ATTOUCH-THÉRA DUALITY REVISITED: PARAMONOTONICITY AND OPERATOR SPLITTING

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Abstract

The problem of finding the zeros of the sum of two maximally monotone operators is of fundamental importance in optimization and variational analysis. In this paper, we systematically study Attouch-Théra duality for this problem. We provide new results related to Passty's parallel sum, to Eckstein and Svaiter's extended solution set, and to Combettes' fixed point description of the set of primal solutions. Furthermore, paramonotonicity is revealed to be a key property because it allows for the recovery of *all* primal solutions given just *one arbitrary* dual solution. As an application, we generalize the best approximation results by Bauschke, Combettes and Luke [J. Approx. Theory 141 (2006), 63–69] from normal cone operators to paramonotone operators. Our results are illustrated through numerous examples.

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1 Introduction

Throughout this paper,

$$\boxed{(1) \qquad X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle}$$

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and induced norm $\|\cdot\|$.

Let $A\colon X \rightrightarrows X$ be a set-valued operator, i.e., $(\forall x \in X)\ Ax \subseteq X$. Recall that A is *monotone* if

$$(\forall (x,x^*) \in \operatorname{gr} A)(\forall (y,y^*) \in \operatorname{gr} A) \quad \langle x-y, x^*-y^*\rangle \geq 0 \tag{2}$$

and that A is *maximally monotone* if it is impossible to properly enlarge the graph of A while keeping monotonicity. Monotone operators continue to play an important role in modern optimization and variational analysis; see, e.g., [5], [11], [14], [17], [42], [43], [44], [45], [48], [50], [51], and [52]. This is due to the fact that subdifferential operators of proper lower semicontinuous convex functions are maximally monotone, as are continuous linear operators with a monotone symmetric part. The sum of two maximally monotone operators is monotone, and often maximally monotone if an appropriate constraint qualification is imposed. Finding the zeros of two maximally monotone operators A and B, i.e., determining

$$(A+B)^{-1}0 = \big\{x \in X \bigm| 0 \in Ax+Bx\big\}, \tag{3}$$

is a problem of great interest because it covers constrained convex optimization, convex feasibility, and many others. Attouch and Théra provided [1] a comprehensive study of this (primal) problem in terms of *duality*. Specifically, they associated with the primal problem a dual problem. We set $B^{-\circledcirc} = (-\operatorname{Id})\circ B^{-1}\circ(-\operatorname{Id})$ where $\operatorname{Id}\colon X \to X : x \mapsto x$ is the identity operator. The *Attouch-Théra dual problem* is then to determine

$$\big(A^{-1}+B^{-\circledcirc}\big)^{-1}0 = \big\{x^* \in X \bigm| 0 \in A^{-1}x^* + B^{-\circledcirc}x^*\big\}. \tag{4}$$

(See [36] for the special case of variational inequalities, and also [19] for work on Toland duality.) This duality is very beautiful; e.g., the dual of the dual problem is the primal problem, and the primal problem possesses at least one solution if and only if the same is true for the dual problem.

Our goal in this paper is to systematically study Attouch-Théra duality, to derive new results, and to expose new applications.

Let us now summarize our main results.

- We observe a curious convexity property of the intersection of two sets involving the graphs of A and B (see Theorem 3.3). This relates to Passty's work on the parallel sum as well as to Eckstein and Svaiter's work on the extended solution set.
- We provide a new description of the fixed point set of the Douglas-Rachford splitting operator (see Theorem 4.5); this refines Combettes' description of $(A+B)^{-1}0$.
- We reveal the importance of paramonotonicity: in this case, the fixed point set of the Douglas-Rachford splitting operator is a *rectangle* (see Corollary 5.6) and it is possible to recover *all* primal solutions from *one* dual solution (see Theorem 5.3).
- We generalize the best approximation results by Bauschke-Combettes-Luke from normal cone operators to paramonotone operators with a common zero (see Corollary 6.8 and Theorem 8.1).

The remainder of this paper is organized as follows. In Section 2, we review and slightly refine the basic results on Attouch-Théra duality. The solution mappings between primal and dual solutions are studied in Section 3. Section 4 deals with the Douglas-Rachford splitting operator. The results in Section 5 and Section 6 underline the importance of paramonotonicity in the understanding of the zeros of the sum. Applications to best approximation as well as comments on other duality framework are the topic of the final Section 8.

We conclude this introductory section with some notational comments. The set of zeros of A is written as $\operatorname{zer} A = A^{-1}0$. The *resolvent* and *reflected resolvent* is defined by

$$J_A = (\mathrm{Id}+A)^{-1} \text{ and } R_A = 2J_A - \mathrm{Id}, \tag{5}$$

respectively. It is well known that $\operatorname{zer} A = \operatorname{Fix} J_A := \{x \in X \mid J_A x = x\}$. Moreover, J_A is firmly nonexpansive if and only if R_A is nonexpansive; see, e.g., [23], [32], or [35]. We also have the *inverse resolvent identity*

$$J_A + J_{A^{-1}} = \mathrm{Id} \tag{6}$$

and the following very useful Minty parametrization.

Fact 1.1 (Minty parametrization) *Let $A\colon X \rightrightarrows X$ be maximally monotone. Then $\operatorname{gr} A \to X\colon (a,a^*) \mapsto a + a^*$ is a continuous bijection with continuous inverse $x \mapsto (J_A x, x - J_A x)$; thus,*

$$\operatorname{gr} A = \big\{(J_A x, x - J_A x) \mid x \in X\big\}. \tag{7}$$

Without explicit mentioning it, we employ standard notation from convex analysis (see [42], [43], or [48]). Most importantly, f^* denotes the *Fenchel conjugate* of a function f, and ∂f its *subdifferential operator*. The set of all convex lower semicontinuous proper functions on X is denoted by Γ (or Γ_X if we need to emphasize the space). Finally, we set $f^{\vee} := f \circ (-\mathrm{Id})$, which yields $\partial(f^{\vee}) = (\partial f)^{\circledcirc}$.

2 Duality for monotone operators

In this paper, we study the problem of finding zeros of the sum of maximally monotone operators. More specifically, we assume that

$$A \text{ and } B \text{ are maximally monotone operators on } X. \tag{8}$$

Definition 2.1 (primal problem) *The* primal problem*, for the ordered pair (A,B), is to find the zeros of $A+B$.*

At first, it looks strange to define the primal problem with respect to the (ordered) pair (A,B). The reason we must do this is to associate a *unique* dual problem. (The ambiguity arises because addition is commutative.) It will be quite convenient to set

$$A^{\circledcirc} = (-\mathrm{Id}) \circ A \circ (-\mathrm{Id}). \tag{9}$$

An easy calculation shows that $(A^{-1})^{\ovee} = (A^{\ovee})^{-1}$, which motivates the notation

$$A^{-\ovee} := \big(A^{-1}\big)^{\ovee} = \big(A^{\ovee}\big)^{-1}. \tag{10}$$

(This is similar to the linear-algebraic notation A^{-T} for invertible square matrices.)

Now since A and B form a pair of maximally monotone operators, so do A^{-1} and $B^{-\ovee}$: We thus define the *dual pair*

$$(A,B)^* := (A^{-1}, B^{-\ovee}). \tag{11}$$

The biduality

$$(A,B)^{**} = (A,B) \tag{12}$$

holds, since $(A^{-1})^{-1} = A$, $(B^{\ovee})^{\ovee} = B$, and $(B^{\ovee})^{-1} = (B^{-1})^{\ovee}$.

We are now in a position to formulate the dual problem.

