Rectangularity and paramonotonicity of maximally monotone operators

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Abstract

Maximally monotone operators play a key role in modern optimization and variational analysis. Two useful subclasses are rectangular (also known as star monotone) and paramonotone operators, which were introduced by Brezis and Haraux, and by Censor, Iusem and Zenios, respectively. The former class has useful range properties while the latter class is of importance for interior point methods and duality theory. Both notions are automatic for subdifferential operators and known to coincide for certain matrices; however, more precise relationships between rectangularity and paramonotonicity were not known.

Our aim is to provide new results and examples concerning these notions. It is shown that rectangularity and paramonotonicity are actually independent. Moreover, for linear relations, rectangularity implies paramonotonicity but the converse implication requires additional assumptions. We also consider continuous linear monotone operators, and we point out that in Hilbert space both notions are automatic for certain displacement mappings.

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1 Introduction

Monotone operators continue to play a fundamental role in optimization and nonlinear analysis as the books [6], [12], [13], [16], [20], [21], [25], [28], [29], [27], [32], [33], and [34] clearly demonstrate. (See also the very recent thesis [30].) Let us start by reminding the reader of the various notions and some key results. To this end, we assume throughout the paper that

$$X \text{ is a real Banach space with norm } \|\cdot\|,$$

that X^* is the (continuous) dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. It will be convenient at times to identify X with its canonical image in the bidual space X^{**}. Furthermore, $X\times X^*$ and $(X\times X^*)^* = X^*\times X^{**}$ are paired via $\langle (x,x^*),(y^*,y^{**})\rangle = \langle x,y^*\rangle + \langle x^*,y^{**}\rangle$, for all $(x,x^*)\in X\times X^*$ and $(y^*,y^{**})\in X^*\times X^{**}$. Now let

$$A\colon X \rightrightarrows X^*$$

be a *set-valued operator* from X to X^*, i.e., $(\forall x\in X)\ Ax\subseteq X^*$. We write $\operatorname{gra} A = \big\{(x,x^*)\in X\times X^* \mid x^*\in Ax\big\}$ for the *graph* of A. The *inverse* of A, written A^{-1} is given by $\operatorname{gra} A^{-1} = \big\{(x^*,x)\in X^*\times X \mid x^*\in Ax\big\}$. The *domain* of A is $\operatorname{dom} A = \big\{x\in X \mid Ax\neq\varnothing\big\}$, while $\operatorname{ran} A = A(X) = \bigcup_{x\in X} Ax$ is the *range* of A, and $\ker A = \{x \mid 0\in Ax\}$ is the *kernel*. Then A is said to be *monotone*, if

$$(1) \qquad \big(\forall (x,x^*)\in\operatorname{gra} A\big)\big(\forall (y,y^*)\in\operatorname{gra} A\big) \quad \langle x-y, x^*-y^*\rangle \geq 0;$$

if A is monotone and it is impossible to properly enlarge A (in the sense of graph inclusion), then A is *maximally monotone*. If the inequality in (1) is strict whenever $x\neq y$, then A is *strictly monotone*. If $(z,z^*)\in X\times X^*$ is a point such that the operator with graph $\{(z,z^*)\}\cup\operatorname{gra} A$ is monotone, then (z,z^*) is *monotonically related* to $\operatorname{gra} A$.

The prime example of a maximally monotone operator is the subdifferential operator of a function on X that is convex, lower semicontinuous and proper. However, not every maximally monotone operator arises in this fashion (e.g., consider a rotator in the Euclidean plane). The books listed above have many results on maximally monotone operators. There are two subclasses of monotone operators that are important especially in optimization, namely rectangular (originally and also known as star or 3^* monotone operators) and paramonotone operators, which were introduced by Brezis and Haraux (see [14]), and by Censor, Iusem and Zenios (see [17] and [23]), respectively. This is due to the fact that the former class has very good range properties while the latter class is important in the study of interior point methods for variational inequalities. (For very recent papers in which these notions play a central role, we refer the reader to [4], [7], and [8].) Before we turn to precise definitions of these notion, we recall key properties of the Fitzpatrick function, which has proven to be a crucial tool in the study of (maximally) monotone operators.

