (23c)

$$\begin{aligned}
(23a) \quad & y_i = J_{A_i} x \Leftrightarrow x \in y_i + A_i y_i \Leftrightarrow x - \lambda_i y_i \in \mu_i y_i + A_i y_i \\
(23b) \quad & \Leftrightarrow \mu_i^{-1} (x - \lambda_i y_i) \in (\text{Id} + \mu_i^{-1} A_i) y_i \\
(23c) \quad & \Leftrightarrow \mu_i^{-1} \sum_{j \in I \setminus \{i\}} \lambda_j y_j \in (\text{Id} + \mu_i^{-1} A_i) y_i \\
(23d) \quad & \Leftrightarrow y_i = J_{\mu_i^{-1} A_i} \left(\sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} y_j \right).
\end{aligned}$$

Thus, $(y_i)_{i \in I} \in \mathbf{S}$ and Claim 2 is verified.

Having verified the two claims above, we now turn to proving the statements announced.

First, let $x \in \text{Fix } J_A$. By Claim 2, $(J_{A_i} x)_{i \in I} \in \mathbf{S}$. Hence $L(J_{A_i} x)_{i \in I} = \sum_{i \in I} \lambda_i J_{A_i} x = J_A x = x$ by (15). Thus, L is surjective.

Second, assume that $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ belong to \mathbf{S} and that $L\mathbf{x} = L\mathbf{y}$. Then, using Claim 1, we see that $\mathbf{x} = (J_{A_i} L\mathbf{x})_{i \in I} = (J_{A_i} L\mathbf{y})_{i \in I} = \mathbf{y}$ and thus L is injective. Altogether, this shows that L is bijective and we also obtain the formula for L^{-1} .

Third, again let $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ be in S . Using the convexity of $\|\cdot\|^2$, we obtain

$$(24a) \quad \|L\mathbf{x} - L\mathbf{y}\|^2 = \left\| \sum_{i \in I} \lambda_i (x_i - y_i) \right\|^2 \leq \sum_{i \in I} \lambda_i \|x_i - y_i\|^2$$

$$(24b) \quad \leq \sum_{i \in I} \|x_i - y_i\|^2 = \|\mathbf{x} - \mathbf{y}\|^2.$$

Thus, L is Lipschitz continuous with constant 1.

Finally, let x and y be in $\text{Fix } J_A$. Since J_{A_i} is (firmly) nonexpansive for all $i \in I$, we estimate

$$(25a) \quad \|L^{-1}x - L^{-1}y\|^2 = \left\| (J_{A_i}x)_{i \in I} - (J_{A_i}y)_{i \in I} \right\|^2 = \sum_{i \in I} \|J_{A_i}x - J_{A_i}y\|^2$$

$$(25b) \quad \leq \sum_{i \in I} \|x - y\|^2 = m\|x - y\|^2.$$

Therefore, L^{-1} is Lipschitz continuous with constant \sqrt{m} . ■

Remark 2.2 Some comments regarding Theorem 2.1 are in order.

- (i) Because of the simplicity of the bijection L provided in Theorem 2.1, the task of finding $\text{Fix } J_A$ is essentially the same as finding \mathbf{S} .
- (ii) Note that when each A_i is a normal cone operator N_{C_i} , then the resolvents J_{A_i} and $J_{\mu_i^{-1}A_i}$ simplify to the projections P_{C_i} , for every $i \in I$.
- (iii) When $m = 2$, the set \mathbf{S} turns into

$$(26) \quad \mathbf{S} = \{(x_1, x_2) \in \mathbf{X} \mid x_1 = J_{\lambda_2^{-1}A_1}x_2 \text{ and } x_2 = J_{\lambda_1^{-1}A_2}x_1\},$$

and Theorem 2.1 coincides with [34, Theorem 3.6]. Note that $(x_1, x_2) \in \mathbf{S}$ if and only if $x_2 \in \text{Fix}(J_{\lambda_1^{-1}A_2}J_{\lambda_2^{-1}A_1})$ and $x_1 = J_{\lambda_2^{-1}A_1}x_2$, which makes the connection between the fixed point set of the composition of the two resolvents and \mathbf{S} . It appears that this is a particularity of the case $m = 2$; it seems that there is no simple connection between fixed points of $J_{\mu_m^{-1}A_m}J_{\mu_{m-1}^{-1}A_{m-1}} \cdots J_{\mu_1^{-1}A_1}$ and $\text{Fix } J_A$ when $m \geq 3$.

3 Fixed Points of a Composition

From now on, we let

$$(27) \quad \mathbf{R}: \mathbf{X} \rightarrow \mathbf{X}: \mathbf{x} = (x_i)_{i \in I} \mapsto \left(\sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} x_j \right)_{i \in I}$$

and

$$(28) \quad \mathbf{J}: \mathbf{X} \rightarrow \mathbf{X}: \mathbf{x} = (x_i)_{i \in I} \mapsto \left(J_{\mu_i^{-1} A_i} x_i \right)_{i \in I}.$$

It is immediate from the definition of the set \mathbf{S} (see (18)) that

$$(29) \quad \mathbf{S} = \text{Fix}(\mathbf{J} \circ \mathbf{R}).$$

We are thus ultimately interested in developing algorithms for finding a fixed point of $\mathbf{J} \circ \mathbf{R}$. We start by collecting relevant information about the operator \mathbf{R} .