Definition 2.2 ((Attouch-Théra) dual problem) *The* (Attouch-Théra) dual problem*, for the ordered pair* (A,B)*, is to find the zeros of* $A^{-1} + B^{-\ovee}$*. Put differently: the dual problem for* (A,B) *is precisely the primal problem for* $(A,B)^*$*.*

This duality was systematically studied by Attouch and Théra [1]; although, it is worth nothing that it was also touched upon earlier by Mercier [34, page 40]. Additional relevant work can be found in [15] and [21]. In view of (12), it is the clear that the primal problem is precisely the dual of the dual problem, as expected. One central aim of this paper is to understand the interplay between the primal and dual solutions that we formally define next.

Definition 2.3 (primal and dual solutions) *The* primal solutions *are the solutions to the primal problem and analogously for the* dual solutions*. We shall abbreviate these sets by*

$$Z := (A+B)^{-1}(0) \quad \textit{and} \quad K := \big(A^{-1} + B^{-\ovee}\big)^{-1}(0), \tag{13}$$

respectively.

As observed by Attouch and Théra in [1, Corollary 3.2], one has:

$$Z \neq \varnothing \quad \Leftrightarrow \quad K \neq \varnothing. \tag{14}$$

Let us make this simple but important equivalence a little more precise. In order to do so, we define

$$(\forall z \in X) \quad K_z := (Az) \cap (-Bz) \tag{15}$$

and

$$(16) \qquad (\forall k \in X) \quad Z_k := (A^{-1}k) \cap (-B^{-\circledvee}k) = (A^{-1}k) \cap \big(B^{-1}(-k)\big).$$

As the next proposition illustrates, these objects are intimately tied to primal and dual solutions defined in Definition 2.3. This result is elementary and implicitly contained in [1] and [34].

Proposition 2.4 *Let $z \in X$ and let $k \in X$. Then the following hold.*

(i) *K_z and Z_k are closed convex (possibly empty) subsets of X.*

(ii) $k \in K_z \Leftrightarrow z \in Z_k$;

(iii) $z \in Z \Leftrightarrow K_z \neq \varnothing$.

(iv) $\bigcup_{z\in Z} K_z = K$.

(v) $Z \neq \varnothing \Leftrightarrow K \neq \varnothing$.

(vi) $k \in K \Leftrightarrow Z_k \neq \varnothing$.

(vii) $\bigcup_{k\in K} Z_k = Z$.

Proof. (i): Because A and B are maximally monotone, the sets Az and Bz are closed and convex. Hence K_z is also closed and convex. We see analogously that Z_k is closed and convex as well.

(ii): This is easily verified from the definitions.

(iii): Indeed, $z \in Z \Leftrightarrow 0 \in (A+B)z \Leftrightarrow (\exists a^* \in Az \cap (-Bz)) \Leftrightarrow (\exists a^* \in K_z) \Leftrightarrow K_z \neq \varnothing$.

(iv): Take $k \in \bigcup_{z\in Z} K_z$. Then there exists $z \in Z$ such that $k \in K_z = Az \cap (-Bz)$. Hence $z \in A^{-1}k$ and $z \in (-B)^{-1}k = B^{-1}(-\mathrm{Id})^{-1}k = B^{-1}(-k)$. Thus $z \in A^{-1}k$ and $-z \in B^{-\circledvee}k$. Hence $0 \in (A^{-1} + B^{-\circledvee})k$ and so $k \in K$. The reverse inclusion is proved analogously.

(v): Combine (iii) and (iv).

(vi)&(vii): The proofs are analogous to the ones of (iii)&(iv). ■

Let us provide some examples illustrating these notions.

Example 2.5 Suppose that $X = \mathbb{R}^2$, and that we consider the rotators by $\mp\pi/2$, i.e.,

$$(17) \qquad A\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (x_1,x_2) \mapsto (x_2,-x_1) \quad \text{and} \quad B\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (x_1,x_2) \mapsto (-x_2,x_1).$$

Note that $B = -A = A^{-1} = A^*$, where A^* denote the adjoint operator. Hence $A+B \equiv 0$, $Z = X$, and $(\forall z \in Z)$ $K_z = \{Az\} = \{-Bz\}$. Furthermore, $A^{-1} = B$ while the linearity of B implies that $B^{-\circledvee} = B^{-1} = -B = A$. Therefore, $(A,B)^* = (B,A)$. Hence $K = Z$, while $(\forall k \in K)$ $Z_k = \{A^{-1}k\} = \{Bk\}$.

Example 2.6 Suppose that $X = \mathbb{R}^2$, that A is the normal cone operator of $\mathbb{R}^2_+$, and that $B\colon X \to X\colon (x_1,x_2) \mapsto (-x_2,x_1)$ is the rotator by $\pi/2$. As already observed in Example 2.5, we have $B^{-1} = -B$ and $B^{-\circledvee} = B^{-1} = -B$. A routine calculation yields

$$Z = \mathbb{R}_+ \times \{0\}; \tag{18}$$

thus, since B is single-valued,

$$(\forall z = (z_1,0) \in Z) \quad K_z = \big\{-Bz\big\} = \big\{(0,-z_1)\big\}. \tag{19}$$

Thus,

$$K = \bigcup_{z\in Z} K_z = \{0\} \times \mathbb{R}_- \tag{20}$$

and so

$$(\forall k = (0,k_2) \in K) \quad Z_k = \big\{-B^{-\circledvee}k\big\} = \big\{Bk\big\} = \big\{(-k_2,0)\big\}. \tag{21}$$

The dual problem is to find the zeros of $A^{-1} + B^{-\circledvee}$, i.e., the zeros of the sum of the normal cone operator of the negative orthant and the rotator by $-\pi/2$.

Example 2.7 (convex feasibility) Suppose that $A = N_U$ and $B = N_V$, where U and V are closed convex subsets of X such that $U \cap V \neq \varnothing$. Then clearly $Z = U \cap V$. Using [7, Proposition 2.4.(i)], we deduce that $(\forall z \in Z)$ $K_z = N_{\overline{U-V}}(0) = K$. Note that we do know at least one dual solution: $0 \in K$. Thus, by Proposition 2.4(ii)&(vii), $(\forall k \in K)$ $Z_k = Z$.

Remark 2.8 The preceding examples give some credence to the conjecture that

$$\left.\begin{array}{r} z_1 \in Z \\ z_2 \in Z \\ z_1 \neq z_2 \end{array}\right\} \quad \Rightarrow \quad \text{either } K_{z_1} = K_{z_2} \text{ or } K_{z_1} \cap K_{z_2} = \varnothing. \tag{22}$$

Note that (22) is trivially true whenever A or B is at most single-valued. While this conjecture fails in general (see Example 2.9 below), it does, however, hold true for the large class of paramonotone operators (see Theorem 5.3).

Example 2.9 Suppose that $X = \mathbb{R}^2$, and set $U := \mathbb{R} \times \mathbb{R}_+$, $V = \mathbb{R} \times \{0\}$, and $R\colon X \to X\colon (x_1,x_2) \mapsto (-x_2,x_1)$. Now suppose that $A = N_U + R$ and that $B = N_V$. Then $\operatorname{dom} A = U$ and $\operatorname{dom} B = V$; hence, $\operatorname{dom}(A+B) = U \cap V = V$. Let $x = (\xi,0) \in V$. Then $Ax = \{0\} \times \left]-\infty,\xi\right]$ and $Bx = \{0\} \times \mathbb{R}$. Hence $Ax \subset \pm Bx$, $(A+B)x = \{0\} \times \mathbb{R}$ and therefore $Z = V$. Furthermore, $K_x = Ax \cap (-Bx) = Ax$. Now take $y = (\eta,0) \in V = Z$ with $\xi < \eta$. Then $K_x = Ax \subsetneqq Ay = K_y$ and thus (22) fails.

Proposition 2.10 (common zeros) $\operatorname{zer} A \cap \operatorname{zer} B \neq \varnothing \Leftrightarrow 0 \in K$.