Fact 1.1 (Fitzpatrick) (See [19, Proposition 3.2 and Corollary 3.9].) *Let* $A\colon X \rightrightarrows X^*$ *be monotone such that* $\operatorname{gra} A \neq \varnothing$. *Define the* Fitzpatrick function *associated with* A *by*

$$F_A\colon X \times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \sup_{(a,a^*)\in\operatorname{gra} A} \big(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\big).$$

Then F_A *is proper, lower semicontinuous and convex, and* $F_A = \langle\cdot,\cdot\rangle$ *on* $\operatorname{gra} A$. *If* A *is maximally monotone, then* $\langle\cdot,\cdot\rangle \leq F_A$ *with equality precisely on* $\operatorname{gra} A$.

We now present the formal definitions of rectangularity and paramonotonicity.

Definition 1.2 (rectangular and paramonotone) *Let* $A\colon X \rightrightarrows X$ *be a monotone operator such that* $\operatorname{gra} A \neq \varnothing$. *Then:*

(i) A *is* rectangular (which is also known as $*$ or 3^* monotone; see [14], [29, Definition 31.5], and [34, Definition 32.40(c) on page 901]) *if*

$$\operatorname{dom} A \times \operatorname{ran} A \subseteq \operatorname{dom} F_A.$$

(ii) A *is* paramonotone (see [17], [23] and also [15]) *if the implication*

$$\left.\begin{array}{c}(x,x^*)\in\operatorname{gra} A\\ (y,y^*)\in\operatorname{gra} A\\ \langle x-y,x^*-y^*\rangle = 0\end{array}\right\} \;\Rightarrow\; \big\{(x,y^*),(y,x^*)\big\} \subseteq \operatorname{gra} A$$

holds.

The following two results illustrate that rectangularity and paramonotonicity are automatic for subdifferential operators from convex analysis. The first result is due to Brezis and Haraux, who considered the Hilbert space case in [14] (see [34, Proposition 32.42] for the Banach space version).

Fact 1.3 (Brezis-Haraux) *Let* $f\colon X \to \left]-\infty,+\infty\right]$ *be convex, lower semicontinuous, and proper. Then* ∂f *is rectangular.*

The second fact—in a finite-dimensional setting—is due to Censor, Iusem and Zenios (see [17] and [23], as well as [11] for an extension to Banach space with a different proof).

Fact 1.4 (Censor-Iusem-Zenios) *Let* $f\colon X \to \left]-\infty,+\infty\right]$ *be convex, lower semicontinuous, and proper. Then* ∂f *is paramonotone.*

Besides these results for subdifferential operators, it was known that rectangularity and paramonotonicity coincide for certain matrices (see [2, Remark 4.11]). Thus, previously, it was not clear whether these notions are different.

The aim of this work is to study systematically rectangular and paramonotone operators.

Let us summarize our key findings.

- Rectangularity and paramonotonicity are *independent* notions.
- For linear relations, *rectangularity implies paramonotonicity but not vice versa.*
- For linear relations satisfying certain closure assumptions in reflexive spaces, *rectangularity and paramonotonicity coincide and also hold for the adjoint.*
- For displacement mappings of nonexpansive operators in Hilbert space, *rectangularity and paramonotonicity are automatic.*

The remainder of this paper is organized as follows. In Section 2, we collect auxiliary results for the reader's convenience and future use. General monotone operators are the topic of Section 3 where we also present an example (see Example 3.7) illustrating that rectangularity does not imply paramonotonicity. In Section 4, we focus on linear relations. For this important subclass of monotone operators, we obtain characterizations and relationships to corresponding properties of the adjoint. It is true that rectangularity implies paramonotonicity in this setting (see Proposition 4.5). Our main result (Theorem 4.9) presents a pleasing characterization in the reflexive setting under a mild closedness assumption. The setting of continuous linear monotone operators is considered in Section 5 where we present characterizations of rectangularity and an example of a paramonotone operator that is not rectangular (Example 5.7). In the final Section 6, we consider monotone operators that are displacement mappings of nonexpansive operators in Hilbert space. For this class of operators, rectangularity and paramonotonicity are automatic.