Proposition 3.1 *The adjoint of \mathbf{R} is given by*

$$(30) \quad \mathbf{R}^*: \mathbf{X} \rightarrow \mathbf{X}: \mathbf{x} = (x_i)_{i \in I} \mapsto \left(\sum_{j \in I \setminus \{i\}} \frac{\lambda_i}{\mu_j} x_j \right)_{i \in I}$$

and the set of fixed points of \mathbf{R} is the “diagonal” in \mathbf{X} , i.e.,

$$(31) \quad \text{Fix } \mathbf{R} = \{(x)_{i \in I} \in \mathbf{X} \mid x \in X\}.$$

Proof. Denote the operator defined in (30) by \mathbf{L} , and take $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ in \mathbf{X} . Then

$$(32a) \quad \langle \mathbf{x}, \mathbf{L}\mathbf{y} \rangle = \sum_{i \in I} \langle x_i, (\mathbf{L}\mathbf{y})_i \rangle = \sum_{i \in I} \sum_{j \in I \setminus \{i\}} \frac{\lambda_i}{\mu_j} \langle x_i, y_j \rangle$$

$$(32b) \quad = \sum_{\{(i,j) \in I \times I \mid i \neq j\}} \frac{\lambda_i}{\mu_j} \langle x_i, y_j \rangle$$

$$(32c) \quad = \sum_{j \in I} \sum_{i \in I \setminus \{j\}} \frac{\lambda_i}{\mu_j} \langle x_i, y_j \rangle = \sum_{j \in I} \langle (\mathbf{R}\mathbf{x})_j, y_j \rangle$$

$$(32d) \quad = \langle \mathbf{R}\mathbf{x}, \mathbf{y} \rangle,$$

which shows that $\mathbf{R}^* = \mathbf{L}$ as claimed.

Next, let $x \in X$ and denote the right side of (31) by Δ . Since

$$(33) \quad (\forall i \in I) \quad \sum_{j \in I \setminus \{i\}} \mu_i^{-1} \lambda_j = 1,$$

it is clear that

$$(34) \quad \Delta \subseteq \text{Fix } \mathbf{R}.$$

Now let $\mathbf{x} = (x_i)_{i \in I} \in \text{Fix } \mathbf{R}$ and set $\bar{x} = \sum_{i \in I} \lambda_i x_i$. Then $\mathbf{x} = \mathbf{R}\mathbf{x}$, i.e., for every $i \in I$, we have

$$(35a) \quad x_i = (\mathbf{x})_i = (\mathbf{R}\mathbf{x})_i \Leftrightarrow x_i = \sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} x_j \Leftrightarrow \mu_i x_i = \sum_{j \in I \setminus \{i\}} \lambda_j x_j$$

$$(35b) \quad \Leftrightarrow (1 - \lambda_i) x_i = \sum_{j \in I \setminus \{i\}} \lambda_j x_j \Leftrightarrow x_i = \sum_{j \in I} \lambda_j x_j$$

$$(35c) \quad \Leftrightarrow x_i = \bar{x}$$

by (16). Hence $\mathbf{x} = (\bar{x})_{i \in I} \in \Delta$ and thus

$$(36) \quad \text{Fix } \mathbf{R} \subseteq \Delta.$$

Combining (34) and (36), we obtain (31). ■

Remark 3.2 If $m = 2$, then $\mathbf{R}^* = \mathbf{R}$. However, when $m \geq 3$, one has the equivalence $\mathbf{R}^* = \mathbf{R} \Leftrightarrow (\lambda_i)_{i \in I} = (\frac{1}{m})_{i \in I}$.

The following observation will be useful when discussing nonexpansiveness of \mathbf{R} .

Lemma 3.3 We have $1 \leq m \sum_{i \in I} \lambda_i^2$; furthermore, equality holds if and only if $(\forall i \in I) \lambda_i = \frac{1}{m}$.

Proof. Indeed,

$$(37) \quad 1 = \sum_{i \in I} \lambda_i \cdot 1 \leq \left(\sum_{i \in I} \lambda_i^2 \right)^{1/2} \left(\sum_{i \in I} 1^2 \right)^{1/2} \Leftrightarrow 1 = 1^2 \leq \left(\sum_{i \in I} \lambda_i^2 \right) m,$$

and the result follows from the Cauchy–Schwarz inequality and its characterization of equality. ■

The next result is surprising as it shows that the actual values of the convex parameters λ_i matter when $m \geq 3$.

Proposition 3.4 (nonexpansiveness of \mathbf{R}) *The following hold.*

(i) *If $m = 2$, then $\mathbf{R}: (x_1, x_2) \mapsto (x_2, x_1)$; thus, \mathbf{R} is an isometry and nonexpansive.*

(ii) *If $m \geq 3$, then: \mathbf{R} is nonexpansive if and only if $(\forall i \in I) \lambda_i = \frac{1}{m}$, in which case $\|\mathbf{R}\| = 1$.*

Proof. (i): When $m = 2$, we have $\lambda_1 = \mu_2$ and $\lambda_2 = \mu_1$; thus, the definition of \mathbf{R} (see (27)) yields the announced formula and it is clear that then \mathbf{R} is an isometry and hence nonexpansive.