Proof. Suppose first that $z \in \operatorname{zer} A \cap \operatorname{zer} B$. Then $0 \in Az$ and $0 \in Bz$, so $0 \in Az \cap (-Bz) = K_z \subseteq K$. Now assume that $0 \in K$. Then $0 \in K_z$, for some $z \in Z$ and so $0 \in Az \cap (-Bz)$. Therefore, $0 \in \operatorname{zer} A \cap \operatorname{zer} B$. ■

Example 2.11 Suppose that $B = A$. Then $Z = \operatorname{zer} A$, and $\operatorname{zer} A \neq \varnothing \Leftrightarrow 0 \in K$.

Proof. Since $2A$ is maximally monotone and $A+A$ is a monotone extension of $2A$, we deduce that $A+A = 2A$. Hence $Z = \operatorname{zer}(2A) = \operatorname{zer} A$ and the result follows from Proposition 2.10. ■

The following result, observed first by Passty, is very useful. For the sake of completeness, we include its short proof.

Proposition 2.12 (Passty) *Suppose that, for every $i \in \{0,1\}$, $w_i \in Ay_i \cap B(x-y_i)$. Then $\langle y_0 - y_1, w_0 - w_1\rangle = 0$.*

Proof. (See [39, Lemma 14].) Since A is monotone, $0 \le \langle y_0 - y_1, w_0 - w_1\rangle$. On the other hand, since B is monotone, $0 \le \langle (x-y_0)-(x-y_1), w_0 - w_1\rangle = \langle y_1 - y_0, w_0 - w_1\rangle$. Altogether, $\langle y_0 - y_1, w_0 - w_1\rangle = 0$. ■

Corollary 2.13 *Suppose that z_1 and z_2 belong to Z, that $k_1 \in K_{z_1}$, and that $k_2 \in K_{z_2}$. Then $\langle k_1 - k_2, z_1 - z_2\rangle = 0$.*

Proof. Apply Proposition 2.12 (with B replaced by $B^{\circledcirc}$ and at $x = 0$). ■

3 Solution mappings K and Z

We now interpret the families of sets $(K_z)_{z\in X}$ and $(Z_k)_{k\in X}$ as set-valued operators by setting

$$\boxed{\mathbf{K}\colon X \rightrightarrows X : z \mapsto K_z \quad\text{and}\quad \mathbf{Z}\colon X \rightrightarrows X : k \mapsto Z_k.} \tag{23}$$

Let us record some basic properties of these fundamental operators.

Proposition 3.1 *The following hold.*

(i) $\operatorname{gr}\mathbf{K} = \operatorname{gr} A \cap \operatorname{gr}(-B)$ *and* $\operatorname{gr}\mathbf{Z} = \operatorname{gr} A^{-1} \cap \operatorname{gr}(-B^{-\circledcirc})$.

(ii) $\operatorname{dom}\mathbf{K} = Z$, $\operatorname{ran}\mathbf{K} = K$, $\operatorname{dom}\mathbf{Z} = K$, *and* $\operatorname{ran}\mathbf{Z} = Z$.

(iii) $\operatorname{gr}\mathbf{K}$ *and* $\operatorname{gr}\mathbf{Z}$ *are closed sets.*

(iv) *The operators* $\mathbf{K}, -\mathbf{K}, \mathbf{Z}, -\mathbf{Z}$ *are monotone.*

(v) $\mathbf{K}^{-1} = \mathbf{Z}$.

Proof. (i): This is clear from the definitions.

(ii): This follows from Proposition 2.4.

(iii): Since A and B are maximally monotone, the sets $\mathrm{gr}A$ and $\mathrm{gr}B$ are closed. Hence, by (i), $\mathrm{gr}\,\mathbf{K}$ is closed and similarly for $\mathrm{gr}\,\mathbf{Z}$.

(iv): Since $\mathrm{gr}\,\mathbf{K} \subseteq \mathrm{gr}A$ and A is monotone, we see that $\mathbf{K}$ is monotone. Similarly, since B is monotone and $\mathrm{gr}(-\mathbf{K}) \subseteq \mathrm{gr}\,B$, we obtain the monotonicity of $-\mathbf{K}$. The proofs for $\pm\mathbf{Z}$ are analogous.

(v): Clear from Proposition 2.4(ii). ■

In Proposition 2.4(iii) we observed the closedness and convexity of K_z and Z_k. In view of Proposition 2.4(iii)&(vii), the sets of primal and dual solutions are both unions of closed convex sets. It would seem that we cannot a priori deduce convexity of these solution sets because unions of convex sets need not be convex. However, not only are Z and K indeed convex, but so are $\mathrm{gr}\,\mathbf{Z}$ and $\mathrm{gr}\,\mathbf{K}$. This surprising result, which is basically contained in works by Passty [39] and by Eckstein and Svaiter [25, 26], is best stated by using the parallel sum, a notion systematically explored by Passty in [39]. See also [31].

Definition 3.2 (parallel sum) *The* parallel sum *of A and B is*

$$A \Box B := (A^{-1} + B^{-1})^{-1}. \tag{24}$$

The notation we use for the parallel sum (see [5, Section 24.4]) is nonstandard but highly convenient: indeed, for sufficiently nice convex functions f and g, one has $\partial(f \Box g) = (\partial f)\Box(\partial g)$ (see [39, Theorem 28], [37, Proposition 4.2.2], or [5, Proposition 24.27]).

The proof of the following result is contained in the proof of [39, Theorem 21], although Passty stated a much weaker conclusion. For the sake of completeness, we present his proof.

Theorem 3.3 *For every $x \in X$, the set*

$$\big(\mathrm{gr}A\big) \cap \big((x,0) - \mathrm{gr}(-B)\big) = \big\{(y,w) \in \mathrm{gr}A \;\big|\; (x-y,w) \in \mathrm{gr}B\big\} \tag{25}$$

is convex.

Proof. (See also [39, Proof of Theorem 21].) The identity (25) is easily verified. To tackle convexity, for every $i \in \{0,1\}$ take (y_i,w_i) from the intersection (25); equivalently,

$$(\forall i \in \{0,1\}) \quad w_i \in Ay_i \cap B(x-y_i). \tag{26}$$

By Proposition 2.12,

$$\langle y_0 - y_1, w_0 - w_1 \rangle = 0. \tag{27}$$

Now let $t \in [0,1]$, set $(y_t,w_t) = (1-t)(y_0,w_0) + t(y_1,w_1)$, and take $(a,a^*) \in \mathrm{gr}A$. Using (26) and the monotonicity of A in (28d), we obtain

$$\langle y_t - a, w_t - a^* \rangle = \langle (1-t)(y_0-a) + t(y_1-a), (1-t)(w_0-a^*) + t(w_1-a^*) \rangle \tag{28a}$$

$$(28b)\qquad = (1-t)^2\langle y_0-a, w_0-a^*\rangle + t^2\langle y_1-a, w_1-a^*\rangle$$
$$(28c)\qquad + (1-t)t\big(\langle y_0-a, w_1-a^*\rangle + \langle y_1-a, w_0-a^*\rangle\big)$$
$$(28d)\qquad \geq (1-t)t\big(\langle y_0-a, w_1-a^*\rangle + \langle y_1-a, w_0-a^*\rangle\big).$$

Thus, using again monotonicity of A and recalling (27), we obtain

$$(29a)\qquad \langle y_0-a, w_1-a^*\rangle + \langle y_1-a, w_0-a^*\rangle = \langle y_0-a, w_1-w_0\rangle + \langle y_0-a, w_0-a^*\rangle$$
$$(29b)\qquad + \langle y_1-a, w_0-w_1\rangle + \langle y_1-a, w_1-a^*\rangle$$
$$(29c)\qquad = \langle y_1-y_0, w_0-w_1\rangle$$
$$(29d)\qquad + \langle y_0-a, w_0-a^*\rangle + \langle y_1-a, w_1-a^*\rangle$$
$$(29e)\qquad \geq \langle y_1-y_0, w_0-w_1\rangle$$
$$(29f)\qquad = 0.$$

Combining (28) and (29), we obtain $\langle y_t-a, w_t-a^*\rangle \geq 0$. Since (a,a^*) is an arbitrary element of $\operatorname{gr}A$ and A is maximally monotone, we deduce that $(y_t,w_t) \in \operatorname{gr}A$. A similar argument yields $(x-y_t,w_t) \in \operatorname{gr}B$. Therefore, (y_t,w_t) is an element of the intersection (25). ■

Before returning to the objects of interest, we record Passty's [39, Theorem 21] as a simple corollary.