For the convenience of the reader, let us recall some (mostly standard) terminology from convex analysis and the theory of linear relations.

Given a subset C of X, $\operatorname{int} C$ and $\overline{C}$ denote the *interior* and *closure* of C, respectively. For every $x \in X$, the *normal cone* of C at x is defined by $N_C(x) = \big\{x^* \in X^* \mid \sup_{c\in C}\langle c-x, x^*\rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$. Let $f\colon X \to [-\infty,+\infty]$. Then $\operatorname{dom} f = \big\{x \in X \mid f(x) < +\infty\big\})$ is the (essential) *domain* of f, and $f^*\colon X^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}(\langle x, x^*\rangle - f(x))$ is the *Fenchel conjugate* of f. The *epigraph* of f is $\operatorname{epi} f = \big\{(x,r) \in X \times \mathbb{R} \mid f(x) \leq r\big\}$. The *lower semicontinuous hull* of f, denoted by $\overline{f}$, is the function defined at every $x \in X$ by $\overline{f}(x) = \inf\big\{t \in \mathbb{R} \mid (x,t) \in \overline{\operatorname{epi} f}\big\}$. We say that f

is *proper* if $\operatorname{dom} f \neq \varnothing$ and $-\infty \notin \operatorname{ran} f$. Let f be proper. The *subdifferential operator* of f is defined by $\partial f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y)\big\}$.

Let Z be a real Banach space with continuous dual Z^*, let $C \subseteq Z$ and $D \subseteq Z^*$. We write $C^\perp := \big\{z^* \in Z^* \mid (\forall c \in C)\ \ \langle c, z^*\rangle = 0\big\}$ and $D_\perp := D^\perp \cap Z$.

Now let $A\colon X \rightrightarrows X^*$ be a *linear relation*, i.e., $\operatorname{gra} A$ is a linear subspace of X. (See [18] for background material on linear relations.) The *adjoint* of A, written as A^*, is defined by

$$\begin{aligned} \operatorname{gra} A^* &= \big\{(x^{**}, x^*) \in X^{**} \times X^* \mid (x^*, -x^{**}) \in (\operatorname{gra} A)^\perp\big\} \\ &= \big\{(x^{**}, x^*) \in X^{**} \times X^* \mid (\forall (a, a^*) \in \operatorname{gra} A)\ \langle a, x^*\rangle = \langle a^*, x^{**}\rangle\big\}. \end{aligned}$$

Furthermore, A is *symmetric* if $\operatorname{gra} A \subseteq \operatorname{gra} A^*$; similarly, A is *skew* if $\operatorname{gra} A \subseteq \operatorname{gra}(-A^*)$. The *symmetric part* of A is defined by $A_+ = \frac{1}{2}A + \frac{1}{2}A^*$, while the *skew part* of A is $A_\circ = \frac{1}{2}A - \frac{1}{2}A^*$. We denote by Id the *identity mapping* on X. Finally, the closed unit ball in X is denoted by $B_X = \big\{x \in X \mid \|x\| \leq 1\big\}$, and the positive integers by $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2 Auxiliary results

We start with an elementary observation that turned out to be quite useful in [26].

Fact 2.1 *Let α, β, and γ be in $\mathbb{R}$. Then*

$$(\forall t \in \mathbb{R}) \quad \alpha t^2 + \beta t + \gamma \geq 0$$

if and only if $\alpha \geq 0$, $\gamma \geq 0$, and $\beta^2 \leq 4\alpha\gamma$.

Most of our results involve linear relations. The next result holds even without monotonicity.

Fact 2.2 (Cross) *Let $A : X \rightrightarrows X^*$ be a linear relation. Then the following hold:*

(i) $(\forall (x, x^*) \in \operatorname{gra} A)\ Ax = x^* + A0$.