(ii): Suppose that $m \geq 3$. Assume first that $(\forall i \in I) \lambda_i = \frac{1}{m}$; hence, $\mu_i = 1 - \frac{1}{m} = (m-1)/m$. Then

$$(38) \quad (\forall j \in I) \quad \lambda_j \sum_{i \in I \setminus \{j\}} \frac{1}{\mu_i} = \frac{1}{m} \sum_{i \in I \setminus \{j\}} \frac{1}{(m-1)/m} = 1.$$

Now let $\mathbf{x} = (x_i)_{i \in I} \in \mathbf{X}$. Using the definition of \mathbf{R} (see (27)), the convexity of $\|\cdot\|^2$ in (39b), and (38) in (39e), we obtain

$$(39a) \quad \|\mathbf{R}\mathbf{x}\|^2 = \sum_{i \in I} \|(\mathbf{R}\mathbf{x})_i\|^2 = \sum_{i \in I} \left\| \sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} x_j \right\|^2$$

$$(39b) \quad \leq \sum_{i \in I} \sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} \|x_j\|^2$$

$$(39c) \quad = \sum_{\{(i,j) \in I \times I \mid i \neq j\}} \frac{\lambda_j}{\mu_i} \|x_j\|^2$$

$$(39d) \quad = \sum_{j \in I} \lambda_j \|x_j\|^2 \sum_{i \in I \setminus \{j\}} \frac{1}{\mu_i}$$

$$(39e) \quad = \sum_{j \in I} \|x_j\|^2$$

$$(39f) \quad = \|\mathbf{x}\|^2.$$

Since \mathbf{R} is linear, it follows that \mathbf{R} is nonexpansive; furthermore, since $\text{Fix } \mathbf{R} \neq \{0\}$ by (31), we then have $\|\mathbf{R}\| = 1$.

To prove the remaining implication, we demonstrate the contrapositive and thus assume that

$$(40) \quad (\lambda_i)_{i \in I} \neq \left(\frac{1}{m}\right)_{i \in I}.$$

Take $u \in X$ such that $\|u\| = 1$ and set $(\forall i \in I) x_i = \mu_i u$ and $\mathbf{x} = (x_i)_{i \in I}$. We compute

$$(41a) \quad \|\mathbf{x}\|^2 = \sum_{i \in I} \|x_i\|^2 = \sum_{i \in I} \|\mu_i u\|^2 = \sum_{i \in I} \mu_i^2$$

$$(41b) \quad = \sum_{i \in I} (1 - \lambda_i)^2 = \sum_{i \in I} (1 - 2\lambda_i + \lambda_i^2)$$

$$(41c) \quad = m - 2 + \sum_{i \in I} \lambda_i^2.$$

Using (30), the fact that $\|u\| = 1$, we obtain

$$(42a) \quad \|\mathbf{R}^* \mathbf{x}\|^2 = \sum_{i \in I} \lambda_i^2 \left\| \sum_{j \in I \setminus \{i\}} \mu_j^{-1} x_j \right\|^2 = \sum_{i \in I} \lambda_i^2 \left\| \sum_{j \in I \setminus \{i\}} \mu_j^{-1} \mu_j u \right\|^2$$

$$(42b) \quad = \sum_{i \in I} \lambda_i^2 \|(m-1)u\|^2 = (m-1)^2 \sum_{i \in I} \lambda_i^2$$

Altogether,

$$(43a) \quad \|\mathbf{R}^* \mathbf{x}\|^2 - \|\mathbf{x}\|^2 = 2 - m + ((m-1)^2 - 1) \sum_{i \in I} \lambda_i^2$$

$$(43b) \quad = (m-2) \left(-1 + m \sum_{i \in I} \lambda_i^2 \right).$$

Now $m \geq 3$ implies that $m - 2 > 0$; furthermore, by (40) and Lemma 3.3, $-1 + m \sum_{i \in I} \lambda_i^2 > 0$. Therefore,

$$(44) \quad \|\mathbf{R}^* \mathbf{x}\| > \|\mathbf{x}\|.$$

This implies $\|\mathbf{R}^*\| > 1$ and hence $\|\mathbf{R}\| > 1$ by [25, Theorem 3.9-2]. Since \mathbf{R} is linear, it cannot be nonexpansive. \blacksquare

For algorithmic purposes, nonexpansiveness is a desirable property but it does not guarantee the convergence of the iterates to a fixed point (consider, e.g., $-\text{Id}$). The very useful notion of an averaged mapping, which is intermediate between nonexpansiveness and firm nonexpansiveness, was introduced by Baillon, Bruck, and Reich in [2].

Definition 3.5 (averaged mapping) *Let $T: X \rightarrow X$. Then T is **averaged** if there exist a non-expansive mapping $N: X \rightarrow X$ and $\alpha \in [0, 1[$ such that*

$$(45) \quad T = (1 - \alpha) \text{Id} + \alpha N;$$

if we wish to emphasis the constant α , we say that T is α -averaged.

It is clear from the definition that every averaged mapping is nonexpansive; the converse, however, is false: indeed, $-\text{Id}$ is nonexpansive, but not averaged. It follows from Fact 1.1 that every firmly nonexpansive mapping is $\frac{1}{2}$ -averaged.

The class of averaged mappings is closed under compositions; this is not true for firmly nonexpansive mappings: e.g., consider two projections onto two lines that meet at 0 at a $\pi/4$ angle. Let us record the following well known key properties.

Fact 3.6 *Let $T, T_1,$ and T_2 be mappings from X to X , let α_1 and α_2 be in $[0, 1[$, and let $x_0 \in X$. Then the following hold.*

- (i) *T is firmly nonexpansive if and only if T is $\frac{1}{2}$ -averaged.*
- (ii) *If T_1 is α_1 -averaged and T_2 is α_2 -averaged, then $T_1 \circ T_2$ is α -averaged, where*

$$(46) \quad \alpha = \begin{cases} 0, & \text{if } \alpha_1 = \alpha_2 = 0; \\ \frac{2}{1 + 1/\max\{\alpha_1, \alpha_2\}}, & \text{otherwise} \end{cases}$$

is the harmonic mean of 1 and $\max\{\alpha_1, \alpha_2\}$.