Corollary 3.4 (Passty) *For every $x \in X$, the set $(A\square B)x$ is convex.*

Proof. Let $x \in X$. Since $(y,w) \to w$ is linear and $\big\{(y,w) \in \operatorname{gr}A \;\big|\; (x-y,w) \in \operatorname{gr}B\big\}$ is convex (Theorem 3.3), we deduce that

$$(30)\qquad \big\{w \in X \;\big|\; (\exists\, y \in X)\; w \in Ay \cap B(x-y)\big\} \text{ is convex.}$$

On the other hand, a direct computation or [39, Lemma 2] implies that $(A\square B)x = \bigcup_{y\in X}\big(Ay \cap B(x-y)\big)$. Altogether, $(A\square B)x$ is convex. ■

Corollary 3.5 *For every $x \in X$, the set $(\operatorname{gr}A) \cap ((x,0) + \operatorname{gr}(-B))$ is convex.*

Proof. On the one hand, $-\operatorname{gr}(-B^{\circledcirc}) = \operatorname{gr}(-B)$. On the other hand, $B^{\circledcirc}$ is maximally monotone. Altogether, Theorem 3.3 (applied with $B^{\circledcirc}$ instead of B) implies that $(\operatorname{gr}A) \cap ((x,0) - \operatorname{gr}(-B^{\circledcirc})) = (\operatorname{gr}A) \cap ((x,0) + \operatorname{gr}(-B))$ is convex. ■

Remark 3.6 Theorem 3.3 and Corollary 3.5 imply that the intersections $(\operatorname{gr}A) \cap \pm(\operatorname{gr} -B)$ are convex. This somewhat resembles works by Martínez-Legaz (see [33, Theorem 2.1]) and by Zălinescu [49], who encountered convexity when studying the Minkowski sum/difference $(\operatorname{gr}A) \pm (\operatorname{gr} -B)$.

Corollary 3.7 (convexity) *The sets $\operatorname{gr}\mathbf{Z}$ and $\operatorname{gr}\mathbf{K}$ are convex; consequently, Z and K are convex.*

Proof. Combining Proposition 3.1(i) and Corollary 3.5 (with $x = 0$), we obtain the convexity of $\operatorname{gr}\mathbf{K}$. Hence $\operatorname{gr}\mathbf{Z}$ is convex by Proposition 3.1(v). It thus follows that Z and K are convex as images of convex sets under linear transformations. ■

Remark 3.8 Since $Z = (A^{-1}\Box B^{-1})(0)$ and $K = (A\Box B^{\circledcirc})(0)$, the convexity of Z and K also follows from Corollary 3.4.

Remark 3.9 (connection to Eckstein and Svaiter's "extended solution set") In [25, Section 2.1], Eckstein and Svaiter defined in 2008 the *extended solution set* (for the primal problem) by

$$S_e(A,B) := \big\{(z,w) \in X \times X \;\big|\; w \in Bz,\, -w \in Az\big\}. \tag{31}$$

It is clear that $\operatorname{gr}\mathbf{Z}^{-1} = \operatorname{gr}\mathbf{K} = S_e(B,A) = -S_e(A,B)$. Unaware of Passty's work, they proved in [25, Lemma 1 and Lemma 2] (in the present notation) that $Z = \operatorname{ran}\mathbf{Z}$, and that $\operatorname{gr}\mathbf{Z}$ is closed and convex. Their proof is very elegant and completely different from the above Passty-like proof. In their 2009 follow-up paper [26], Eckstein and Svaiter generalize the notion of the extended solution set to three or more operators; their corresponding proof of convexity in [26, Proposition 2.2] is more direct and along Passty's lines. We are grateful to the associate editor for pointing out that the extension to three or more operators in [26] is a product space reformulation of the corresponding result in [25] and that such results are significantly extended in [15] and [21].

Remark 3.10 (convexity of Z and K) If Z is nonempty and a constraint qualification holds, then $A+B$ is maximally monotone (see, e.g., [5, Section 24.1]) and therefore $Z = \operatorname{zer}(A+B)$ is convex. It is somewhat surprising that Z is always convex even without the maximal monotonicity of $A+B$.

One may inquire whether or not Z is also closed, which is another standard property of zeros of maximally monotone operators. The next example illustrates that Z may fail to be closed.

Example 3.11 (Z need not be closed!) Suppose that $X = \ell_2$, the real Hilbert space of square-summable sequences. In [10, Example 3.17], the authors provide a monotone discontinuous linear at most single-valued operator S on X such that S is maximally monotone and its adjoint S^* is a maximally monotone single-valued extension of $-S$. Hence $\operatorname{dom} S$ is not closed. Now assume that $A = S$ and $B = S^*$. Then $A+B$ is operator that is zero on the dense proper subspace $Z = \operatorname{dom}(A+B) = \operatorname{dom} S$ of X. Thus Z fails to be closed. Furthermore, in the language of Passty's parallel sums (see Remark 3.8), this also illustrates that the parallel sum need not map a point to a closed set.

Remark 3.12 We do not know whether or not such counterexamples can reside in finite-dimensional Hilbert spaces when $\operatorname{dom} A \cap \operatorname{dom} B \neq \varnothing$. On the one hand, in view of the forthcoming Corollary 5.5(i), any counterexample must feature at least one operator that is not paramonotone, which means that the operators *cannot* be simultaneously subdifferential operators of functions in Γ. On the other hand, one has to avoid the situation when $A+B$ is maximally monotone, which happens when $\operatorname{ri}\operatorname{dom} A \cap \operatorname{ri}\operatorname{dom} B \neq \varnothing$. This means that neither is one of the operators allowed to have full domain, nor can they simultaneously have relatively open domains, which excludes the situation when both operators are maximally monotone linear relations (i.e., maximally monotone operators with graphs that are linear subspaces, see [9]).

Remark 3.13 We note that $\mathbf{K}$ and $\mathbf{Z}$ are in general *not* maximally monotone. Indeed if $\mathbf{Z}$, say, is maximally monotone, then Corollary 3.7 and [9, Theorem 4.2] imply that $\operatorname{gr}\mathbf{Z}$ is actually *affine* (i.e., a translate of a subspace) and so are Z and K (as range and domain of $\mathbf{Z}$). However, the set Z of Example 2.7 need not be an affine subspace (e.g., when U, V and Z coincide with the closed unit ball in X).

4 Reflected Resolvents and Splitting Operators

We start with some useful identities involving resolvents and reflected resolvents (recall (5)).

Proposition 4.1 *Let $C\colon X \rightrightarrows X$ be maximally monotone. Then the following hold.*

(i) $R_{C^{-1}} = -R_C$.

(ii) $J_{C^{\ovee}} = J_C^{\ovee}$.

(iii) $R_{C^{-\ovee}} = \operatorname{Id} - 2J_C^{\ovee}$.

Proof. (i): By (6), we have $R_{C^{-1}} = 2J_{C^{-1}} - \operatorname{Id} = 2(\operatorname{Id} - J_C) - \operatorname{Id} = \operatorname{Id} - 2J_C = -(2J_C - \operatorname{Id}) = -R_C$.