(ii) $(\forall x \in \operatorname{dom} A)(\forall y \in \operatorname{dom} A^*)\ \langle x, A^*y\rangle = \langle y, Ax\rangle$ *is a singleton.*

(iii) $\ker A^* = (\operatorname{ran} A)^\perp$ *and* $\ker \bar{A} = (\operatorname{ran} A^*)_\perp$, *where* $\operatorname{gra} \bar{A} = \overline{\operatorname{gra} A}$.

Proof. (i): See [18, Proposition I.2.8(a)]. (ii): See [18, Proposition III.1.2]. (iii): See [18, Proposition III.1.4(a)&(c)]. ■

For a monotone linear relation $A\colon X \rightrightarrows X^*$ it will be convenient to define—as in, e.g., [2]—the associated quadratic form

$$(\forall x \in X) \quad q_A(x) = \begin{cases} \frac{1}{2}\langle x, Ax\rangle, & \text{if } x \in \operatorname{dom} A; \\ +\infty, & \text{otherwise.} \end{cases}$$

By Fact 2.3(iii), q_A is a well-defined convex function.

Let us record some useful properties of monotone linear relations for future reference.

Fact 2.3 (See [3, Proposition 5.2(iv)&(v), Proposition 5.3, and Proposition 5.4(iv)&(iii)].) *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Then the following hold:*

(i) $A0 = A^*0 = A_+0 = (\operatorname{dom} A)^\perp$ *is (weak*) closed.*

(ii) *If* $\operatorname{dom} A$ *is closed, then* $X \cap \operatorname{dom} A^* = \operatorname{dom} A$.

(iii) q_A *is single-valued and convex.*

(iv) $A^*|_X$ *is monotone.*

(v) *If* $\operatorname{dom} A$ *is closed, then* $A_+ = \partial\overline{q_A}$, *where* $\overline{q_A}$ *is the lower semicontinuous hull of* q_A.

The next result provides a formula connecting the Fitzpatrick function of a given linear relation with that of its adjoint.

Lemma 2.4 *Let $A\colon X \rightrightarrows X^*$ be a linear relation such that* $\operatorname{dom} A = \operatorname{dom}(A^*|_X)$, *and let $z^* \in X^*$. Then*

$$\big(\forall (x, x^*) \in \operatorname{gra} A\big) \quad F_{A^*|_X}(x, z^*) = F_A(0, x^* + z^*) \tag{3a}$$

$$\big(\forall (x, y^*) \in \operatorname{gra}(A^*|_X)\big) \quad F_A(x, z^*) = F_{A^*|_X}(0, y^* + z^*). \tag{3b}$$

Proof. Let $(x, x^*) \in \operatorname{gra} A$. Then

$$\begin{aligned} F_{A^*|_X}(x, z^*) &= \sup_{(a,b^*)\in\operatorname{gra}(A^*|_X)} \big(\langle x, b^*\rangle + \langle a, z^*\rangle - \langle a, b^*\rangle\big) \\ &= \sup_{(a,a^*)\in\operatorname{gra} A} \big(\langle a, x^*\rangle + \langle a, z^*\rangle - \langle a, a^*\rangle\big) \quad \text{(by Fact 2.2(ii))} \\ &= F_A(0, x^* + z^*). \end{aligned}$$

This establishes (3a). The proof of (3b) is similar. ■

Finally, "self-orthogonal" elements of the graph of $-A$ must belong to the graph of A^*.

Lemma 2.5 *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation and suppose that $(a,a^*) \in \operatorname{gra} A$ satisfies $\langle a, a^*\rangle = 0$. Then $(a,-a^*) \in \operatorname{gra} A^*$.*

Proof. Take $(b,b^*) \in \operatorname{gra} A$ and set

$$\delta := \langle a, b^*\rangle + \langle b, a^*\rangle. \tag{5}$$

The monotonicity of A and linearity of $\operatorname{gra} A$ yield

$$(\forall t \in \mathbb{R}) \quad t^2\langle b, b^*\rangle - t\delta = \langle a - tb, a^* - tb^*\rangle \geq 0.$$

By Fact 2.1, $\delta^2 \leq 0$ and thus $\delta = 0$. Therefore, $(a^*, a) \in (\operatorname{gra} A)^{\perp}$ by (5). Hence it follows that $(a, -a^*) \in \operatorname{gra} A^*$. ■

3 General results

The results in this section pertain to general operators. We start with a simple observation concerning the sum of two paramonotone operators.