- (iii) *If T_1 and T_2 are averaged, and $\text{Fix}(T_1 \circ T_2) \neq \emptyset$, then $\text{Fix}(T_1 \circ T_2) = \text{Fix}(T_1) \cap \text{Fix}(T_2)$.*
- (iv) *If T is averaged and $\text{Fix } T \neq \emptyset$, then the sequence of iterates $(T^n x_0)_{n \in \mathbf{N}}$ converges weakly² to a point in $\text{Fix } T$; otherwise, $\|T^n x_0\| \rightarrow +\infty$.*

Proof. (i): This is well known and immediate from Fact 1.1.

(ii): The fact that the composition of averaged mappings is again averaged is well known and implicit in the proof of [2, Corollary 2.4]. For the exact constants, see [18, Lemma 2.2] or [4, Proposition 4.32].

(iii): This follows from [12, Proposition 1.1, Proposition 2.1, and Lemma 2.1]. See also [28, Theorem 3] for the case when $\text{Fix } T \neq \emptyset$.

(iv): This follows from [12, Corollary 1.3 and Corollary 1.4]. ■

Theorem 3.7 (averagedness of \mathbf{R}) *The following hold.*

- (i) *If $m = 2$, then \mathbf{R} is not averaged.*
- (ii) *If $m \geq 3$ and $(\forall i \in I) \lambda_i = \frac{1}{m}$, then $\mathbf{R} = (1 - \alpha) \text{Id} + \alpha \mathbf{N}$, where $\alpha = \frac{m}{2m-2}$ and \mathbf{N} is an isometry; in particular, \mathbf{R} is α -averaged.*

²When T is firmly nonexpansive, the weak convergence goes back at least to [11].

Proof. (i): Assume that $m = 2$. By Proposition 3.4(i), $\mathbf{R}: (x_1, x_2) \mapsto (x_2, x_1)$. We argue by contradiction and thus assume that \mathbf{R} is averaged, i.e., there exist a nonexpansive mapping $\mathbf{N}: \mathbf{X} \rightarrow \mathbf{X}$ and $\alpha \in]0, 1[$ such that $\mathbf{R} = (1 - \alpha) \text{Id} + \alpha \mathbf{N}$. Since $\mathbf{R} \neq \text{Id}$, it is clear that $\alpha > 0$. Thus,

$$(47) \quad \mathbf{N}: \mathbf{X} \rightarrow \mathbf{X}: (x_1, x_2) \mapsto \alpha^{-1}(x_2 - x_1 + \alpha x_1, x_1 - x_2 + \alpha x_2).$$

Now take $u \in X$ such that $\|u\| = 1$ and set $\mathbf{x} = (x_1, x_2) = (0, \alpha u)$. Then

$$(48) \quad \|\mathbf{x}\|^2 = \|0\|^2 + \|\alpha u\|^2 = \alpha^2$$

and $\mathbf{N}\mathbf{x} = (u, (\alpha - 1)u)$. Thus,

$$(49) \quad \|\mathbf{N}\mathbf{x}\|^2 = \|u\|^2 + \|(\alpha - 1)u\|^2 = 1 + (1 - \alpha)^2 = \alpha^2 + 2(1 - \alpha) > \alpha^2 = \|\mathbf{x}\|^2.$$

Hence $\|\mathbf{N}\| > 1$ and, since \mathbf{N} is linear, \mathbf{N} cannot be nonexpansive. This contradiction completes the proof of (i).

(ii): Assume that $m \geq 3$ and that $(\forall i \in I) \lambda_i = \frac{1}{m}$. For future reference, we observe that

$$(50) \quad (\forall i \in I)(\forall j \in I) \quad \frac{\lambda_j}{\mu_i} = \frac{\frac{1}{m}}{1 - \frac{1}{m}} = \frac{1}{m - 1}.$$

We start by defining

$$(51) \quad L: \mathbf{X} \rightarrow X: (x_i)_{i \in I} \mapsto \sum_{i \in I} x_i.$$

Then it is easily verified that

$$(52) \quad L^*: X \rightarrow \mathbf{X}: x \mapsto (x)_{i \in I}$$

and hence that

$$(53) \quad L^* L L^* L = m L^* L.$$

Now set

$$(54) \quad \alpha = \frac{m}{2m - 2} \quad \text{and} \quad \mathbf{N} = \alpha^{-1}(\mathbf{R} - (1 - \alpha) \text{Id}).$$

Then $\alpha \in]0, 1[$ and $\mathbf{R} = \alpha \mathbf{N} + (1 - \alpha) \text{Id}$; thus, it suffices to show that \mathbf{N} is an isometry. Note that

$$(55) \quad \alpha - 1 = -\alpha + \frac{1}{m - 1}.$$

Take $\mathbf{x} = (x_i)_{i \in I} \in \mathbf{X}$. Using (27), (50), and (55), we obtain for every $i \in I$,

$$\begin{aligned}
(56a) \quad (\mathbf{N}\mathbf{x})_i &= \alpha^{-1} \left(- (1 - \alpha)x_i + (\mathbf{R}\mathbf{x})_i \right) \\
(56b) \quad &= \alpha^{-1} \left((\alpha - 1)x_i + \sum_{j \in I \setminus \{i\}} \frac{\lambda_j}{\mu_i} x_j \right) \\
(56c) \quad &= \alpha^{-1} \left(-\alpha x_i + \frac{1}{m-1} x_i + \sum_{j \in I \setminus \{i\}} \frac{1}{m-1} x_j \right) \\
(56d) \quad &= \alpha^{-1} \left(-\alpha x_i + \sum_{j \in I} \frac{1}{m-1} x_j \right) \\
(56e) \quad &= -x_i + \frac{\alpha^{-1}}{m-1} L\mathbf{x} \\
(56f) \quad &= -x_i + \frac{2}{m} L\mathbf{x};
\end{aligned}$$