(ii): Indeed,

$$
\begin{aligned}
J_{C^{\ovee}} &= \big(\operatorname{Id} + (-\operatorname{Id})\circ C \circ (-\operatorname{Id})\big)^{-1} && \text{(32a)}\\
&= \big((-\operatorname{Id})\circ(\operatorname{Id}+C)\circ(-\operatorname{Id})\big)^{-1} && \text{(32b)}\\
&= (-\operatorname{Id})^{-1}\circ(\operatorname{Id}+C)^{-1}\circ(-\operatorname{Id})^{-1} && \text{(32c)}\\
&= (-\operatorname{Id})\circ J_C \circ (-\operatorname{Id}) && \text{(32d)}\\
&= J_C^{\ovee}. && \text{(32e)}
\end{aligned}
$$

(iii): Using (6) and (ii), we have that $R_{C^{-\ovee}} = 2J_{C^{-\ovee}} - \operatorname{Id} = 2(\operatorname{Id} - J_{C^{\ovee}}) - \operatorname{Id} = \operatorname{Id} - 2J_C^{\ovee}$. ■

Corollary 4.2 (Peaceman-Rachford operator is self-dual) (See **Eckstein**'s [22, Lemma 3.5 on page 125].) *The Peaceman-Rachford operators for (A,B) and $(A,B)^* = (A^{-1}, B^{-\ovee})$ coincide, i.e., we have* self-duality *in the sense that*

$$R_B R_A = R_{B^{-\ovee}} R_{A^{-1}}. \tag{33}$$

Consequently,

$$(\forall \lambda \in [0,1]) \quad (1-\lambda)\operatorname{Id} + \lambda R_B R_A = (1-\lambda)\operatorname{Id} + \lambda R_{B^{-\ovee}} R_{A^{-1}}. \tag{34}$$

Proof. Using Proposition 4.1(i)&(iii), we obtain (33) $R_{B^{-\ovee}} R_{A^{-1}} = (\operatorname{Id} - 2J_B^{\ovee})(-R_A) = -R_A + 2J_B R_A = (2J_B - \operatorname{Id})R_A = R_B R_A$. Now (34) follows immediately from (33). ■

Corollary 4.3 (Douglas-Rachford operator is self-dual) (See **Eckstein**'s [22, Lemma 3.6 on page 133].) *For the* Douglas-Rachford operator

$$T_{(A,B)} := \tfrac{1}{2}\operatorname{Id} + \tfrac{1}{2} R_B R_A \tag{35}$$

we have

$$T_{(A,B)} = J_B R_A + \operatorname{Id} - J_A = T_{(A^{-1}, B^{-\ovee})}. \tag{36}$$

Proof. The left equality is a simple expansion while self-duality is (34) with $\lambda = \frac{1}{2}$. ■

Remark 4.4 (backward-backward operator is not self-dual) In contrast to Corollary 4.2, the backward-backward operator is not self-dual: indeed, using (6) and Proposition 4.1(iii), we deduce that

$$J_{B^{-\circledvee}} J_{A^{-1}} = (\mathrm{Id} - J_B^{\circledvee})(\mathrm{Id} - J_A) = \mathrm{Id} - J_A + J_B(J_A - \mathrm{Id}) = (J_B - \mathrm{Id})(J_A - \mathrm{Id}). \tag{37}$$

Thus if $A \equiv 0$ and $\mathrm{dom}\, B$ is not a singleton (equivalently, $J_A = \mathrm{Id}$ and $\mathrm{ran}\, J_B$ is not a singleton), then $J_{B^{-\circledvee}} J_{A^{-1}} \equiv (J_B - \mathrm{Id})0 \equiv J_B 0 \neq J_B = J_B J_A$.

For the rest of this paper, we set

$$\boxed{T = \tfrac{1}{2}\mathrm{Id} + \tfrac{1}{2} R_B R_A = J_B R_A + \mathrm{Id} - J_A.} \tag{38}$$

Clearly,

$$\mathrm{Fix}\, T = \mathrm{Fix}\, R_B R_A. \tag{39}$$

Theorem 4.5 *The mapping*

$$\Psi\colon\ \mathrm{gr}\,\mathbf{K} \to \mathrm{Fix}\, T\colon\ (z,k) \mapsto z + k \tag{40}$$

is a well defined bijection that is continuous in both directions, with $\Psi^{-1}\colon x \mapsto (J_A x, x - J_A x)$.

Proof. Take $(z,k) \in \mathrm{gr}\,\mathbf{K}$. Then $k \in K_z = (Az) \cap (-Bz)$. Now $k \in Az \Leftrightarrow z + k \in (\mathrm{Id} + A)z \Leftrightarrow z = J_A(z+k)$, and $k \in (-Bz) \Leftrightarrow -k \in Bz \Leftrightarrow z - k \in (\mathrm{Id} + B)z \Leftrightarrow z = J_B(z-k)$. Set $x := z + k$. Then $J_A x = J_A(z+k) = z$ and hence $R_A x = 2J_A x - x = 2z - (z+k) = z - k$. Thus,

$$Tx = x - J_A x + J_B R_A x = z + k - z + J_B(z-k) = k + z = x, \tag{41}$$

i.e., $x \in \mathrm{Fix}\, T$. It follows that Ψ is *well defined*.

Let us now show that Ψ is *surjective*. To this end, take $x \in \mathrm{Fix}\, T$. Set $z := J_A x$ as well as $k := (\mathrm{Id} - J_A)x = x - z$. Clearly,

$$x = z + k. \tag{42}$$

Now $z = J_A x \Leftrightarrow x \in (\mathrm{Id} + A)z = z + Az \Leftrightarrow k = x - z \in Az$. Thus,

$$k \in Az. \tag{43}$$

We also have $R_A x = 2J_A x - x = 2z - (z+k) = z - k$; hence, $x = Tx = x - J_A x + J_B R_A x \Leftrightarrow J_A x = J_B R_A x \Leftrightarrow z = J_B(z-k) \Leftrightarrow z - k \in (\mathrm{Id} + B)z = z + Bz \Leftrightarrow$

$$k \in -Bz. \tag{44}$$

Altogether, $k \in (Az) \cap (-Bz) = K_z \Leftrightarrow (z,k) \in \operatorname{gr} \mathbf{K}$. Hence Ψ is surjective.

In view of Fact 1.1 and since $\operatorname{gr} \mathbf{K} \subseteq \operatorname{gr} A$, it is clear that Ψ is *injective* with the announced inverse. ■

The following result is a straight-forward consequence of Theorem 4.5.

Corollary 4.6 *We have*

(45) $$(\forall z \in Z) \quad K_z = J_{A^{-1}}\big(J_A^{-1} z \cap \operatorname{Fix} T\big)$$

and

(46) $$(\forall k \in K) \quad Z_k = J_A\big(J_{A^{-1}}^{-1} k \cap \operatorname{Fix} T\big).$$

Corollary 4.7 (Combettes) (see [20, Lemma 2.6(iii)]) $J_A(\operatorname{Fix} T) = Z$.

Proof. Set $Q \colon X \times X \to X \colon (x_1, x_2) \mapsto x_1$. By Theorem 4.5 and Proposition 3.1(ii), $J_A(\operatorname{Fix} T) = Q \operatorname{ran} \Psi^{-1} = Q \operatorname{dom} \Psi = Q(\operatorname{gr} \mathbf{K}) = \operatorname{dom} \mathbf{K} = Z$. ■

Example 4.8 (see also [6, Fact A1]) Suppose that $A = N_U$ and $B = N_V$, where U and V are closed convex subsets of X such that $U \cap V \neq \varnothing$. Then $P_U(\operatorname{Fix} T) = U \cap V$.

Corollary 4.9 $(\operatorname{Id} - J_A)(\operatorname{Fix} T) = K$.

Proof. Either argue similarly to the proof of Corollary 4.7, or apply Corollary 4.7 to the dual and recall that T is self-dual by Corollary 4.3. ■

5 Paramonotonicity

Definition 5.1 *A monotone operator* $C \colon X \rightrightarrows X$ *is* paramonotone*, if*

(47) $$\left.\begin{array}{r} x^* \in Cx \\ y^* \in Cy \\ \langle x - y, x^* - y^* \rangle = 0 \end{array}\right\} \quad \Rightarrow \quad x^* \in Cy \text{ and } y^* \in Cx.$$

Remark 5.2 Paramonotonicity has proven to be a very useful property for finding solution of variational inequalities by iteration; see, e.g., [30], [18], [16], [38], and [27]. Examples of paramonotone operators abound: indeed, each of the following is paramonotone.

(i) ∂f, where $f \in \Gamma$ [30, Proposition 2.2].

(ii) $C \colon X \rightrightarrows X$, where C is strictly monotone.

(iii) $\mathbb{R}^n \to \mathbb{R}^n \colon x \mapsto Cx + b$, where $C \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $C_+ = \frac{1}{2}C + \frac{1}{2}C^T$, $\ker C_+ \subseteq \ker C$, and C_+ is positive semidefinite [30, Proposition 3.1].

For further examples, see [30]. When C is a continuous linear monotone operator, then C is paramonotone if and only if C is rectangular (a.k.a. 3* monotone); see [3, Section 4]. It is straight-forward to check that for $C\colon X \rightrightarrows X$, we have

$$
\begin{aligned}
C \text{ is paramonotone} &\Leftrightarrow C^{-1} \text{ is paramonotone} \\
&\Leftrightarrow C^{\ovee} \text{ is paramonotone} \\
&\Leftrightarrow C^{-\ovee} \text{ is paramonotone.}
\end{aligned} \tag{48}
$$

Theorem 5.3 *Suppose that A and B are paramonotone. Then $(\forall z \in Z)\ K_z = K$ and $(\forall k \in K)\ Z_k = Z$.*

Proof. Suppose that z_1 and z_2 belong to Z and that $z_1 \neq z_2$. Take $k_1 \in K_{z_1} = Az_1 \cap (-Bz_1)$ and $k_2 \in K_{z_2} = Az_2 \cap (-Bz_2)$. By Corollary 2.13,

$$\langle k_1 - k_2, z_1 - z_2 \rangle = 0. \tag{49}$$

Since A and B are paramonotone, we have $k_2 \in Az_1$ and $-k_2 \in Bz_1$; equivalently, $k_2 \in K_{z_1}$. It follows that $K_{z_2} \subseteq K_{z_1}$. Since the reverse inclusion follows in the same fashion, we see that $K_{z_1} = K_{z_2}$. In view of Proposition 2.4(iv), $K_{z_1} = K$, which proves the first conclusion. Since A and B are paramonotone so are A^{-1} and $B^{-\ovee}$ by (48). Therefore, the second conclusion follows from what we already proved (applied to A^{-1} and $B^{-\ovee}$). ■

Remark 5.4 (recovering *all* primal solutions from *one* dual solution) Suppose that A and B are paramonotone and we know *one* (arbitrary) dual solution, say $k_0 \in K$. Then

$$Z_{k_0} = A^{-1}k_0 \cap \big(B^{-1}(-k_0)\big) \tag{50}$$

recovers the set Z of *all* primal solutions, by Theorem 5.3. If $A = \partial f$ and $B = \partial g$, where f and g belong to Γ, then, since $(\partial f)^{-1} = \partial f^*$ and $(\partial g)^{-1} = \partial g^*$, we obtain a formula well known in *Fenchel duality*, namely,

$$Z = \partial f^*(k_0) \cap \partial g^*(-k_0). \tag{51}$$

We shall revisit this setting in more detail in Section 7. In striking contrast, the complete recovery of all primal solutions from one dual solution is generally impossible when at least one of the operators is no longer paramonotone — see, e.g., Example 2.6 where one of the operators is even a normal cone operator.

Corollary 5.5 *Suppose A and B are paramonotone. Then the following hold.*

(i) *Z and K are closed.*

(ii) $\operatorname{gr}\mathbf{K}$ *and* $\operatorname{gr}\mathbf{Z}$ *are the "rectangles" $Z \times K$ and $K \times Z$, respectively.*

(iii) $\operatorname{Fix} T = Z + K$.

(iv) $(Z - Z) \perp (K - K)$.

(v) $\overline{\operatorname{span}}\,(K - K) = X \Rightarrow Z$ *is a singleton.*

(vi) $\overline{\operatorname{span}}\,(Z-Z)=X \Rightarrow K$ *is a singleton.*

Proof. (i): Combine Theorem 5.3 and Proposition 2.4.

(ii): Clear from Theorem 5.3.

(iii): Combine (ii) with Theorem 4.5.

(iv): Combine Corollary 2.13 with Theorem 5.3.

(v): In view of (iv), we have that $0=\langle Z-Z,K-K\rangle=\langle Z-Z,\overline{\operatorname{span}}\,(K-K)\rangle=\langle Z-Z,X\rangle \Rightarrow Z-Z=\{0\}$ $\Leftrightarrow Z$ is a singleton.

(vi): This is verified analogously to the proof of (v). ■

Corollary 5.6 *Suppose that A and B are paramonotone. Then* $\operatorname{Fix} T=Z+K$*,* $Z=J_A(Z+K)$ *and* $K=J_{A^{-1}}(Z+K)=(\operatorname{Id}-J_A)(Z+K)$.

Proof. Combine Corollary 5.5(ii) with Theorem 4.5. ■

Remark 5.7 (paramonotonicity is critical) Various results in this section — e.g., Theorem 5.3, Remark 5.4, Corollary 5.5(ii)–(vi)— fail if the assumption of paramonotonicity is omitted. To generate these counterexamples, assume that A and B are as in Example 2.5 or Example 2.6.

6 Projection operators and solution sets

The following two facts regarding projection operators will be used in the sequel.

Fact 6.1 (See, e.g., [8, Proposition 2.6].) *Let U and V be nonempty closed convex subsets of X such that $U\perp V$. Then $U+V$ is convex and closed, and $P_{U+V}=P_U+P_V$.*

Fact 6.2 *Let S be a nonempty subset of X, and let $y\in X$. Then* $(\forall x\in X)\ P_{y+S}(x)=y+P_S(x-y)$.

Theorem 6.3 *Suppose that A and B are paramonotone, that $(z_0,k_0)\in Z\times K$, and that $x\in X$. Then the following hold.*

(i) *$Z+K$ is convex and closed.*

(ii) $P_{Z+K}(x)=P_Z(x-k_0)+P_K(x-z_0)$.

(iii) *If $(Z-Z)\perp K$, then* $P_{Z+K}(x)=P_Z(x)+P_K(x-z_0)$.

(iv) *If $Z\perp(K-K)$, then* $P_{Z+K}(x)=P_Z(x-k_0)+P_K(x)$.

Proof. (i): The convexity and closedness of Z and K follows from Corollary 3.7 and Corollary 5.5(i). By Corollary 5.5(iv),

$$(Z-z_0)\perp(K-k_0). \tag{52}$$

Using Fact 6.1,

$$Z+K-z_0-k_0 \text{ is convex and closed, and } P_{Z+K-z_0-k_0}=P_{Z-z_0}+P_{K-k_0}. \tag{53}$$

Hence $Z+K$ is convex and closed. (ii): Using (53), Fact 6.1, and Fact 6.2, we obtain

$$\begin{aligned}
P_{Z+K}x &= P_{(z_0+k_0)+(Z+K-z_0-k_0)}x && \text{(54a)}\\
&= z_0+k_0+P_{(Z-z_0)+(K-k_0)}\big(x-(z_0+k_0)\big) && \text{(54b)}\\
&= z_0+P_{Z-z_0}\big((x-k_0)-z_0\big)+k_0+P_{K-k_0}\big((x-z_0)-k_0\big) && \text{(54c)}\\
&= P_Z(x-k_0)+P_K(x-z_0). && \text{(54d)}
\end{aligned}$$

(iii): Using Fact 6.1 and Fact 6.2, we have

$$\begin{aligned}
P_{Z+K}x &= P_{z_0+(Z+K-z_0)}x && \text{(55a)}\\
&= z_0+P_{(Z-z_0)+K}(x-z_0) && \text{(55b)}\\
&= z_0+P_{Z-z_0}(x-z_0)+P_K(x-z_0) && \text{(55c)}\\
&= P_Zx+P_K(x-z_0). && \text{(55d)}
\end{aligned}$$

(iv): Argue analogously to the proof of (iii). ■

Remark 6.4 Suppose that A and B are paramonotone and that $0\in K$. Then Corollary 5.5(iv) implies that $(Z-Z)\perp K-\{0\}=K$ and we thus may employ either item (ii) (with $k_0=0$) or item (iii) to obtain the formula for P_{Z+K}.