hence, $\mathbf{N}\mathbf{x} = -\mathbf{x} + \frac{2}{m} L^* L \mathbf{x}$. It follows that $\mathbf{N} = -\text{Id} + \frac{2}{m} L^* L$ and thus $\mathbf{N}^* = \mathbf{N}$. Using (53), we now obtain

$$\begin{aligned}
(57a) \quad \mathbf{N}^* \mathbf{N} &= \mathbf{N} \mathbf{N} = \left(-\text{Id} + \frac{2}{m} L^* L \right) \left(-\text{Id} + \frac{2}{m} L^* L \right) \\
(57b) \quad &= \text{Id} - \frac{2}{m} L^* L - \frac{2}{m} L^* L + \frac{4}{m^2} (L^* L L^* L) \\
(57c) \quad &= \text{Id} - \frac{4}{m} L^* L + \frac{4}{m^2} (m L^* L) \\
(57d) \quad &= \text{Id}.
\end{aligned}$$

Therefore, $\|\mathbf{N}\mathbf{x}\|^2 = \langle \mathbf{N}\mathbf{x}, \mathbf{N}\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{N}^* \mathbf{N} \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$ and hence \mathbf{N} is an isometry; in particular, \mathbf{N} is nonexpansive and \mathbf{R} is α -averaged. \blacksquare

We are now in a position to describe the set \mathbf{S} as the fixed point set of an averaged mapping.

Corollary 3.8 *Suppose that $m \geq 3$ and that $(\forall i \in I) \lambda_i = \frac{1}{m}$. Then $\mathbf{J} \circ \mathbf{R}$ is $\frac{2m}{3m-2}$ -averaged and $\text{Fix}(\mathbf{J} \circ \mathbf{R}) = \mathbf{S}$.*

Proof. On the one hand, since \mathbf{J} is clearly firmly nonexpansive, \mathbf{J} is $\frac{1}{2}$ -averaged. On the other hand, by Theorem 3.7(ii), \mathbf{R} is $\frac{m}{2m-2}$ -averaged. Since $0 < \frac{1}{2} < \frac{m}{2m-2}$, it follows from Fact 3.6(ii) that $\mathbf{J} \circ \mathbf{R}$ is α -averaged, where

$$(58) \quad \alpha = \frac{2}{1 + 1/(m/(2m-2))} = \frac{2m}{3m-2}$$

as claimed. To complete the proof, recall (29). \blacksquare

4 Algorithmic Consequences

In Section 2, we saw that $\text{Fix } J_A = L(\mathbf{S})$ (see Theorem 2.1), and in Section 3 we discovered that $\mathbf{S} = \text{Fix}(\mathbf{J} \circ \mathbf{R})$ is the fixed point set of an averaged operator. This analysis leads to new algorithms for finding a point in $\text{Fix } J_A$.

Theorem 4.1 *Suppose that $m \geq 3$ and that $(\forall i \in I) \lambda_i = \frac{1}{m}$. Let $\mathbf{x}_0 = (x_{0,i})_{i \in I} \in \mathbf{X}$ and generate the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ by*

$$(59) \quad (\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1} = (\mathbf{J} \circ \mathbf{R})\mathbf{x}_n.$$

Then exactly one of the following holds.

- (i) $\text{Fix } J_A \neq \emptyset$, $(\mathbf{x}_n)_{n \in \mathbb{N}}$ converges weakly to a point $\mathbf{x} = (x_i)_{i \in I}$ in \mathbf{S} and $(\sum_{i \in I} \lambda_i \mathbf{x}_{n,i})_{n \in \mathbb{N}}$ converges weakly to $\sum_{i \in I} \lambda_i x_i \in \text{Fix } J_A$.
- (ii) $\text{Fix } J_A = \emptyset$ and $\|\mathbf{x}_n\| \rightarrow +\infty$.

Proof. By Theorem 2.1, $\mathbf{S} \neq \emptyset$ if and only if $\text{Fix } J_A \neq \emptyset$. Furthermore, Corollary 3.8 shows that $\mathbf{S} = \text{Fix}(\mathbf{J} \circ \mathbf{R})$, where $\mathbf{J} \circ \mathbf{R}$ is averaged. The result thus follows from Fact 3.6(iv), Theorem 2.1, and the weak continuity of the operator L defined in (19). \blacksquare

Remark 4.2 The assumption that $m \geq 3$ in Theorem 4.1 is critical: indeed, suppose that $m = 2$. Then, by Proposition 3.4(i), $\mathbf{R}: (x_1, x_2) \mapsto (x_2, x_1)$. Now assume further that $A_1 = A_2 \equiv 0$. Then $\mathbf{J} = \text{Id}$ and hence $\mathbf{J} \circ \mathbf{R} = \mathbf{R}$. Thus, if y and z are two distinct points in X and the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is generated by iterating $\mathbf{J} \circ \mathbf{R}$ with a starting point $\mathbf{x}_0 = (y, z)$, then

$$(60) \quad (\forall n \in \mathbb{N}) \quad \mathbf{x}_n = \begin{cases} (y, z), & \text{if } n \text{ is even;} \\ (z, y), & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is a bounded sequence that is not weakly convergent. On the other hand, keeping the assumption $m = 2$ but allowing again for general maximally monotone operators A_1 and A_2 , and assuming that $\text{Fix } J_A \neq \emptyset$, we observe that

$$(61) \quad \mathbf{J} \circ \mathbf{R} \circ \mathbf{J} \circ \mathbf{R}: \mathbf{X} \rightarrow \mathbf{X}: (x_1, x_2) \mapsto (J_{\lambda_2^{-1}A_1} J_{\lambda_1^{-1}A_2} x_1, J_{\lambda_1^{-1}A_2} J_{\lambda_2^{-1}A_1} x_2).$$

Hence, by [34, Theorem 5.3], the *even* iterates of $\mathbf{J} \circ \mathbf{R}$ will converge weakly to point (\bar{x}_1, \bar{x}_2) with $\bar{x}_1 = J_{\lambda_2^{-1}A_1} J_{\lambda_1^{-1}A_2} \bar{x}_1$ and $\bar{x}_2 = J_{\lambda_1^{-1}A_2} J_{\lambda_2^{-1}A_1} \bar{x}_2$. However, $(\bar{x}_1, \bar{x}_2) \notin \mathbf{S}$ in general.