However, if $(Z-Z)\perp K$, then the next two examples show—in strikingly different ways since Z is either large or small—that we cannot conclude that $0\in K$:

Example 6.5 Fix $u\in X$ and suppose that $(\forall x\in X)\ Ax=u$ and $B=-A$. Then A and B are paramonotone, $A+B\equiv 0$, and hence $Z=X$. Furthermore, $K=\{u\}$. Thus if $u\neq 0$, then $K\not\perp X=(Z-Z)$.

Example 6.6 Let U and V be closed convex subsets of X such that

$$0\notin U\cap V \text{ and } U-V=X. \tag{56}$$

(For example, suppose that $X=\mathbb{R}$ and set $U=V=[1,+\infty[$.) Now assume that $(A,B)=(N_U,N_V)^*$. In view of Example 2.7, $K=U\cap V$ and $Z=N_{\overline{U-V}}(0)=N_X(0)=\{0\}$. Hence Z is a singleton and thus $Z-Z=\{0\}\perp K$ while $0\notin K$.

Theorem 6.7 *Suppose that A and B are paramonotone, let $k_0\in K$, and let $x\in X$. Then the following hold.*

(i) $J_A P_{Z+K}(x) = P_Z(x-k_0)$.

(ii) *If* $(Z-Z)\perp K$, *then* $J_A\circ P_{Z+K} = P_Z$.

Proof. Take an arbitrary $z_0\in Z$. (i): Set $z := P_Z(x-k_0)$. Using Theorem 6.3(ii) and Theorem 5.3, we have

$$P_{Z+K}x - z = P_{Z+K}x - P_Z(x-k_0) = P_K(x-z_0)\in K = K_z\subseteq Az. \tag{57}$$

Hence $P_{Z+K}x\in(\mathrm{Id}+A)z \Leftrightarrow z = J_A P_{Z+K}x \Leftrightarrow P_Z(x-k_0) = J_A P_{Z+K}x$.

(ii): This time, let us set $z := P_Z x$. Using Theorem 6.3(iii) and Theorem 5.3, we have

$$P_{Z+K}x - z = P_{Z+K}x - P_Z x = P_K(x-z_0)\in K = K_z\subseteq Az. \tag{58}$$

Hence $P_{Z+K}x\in(\mathrm{Id}+A)z \Leftrightarrow z = J_A P_{Z+K}x \Leftrightarrow P_Z x = J_A P_{Z+K}x$. ■

Corollary 6.8 *Suppose that A and B are paramonotone, and that* $0\in K$. *Then*

$$P_Z = J_A P_{Z+K}. \tag{59}$$

Specializing the previous result to normal cone operators, we recover the consistent case of [7, Corollary 3.9].

Example 6.9 Suppose that $A = N_U$ and $B = N_V$, where U and V are closed convex subsets of X such that $U\cap V\neq\varnothing$. Then $Z = U\cap V$, $K = N_{\overline{U-V}}(0)$, and

$$P_Z = P_U P_{Z+K} = P_U P_{\operatorname{Fix} T}. \tag{60}$$

Proof. This follows from Example 2.7, Corollary 5.5(iii), and Corollary 6.8. ■

7 Subdifferential operators

In this section, we assume that

$$A = \partial f \text{ and } B = \partial g, \tag{61}$$

where f and g belong to Γ. We consider the *primal problem*

$$\underset{x\in X}{\text{minimize}}\;\; f(x)+g(x) \tag{62}$$

the associated *Fenchel dual problem*

$$\underset{x^*\in X}{\text{minimize}}\;\; f^*(x^*)+g^*(-x^*), \tag{63}$$

the *primal* and *dual optimal values*

$$\mu = \inf(f+g)(X) \text{ and } \mu^* = \inf(f^*+g^{*\vee})(X). \tag{64}$$

Note that

$$\mu \geq -\mu^*. \tag{65}$$

Following [12] and [13], we say that *total duality* holds if $\mu = -\mu^* \in \mathbb{R}$, the primal problem (62) has a solution, and the dual problem (63) has a solution.

Theorem 7.1 (total duality) *Suppose that $A = \partial f$ and $B = \partial g$, where f and g belong to Γ. Then $Z \neq \varnothing \Leftrightarrow$* total duality *holds, in which case Z coincides with the set of solutions to the primal problem* (62).

Proof. Observe that $(\partial f)^{-1} = \partial f^*$ and that $(\partial g)^{-\circledcirc} = (\partial g^*)^{\circledcirc} = \partial(g^{*\vee})$.

"$\Rightarrow$": Suppose that $Z \neq \varnothing$, and let $z \in Z$. Then $0 \in \partial f(z) + \partial g(z) \subseteq \partial(f+g)(z)$. Hence z solves the primal problem (62), and

$$\mu = f(z) + g(z). \tag{66}$$

Take $k \in K = K_z = (\partial f)(z) \cap (-\partial g)(z)$. First, we note that $0 \in (\partial f)^{-1}(k) + (\partial g)^{-\circledcirc}(k) = \partial f^*(k) + \partial g^{*\vee}(k) \subseteq \partial(f^* + g^{*\vee})(k)$ and so k solves the Fenchel dual problem (63). Thus,

$$\mu^* = f^*(k) + g^{*\vee}(k). \tag{67}$$

Moreover, $k \in \partial f(z)$ and $-k \in \partial g(z)$, i.e., $f(z) + f^*(k) = \langle z, k\rangle$ and $g(z) + g^*(-k) = \langle z, -k\rangle$. Adding these equations gives $0 = f(z) + f^*(k) + g(z) + g^{*\vee}(k) = \mu + \mu^*$. This verifies total duality.

"$\Leftarrow$": Suppose we have total duality. Then there exists $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and $x^* \in \operatorname{dom} f^* \cap \operatorname{dom} g^{*\vee}$ such that

$$f(x) + g(x) = \mu = -\mu^* = -f^*(x^*) - g^{*\vee}(x^*) \in \mathbb{R}. \tag{68}$$

Hence $0 = (f(x) + f^*(x^*)) + (g(x) + g^*(-x^*)) \geq \langle x, x^*\rangle + \langle x, -x^*\rangle = 0$. Therefore, using convex analysis and Proposition 2.4,

$$\big(x^* \in \partial f(x) \text{ and } -x^* \in \partial g(x)\big) \;\Leftrightarrow\; x^* \in K_x \;\Leftrightarrow\; x \in Z_{x^*}. \tag{69}$$

Hence $x \in Z$.