We are grateful to a referee for pointing out that—in the context of proximity operators—the algorithm presented in Theorem 4.1 is a special case of an algorithm developed in [10, Section 5.3]. While the results in [10] are given for proximity operators, an extension to resolvents is possible by reasoning as in [1].

Remark 4.3 Just as the Gauss-Seidel iteration can be viewed as a modification of the Jacobi iteration where new information is immediately utilized (see, e.g., [23, Chapters 4 and 5] and [31, Section 4.1]), we similarly modify the iteration of the operator $\mathbf{J} \circ \mathbf{R}$. To this end, we introduce, for every $k \in I$, the following operators from \mathbf{X} to \mathbf{X} :

$$(62) \quad (\forall \mathbf{x} = (x_i)_{i \in I} \in \mathbf{X}) (\forall i \in I) \quad (\mathbf{R}_k \mathbf{x})_i = \begin{cases} x_i, & \text{if } i \neq k; \\ \sum_{j \in I \setminus \{k\}} \frac{\lambda_j}{\mu_k} x_j, & \text{if } i = k, \end{cases}$$

and

$$(63) \quad (\forall \mathbf{x} = (x_i)_{i \in I} \in \mathbf{X}) (\forall i \in I) \quad (\mathbf{J}_k \mathbf{x})_i = \begin{cases} x_i, & \text{if } i \neq k; \\ J_{\mu_k^{-1} A_k} x_k, & \text{if } i = k. \end{cases}$$

It follows from the definition of \mathbf{S} (see (18)) that $\mathbf{S} = \bigcap_{k \in I} \text{Fix}(\mathbf{J}_k \circ \mathbf{R}_k)$. This implies

$$(64) \quad \mathbf{S} \subseteq \text{Fix}(\mathbf{J}_m \circ \mathbf{R}_m \circ \cdots \circ \mathbf{J}_1 \circ \mathbf{R}_1),$$

and it motivates—but does not justify—to iterate the composition

$$(65) \quad \mathbf{T} = \mathbf{J}_m \circ \mathbf{R}_m \circ \cdots \circ \mathbf{J}_1 \circ \mathbf{R}_1$$

in order to find points in \mathbf{S} . In general, the composition \mathbf{T} is not nonexpansive even when $(\forall k \in I) A_k \equiv 0$ so that $\mathbf{J}_k = \text{Id}$. Then $\mathbf{T} = \mathbf{R}_m \circ \cdots \circ \mathbf{R}_1$ and this composition is not nonexpansive and neither is any \mathbf{R}_k (see [6, Proof of Remark 4.3 in Appendix B]). One may verify that $\mathbf{J}_k \circ \mathbf{R}_k$ is Lipschitz continuous with constant $\sqrt{m/(m-1)}$ when $(\lambda_i)_{i \in I} = (\frac{1}{m})_{i \in I}$ (see [6, Proof of Remark 4.4 in Appendix B]). In turn, this implies that \mathbf{T} is Lipschitz continuous with constants $(m/(m-1))^{m/2}$. Using elementary calculus, one verifies that when $m \rightarrow +\infty$, the Lipschitz constant of \mathbf{T} decreases to $\sqrt{\exp(1)} \approx 1.6487$. Numerical experiments (which are explained in more detail in [6, Section 4]) show that iterating \mathbf{T} is faster than iterating J_A and $\mathbf{J} \circ \mathbf{R}$, which perform quite similarly. Let us now list some open problems. Suppose that $m \geq 3$. We do not know the answers to the following questions.

Q1: Concerning (64), is it actually true that $\mathbf{S} = \text{Fix } \mathbf{T}$?

Q2: Can one give simple sufficient or necessary conditions for the convergence of the heuristic algorithm, i.e., the iteration of \mathbf{T} , when $\text{Fix } \mathbf{T} \neq \emptyset$?

Q3: Under the most general assumption (13), we observed convergence in numerical experiments of the new rigorous algorithm even though there is no underlying theory—see Proposition 3.4(ii) and Theorem 4.1. Can one provide simple sufficient or necessary conditions for the convergence of the sequence defined by (59)?

Finally, **Q1** and **Q2** have affirmative answers when $m = 2$: indeed, one then computes

$$(66) \quad \mathbf{T}: \mathbf{X} \rightarrow \mathbf{X}: (x_1, x_2) \mapsto \left(J_{\lambda_2^{-1}A_1} x_2, J_{\lambda_1^{-1}A_2} J_{\lambda_2^{-1}A_1} x_2 \right)$$

and hence $(x_1, x_2) \in \text{Fix } \mathbf{T}$ if and only if $x_1 = J_{\lambda_2^{-1}A_1} x_2$ and $x_2 = J_{\lambda_1^{-1}A_2} x_1$, which is the same as requiring that $(x_1, x_2) \in \mathbf{S}$. When $\mathbf{S} = \text{Fix } \mathbf{T} \neq \emptyset$, then the iterates of \mathbf{T} converge weakly to a fixed point by [34, Theorem 5.3(i)].

Appendix

Most of this part of the appendix is part of the folklore; however, we include it here for completeness and because we have not quite found a reference that makes all points we wish to stress.

We assume that $m \in \{1, 2, \dots\}$, that $I = \{1, 2, \dots, m\}$ and that $(C_i)_{i \in I}$ is a family of closed hyperplanes given by

$$(67) \quad (\forall i \in I) \quad C_i = \{x \in X \mid \langle a_i, x \rangle = b_i\}, \quad \text{where } a_i \in X \setminus \{0\} \text{ and } b_i \in \mathbb{R},$$

with corresponding projections P_i . Set $A: X \rightarrow \mathbb{R}^m: x \mapsto (\langle a_i, x \rangle)_{i \in I}$ and $b = (b_i)_{i \in I} \in \mathbb{R}^m$. Then $A^*: \mathbb{R}^m \rightarrow X: (y_i)_{i \in I} \mapsto \sum_{i \in I} y_i a_i$. Denote, for every $i \in I$, the i^{th} unit vector in \mathbb{R}^m by e_i , and the projection $\mathbb{R}^m \rightarrow \mathbb{R}^m: y \mapsto \langle y, e_i \rangle e_i$ onto $\mathbb{R} e_i$ by Q_i . Note that

$$(68) \quad \sum_{i \in I} Q_i = \text{Id} \quad \text{and} \quad (\forall (i, j) \in I \times I) \quad Q_i Q_j = \begin{cases} Q_i, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

We now assume that

$$(69) \quad (\forall i \in I) \quad \|a_i\| = 1,$$

which gives rise to the pleasant representation of the projectors as

$$(70) \quad (\forall i \in I) \quad P_i: x \mapsto x - A^* Q_i (Ax - b)$$

and to (see (3))

$$(71) \quad (\forall x \in X) \quad \|Ax - b\|^2 = \sum_{i \in I} |\langle a_i, x \rangle - b_i|^2 = \sum_{i \in I} \|x - P_i x\|^2.$$

Now let $x \in X$. Using (70) and (68), we thus obtain the following characterization of fixed points of averaged projections:

$$(72a) \quad x \in \text{Fix} \left(\sum_{i \in I} \lambda_i P_i \right)$$

$$(72b) \quad \Leftrightarrow x = \sum_{i \in I} \lambda_i P_i x$$

$$(72c) \quad \Leftrightarrow x = \sum_{i \in I} \lambda_i (x - A^* Q_i (Ax - b))$$

$$(72d) \quad \Leftrightarrow x = \left(\sum_{i \in I} \lambda_i x \right) - A^* \sum_{i \in I} \lambda_i Q_i (Ax - b)$$

$$(72e) \quad \Leftrightarrow A^* \left(\sum_{i \in I} \lambda_i Q_i \right) (Ax - b) = 0$$

$$(72f) \quad \Leftrightarrow A^* \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) (Ax - b) = 0$$

$$(72g) \quad \Leftrightarrow A^* \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) Ax = A^* \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) b$$

$$(72h) \quad \Leftrightarrow \left(\left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) A \right)^* \left(\left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) A \right) x = \left(\left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) A \right)^* \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) b$$

$$(72i) \quad \Leftrightarrow x \text{ satisfies the normal equation of the system}$$

$$(72j) \quad \left(\left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) A \right) x = \left(\sum_{i \in I} \sqrt{\lambda_i} Q_i \right) b$$

$$(72k) \quad \Leftrightarrow \left(\left(\sum_{i \in I} \sqrt{m \lambda_i} Q_i \right) A \right)^* \left(\left(\sum_{i \in I} \sqrt{m \lambda_i} Q_i \right) A \right) x$$

$$(72l) \quad = \left(\left(\sum_{i \in I} \sqrt{m \lambda_i} Q_i \right) A \right)^* \left(\sum_{i \in I} \sqrt{m \lambda_i} Q_i \right) b$$

$$(72m) \quad \Leftrightarrow x \text{ satisfies the normal equation of the system}$$

$$(72n) \quad \left(\left(\sum_{i \in I} \sqrt{m \lambda_i} Q_i \right) A \right) x = \left(\sum_{i \in I} \sqrt{m \lambda_i} Q_i \right) b.$$

Note that when $(\lambda_i)_{i \in I} = (\frac{1}{m})_{i \in I}$, i.e., we have equal weights, then (72) and (68) yield

$$(73a) \quad x \in \text{Fix} \left(\frac{1}{m} \sum_{i \in I} P_i \right) \Leftrightarrow \left(\left(\sum_{i \in I} Q_i \right) A \right)^* \left(\left(\sum_{i \in I} Q_i \right) A \right) x = \left(\left(\sum_{i \in I} Q_i \right) A \right)^* \left(\sum_{i \in I} Q_i \right) b$$

$$(73b) \quad \Leftrightarrow A^* Ax = A^* b$$

$$(73c) \quad \Leftrightarrow x \text{ satisfies the normal equation of the system } Ax = b$$

$$(73d) \quad \Leftrightarrow x \text{ is a least squares solution of the system } Ax = b.$$

In other words, the fixed points of the equally averaged projections onto hyperplanes are precisely the classical **least squares solutions** encountered in linear algebra, i.e., the solutions to the classical **normal equation** $A^* Ax = A^* b$ of the system $Ax = b$. The idea of least squares solutions goes back to the famous prediction of the asteroid Ceres due to Carl Friedrich Gauss in 1801 (see [7, Subsection 1.1.1] and also [26, Epilogue in Section 4.6]).