Note that $Z = \operatorname{zer}(\partial f + \partial g) \subseteq \operatorname{zer} \partial(f+g)$ since $\operatorname{gr}(\partial f + \partial g) \subseteq \operatorname{gr} \partial(f+g)$. Hence Z is a subset of the set of primal solutions. Conversely, if x is a primal solution and x^* is a dual solution, then (68) holds and the rest of the proof of "$\Leftarrow$" shows that $x \in Z$. Altogether, Z coincides with the set of primal solutions. ■

Remark 7.2 (sufficient conditions) On the one hand,

(70) the primal problem has at least one solution

if $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$ and one of the following holds (see, e.g., [5, Corollary 11.15]): (i) f is supercoercive; (ii) f is coercive and g is bounded below; (iii) $0 \in \operatorname{sri}(\operatorname{dom} f^* + \operatorname{dom} g^*)$ (by, e.g., [5, Proposition 15.13] and since (62) is the *Fenchel dual problem* of (63)). On the other hand,

$$\text{the sum rule } \partial(f+g) = \partial f + \partial g \text{ holds} \tag{71}$$

whenever one of the following is satisfied (see, e.g., [5, Corollary 16.38]): (i) (Attouch-Brezis condition) $\mathbb{R}_{++}(\operatorname{dom} f - \operatorname{dom} g)$ is a closed linear subspace; (ii) $\operatorname{dom} f \cap \operatorname{int} \operatorname{dom} g \neq \varnothing$; (iii) $\operatorname{dom} g = X$; (iv) X is finite-dimensional and $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} \operatorname{dom} g \neq \varnothing$. If both (70) and (71) hold, then $Z \neq \varnothing$ and Z coincides with the set of primal solutions. Finally, we are grateful to the associate editor for pointing out that [15, Section 4] contained related results in a more general setting.

8 Algorithms and Eckstein-Ferris-Pennanen-Robinson Duality

In this last section, we sketch first algorithmic consequences and then conclude by commenting on the applicability of our work to a more general duality framework. Recall that

$$T = \tfrac{1}{2}\operatorname{Id} + \tfrac{1}{2} R_B R_A = J_B R_A + \operatorname{Id} - J_A, \tag{72}$$

and that the set of primal solutions Z coincides with $J_A(\operatorname{Fix} T)$ (see Corollary 4.7). This explains the interest in finding fixed points of T. Moreover, if the nearest primal solution is of interest (e.g., the problem of finding the projection onto the intersection of two nonempty closed convex sets), then the following result may be helpful:

Theorem 8.1 (abstract algorithm) *Suppose that A and B are paramonotone. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $(x_n)_{n\in\mathbb{N}}$ converges (weakly or strongly) to $x \in \operatorname{Fix} T$ and $(J_A x_n)_{n\in\mathbb{N}}$ converges (weakly or in norm) to $J_A x$. Then the following hold.*

(i) $(\forall k \in K)\ J_A x = P_Z(x-k)$.

(ii) *If $(Z-Z) \perp K$, then $J_A x = P_Z x$.*

Proof. Combine Corollary 5.5(iii) with Theorem 6.7. ■

We provide three examples.

Example 8.2 (Douglas-Rachford algorithm) Suppose that A and B are paramonotone and that the sequence $(x_n)_{n\in\mathbb{N}}$ is generated by $(\forall n \in \mathbb{N})\ x_{n+1} = Tx_n$. The hypothesis in Theorem 8.1 is satisfied, and the convergence of the sequences is with respect to the *weak topology* [46]. See also [2] for a much simpler proof and [5, Theorem 25.6] for a powerful generalization.

Example 8.3 (Halpern-type algorithm) Suppose that A and B are paramonotone and that the sequence $(x_n)_{n\in\mathbb{N}}$ is generated by $(\forall n \in \mathbb{N})\ x_{n+1} = (1-\lambda_n)Tx_n + \lambda_n y$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of parameters in $]0,1[$ and $y \in X$ is given. Under suitable assumptions on $(\lambda_n)_{n\in\mathbb{N}}$, it is known (see, e.g., [28], [47]) that $x_n \to x := P_{\mathrm{Fix}\,T}y$ with respect to the *norm topology*. Since J_A is (firmly) nonexpansive, it is clear that the hypothesis of Theorem 8.1 holds. Furthermore, $J_Ax_n \to J_Ax = J_AP_{\mathrm{Fix}\,T}y$. Thus, if $k_0 \in K$, then $J_Ax_n \to P_Z(y-k_0)$ by Theorem 6.7(i). And if $(Z-Z) \perp K$, then $J_Ax_n \to P_Zy$ by Theorem 6.7(ii).

Example 8.4 (Haugazeau-type algorithm) This is similar to Example 8.3 in that $x_n \to x := P_{\mathrm{Fix}\,T}y$ with respect to the *norm topology* and where $y \in X$ is given. For the precise description of the (somewhat complicated) update formula for $(x_n)_{n\in\mathbb{N}}$, we refer the reader to [5, Section 29.2] or [4]; see also [29]. Once again, we have $J_Ax_n \to J_Ax = J_AP_{\mathrm{Fix}\,T}y$ and thus, if $k_0 \in K$, then $J_Ax_n \to P_Z(y-k_0)$ by Theorem 6.7(i). And if $(Z-Z) \perp K$, then $J_Ax_n \to P_Zy$ by Theorem 6.7(ii). Consequently, in the context of Example 6.9, we obtain $P_Ux_n \to P_{U\cap V}y$; in fact, this is [8, Theorem 3.3], which is the main result of [8].

Turning to Eckstein-Ferris-Pennanen-Robinson duality, let us assume the following:

- Y is a real Hilbert space (and possibly different from X);
- C is a maximally monotone operator on Y;
- $L\colon X \to Y$ is continuous and linear.

Around the turn of the millennium, Eckstein and Ferris [24], Pennanen [40] as well as Robinson [41] considered the problem of finding zeros of

$$A + L^*CL. \tag{73}$$

This framework is more flexible than the Attouch-Théra framework, which corresponds to the case when $Y = X$ and $L = \mathrm{Id}$. (For an even more general framework, see [21].) Note that just as Attouch-Théra duality relates to classical Fenchel duality in the subdifferential case (see Section 7), the Eckstein-Ferris-Pennanen-Robinson duality pertains to classical *Fenchel-Rockafellar duality* for the problem of minimizing $f + h\circ L$ when $f \in \Gamma_X$ and $h \in \Gamma_Y$, and $A = \partial f$ and $C = \partial h$.

The results in the previous sections can be used in the Eckstein-Ferris-Pennanen-Robinson framework thanks to items (ii) and (iii) of the following result, which allows us to set $B = L^*CL$.

Proposition 8.5 *The following hold.*

(i) *If C is paramonotone, then L^*CL is paramonotone.*

(ii) **(Pennanen)** *If $\mathbb{R}_{++}(\operatorname{ran} L - \operatorname{dom} C)$ is a closed subspace of Y, then L^*CL is maximally monotone.*

(iii) *If C is paramonotone and $\mathbb{R}_{++}(\operatorname{ran} L - \operatorname{dom} C)$ is a closed subspace of Y, then L^*CL is maximally monotone and paramonotone.*

Proof. (i): Take x_1 and x_2 in X, and suppose that $x_1^* \in L^*CLx_1$ and $x_2^* \in L^*CLx_2$. Then there exist $y_1^* \in CLx_1$ and $y_2^* \in CLx_2$ such that $x_1^* = L^*y_1^*$ and $x_2^* = L^*y_2^*$. Thus, $\langle x_1 - x_2, x_1^* - x_2^*\rangle = \langle x_1 - x_2, L^*y_1^* - L^*y_2^*\rangle = \langle Lx_1 - Lx_2, y_1^* - y_2^*\rangle \geq 0$ because C is monotone. Hence L^*CL is monotone. Now suppose furthermore that

$\langle x_1 - x_2, x_1^* - x_2^* \rangle = 0$. Then $\langle Lx_1 - Lx_2, y_1^* - y_2^* \rangle = 0$ and the paramonotonicity of C yields $y_2^* \in C(Lx_1)$ and $y_1^* \in C(Lx_2)$. Therefore, $x_2^* = L^* y_2^* \in L^* CLx_1$ and $x_1^* = L^* y_1^* \in L^* CLx_2$.

(ii): See [40, Corollary 4.4.(c)].

(iii): Combine (i) and (ii). ■

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