Example. Consider the following inconsistent linear system of equations

$$(74a) \quad x = 1$$

$$(74b) \quad x = 2,$$

which was also studied by Byrne [15, Subsection 8.3.2 on page 100]. Here $m = 2$ and (69) holds, and the above discussion yields that $\text{Fix}(\frac{1}{2}P_1 + \frac{1}{2}P_2)$ and the set of least squares solutions coincide, namely with the singleton $\{\frac{3}{2}\}$. Now change the representation to

$$(75a) \quad 2x = 2$$

$$(75b) \quad x = 2,$$

so that (69) is violated. The set of fixed points remains unaltered as the two hyperplanes C_1 and C_2 are unchanged and thus it equals $\{\frac{3}{2}\}$. However, the set of least squares solutions is now $\{\frac{6}{5}\}$. Similarly and returning to the first representation in (74), the set of fixed points will change if we consider different weights, say $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{2}{3}$: indeed, we then obtain $\text{Fix}(\frac{1}{3}P_1 + \frac{2}{3}P_2) = \{\frac{5}{3}\}$ while the set of least squares solutions is still $\{\frac{3}{2}\}$.

Acknowledgments

The authors thank the referees for their thoughtful and constructive comments, and for bringing [1], [10], and [23] to our attention. Heinz Bauschke was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program. Xianfu Wang was partially supported by the Natural Sciences and Engineering Research Council of Canada. Calvin Wylie was partially supported by the Irving K. Barber Endowment Fund.

References

- [1] H. Attouch, L.M. Briceño-Arias, and P.L. Combettes, A parallel splitting method for coupled monotone inclusions, *SIAM Journal on Control and Optimization* 48 (2010), 3246–3270.
- [2] J.B. Baillon, R.E. Bruck, and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston Journal of Mathematics* 4 (1978), 1–9.
- [3] H.H. Bauschke, J.M. Borwein, and A.S. Lewis, The method of cyclic projections for closed convex sets in Hilbert space, in *Recent Developments in Optimization Theory and Nonlinear Analysis (Jerusalem 1995)*, Y. Censor and S. Reich (editors), Contemporary Mathematics vol. 204, American Mathematical Society, pp. 1–38, 1997.

- [4] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, 2011.
- [5] H.H. Bauschke and M.R. Edwards, A conjecture by De Pierro is true for translates of regular subspaces, *Journal of Nonlinear and Convex Analysis* 6 (2005), 93–116.
- [6] H.H. Bauschke, X. Wang, and C.J.S. Wylie, Fixed points of averages of resolvents: geometry and algorithms, <http://arxiv.org/pdf/1102.1478v1>, February 2011.
- [7] Å. Björck, *Numerical Methods for Least Squares Problems*, SIAM, 1996.
- [8] J.M. Borwein and J.D. Vanderwerff, *Convex Functions*, Cambridge University Press, 2010.
- [9] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland/Elsevier, 1973.
- [10] L.M. Briceño-Arias and P.L. Combettes, Convex variational formulation with smooth coupling for multicomponent signal decomposition and recovery, *Numerical Mathematics: Theory, Methods, and Applications* 2 (2009), 485–508.
- [11] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Mathematische Zeitschrift* 100 (1967), 201–225.
- [12] R.E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston Journal of Mathematics* 3 (1977), 459–470.
- [13] R.S. Burachik and A.N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer-Verlag, 2008.
- [14] C.L. Byrne, *Signal Processing*, AK Peters, 2005.
- [15] C.L. Byrne, *Applied Iterative Methods*, AK Peters, 2008.
- [16] Y. Censor, P.P.B. Eggermont, and D. Gordon, Strong underrelaxation in Kaczmarz’s method for inconsistent systems, *Numerische Mathematik* 41 (1983), 83–92.
- [17] P.L. Combettes, Inconsistent signal feasibility problems: least-squares solutions in a product space, *IEEE Transactions on Signal Processing* 42 (1994), 2955–2966.
- [18] P.L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization* 53 (2004), 475–504.
- [19] A.R. De Pierro, From parallel to sequential projection methods and vice versa in convex feasibility: results and conjectures, in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa 2000)*, D. Butnariu, Y. Censor, and S. Reich (editors), Elsevier, pp. 187–201, 2001.
- [20] J. Eckstein and D.P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Mathematical Programming Series A* 55 (1992), 293–318.

- [21] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.
- [22] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.
- [23] W. Hackbusch, *Iterative Solution of Large Sparse Systems of Equations*, Springer-Verlag, 1994.
- [24] R.A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, 1960.
- [25] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, 1989.
- [26] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.
- [27] G.J. Minty, Monotone (nonlinear) operators in Hilbert spaces, *Duke Mathematical Journal* 29 (1962), 341–346.
- [28] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bulletin of the American Mathematical Society* 73 (1967), 591–597.
- [29] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
- [30] R.T. Rockafellar and R. J-B Wets, *Variational Analysis*, Springer-Verlag, corrected 3rd printing, 2009.
- [31] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd edition, SIAM, 2003.
- [32] S. Simons, *Minimax and Monotonicity*, Springer-Verlag, 1998.
- [33] S. Simons, *From Hahn-Banach to Monotonicity*, Springer-Verlag, 2008.
- [34] X. Wang and H.H. Bauschke, Compositions and averages of two resolvents: relative geometry of fixed point sets and a partial answer to a question by C. Byrne, *Nonlinear Analysis: Theory, Methods & Applications* 74 (2011), 4550–4572.
- [35] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing, 2002.
- [36] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory I. Projections on convex sets, in *Contributions to Nonlinear Functional Analysis*, E.H. Zarantonello (editor), pp. 237–341, Academic Press, 1971.
- [37] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/A: Linear Monotone Operators*, Springer-Verlag, 1990.
- [38] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators*, Springer-Verlag, 1990.
- [39] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I: Fixed Point Theorems*, Springer-Verlag, 1993.