Proposition 3.1 *Let A and B be paramonotone operators on X. Then $A+B$ is paramonotone.*

Proof. Suppose that $\{(x_1, a_1^*), (x_2, a_2^*)\} \subseteq \operatorname{gra} A$ and $\{(x_1, b_1^*), (x_2, b_2^*)\} \subseteq \operatorname{gra} B$ are such that $\langle x_1 - x_2, (a_1^* + b_1^*) - (a_2^* + b_2^*)\rangle = 0$. Then $\langle x_1 - x_2, a_1^* - a_2^*\rangle + \langle x_1 - x_2, b_1^* - b_2^*\rangle = 0$. Combining with the monotonicity of A and B, we see that

$$\langle x_1 - x_2, a_1^* - a_2^*\rangle = 0 \quad \text{and} \quad \langle x_1 - x_2, b_1^* - b_2^*\rangle = 0. \tag{6}$$

Since A and B are paramonotone and by (6), we have $\{(x_1, a_2^*), (x_2, a_1^*)\} \subseteq \operatorname{gra} A$ and $\{(x_1, b_2^*), (x_2, b_1^*)\} \subseteq \operatorname{gra} B$. Thus $\{(x_1, a_2^* + b_2^*), (x_2, a_1^* + b_1^*)\} \subseteq \operatorname{gra}(A+B)$. Therefore, $A+B$ is paramonotone. ■

The following result, which provides a useful sufficient condition for rectangularity, was first proved by Brezis and Haraux in [14, Example 2] in a Hilbert space setting. In fact, their result holds in general Banach space. For completeness, we include the proof.

Proposition 3.2 *Let $A : X \rightrightarrows X^*$ be monotone and* $\operatorname{gra} A \neq \varnothing$. *Suppose that A is* strongly coercive *in the sense that*

$$(\forall x \in \operatorname{dom} A) \quad \lim_{\rho \to +\infty} \inf_{(a,a^*) \in \operatorname{gra} A \text{ and } \|a\| \geq \rho} \frac{\langle a - x, a^*\rangle}{\|a\|} = +\infty.$$

Then $\operatorname{dom} A \times X^* \subseteq \operatorname{dom} F_A$*; consequently, A is rectangular.*

Proof. Let $(x, y^*) \in \operatorname{dom} A \times X^*$, and set $M := \|y^*\| + 1$. Then there exists $\rho > 0$ such that for every $(a, a^*) \in \operatorname{gra} A$ with $\|a\| \geq \rho$, we have $\langle a - x, a^* \rangle \geq M\|a\|$. Thus,

$$(7) \qquad \big(\forall (a, a^*) \in \operatorname{gra} A\big) \quad \|a\| \geq \rho \ \Rightarrow \ \langle x - a, a^* \rangle \leq -M\|a\|.$$

Fix $x^* \in Ax$. Then, by the monotonicity of A,

$$(8) \qquad \big(\forall (a, a^*) \in \operatorname{gra} A\big) \quad \langle x - a, a^* \rangle \leq \langle x - a, x^* \rangle.$$

Let us now evaluate $F_A(x, y^*)$. Let $(a, a^*) \in \operatorname{gra} A$.

`Case 1:` $\|a\| < \rho$.
Using (8), we estimate $\langle x - a, a^* \rangle + \langle a, y^* \rangle \leq \langle x - a, x^* \rangle + \langle a, y^* \rangle \leq \|x^*\| \cdot \|x - a\| + \|a\| \cdot \|y^*\| \leq \|x^*\|(\|x\| + \|a\|) + \|a\| \cdot \|y^*\| \leq \|x^*\|(\|x\| + \rho) + \rho\|y^*\|$.

`Case 2:` $\|a\| \geq \rho$.
Using (7), we have $\langle x - a, a^* \rangle + \langle a, y^* \rangle \leq -M\|a\| + \|a\| \cdot \|y^*\| = \|a\| \cdot (\|y^*\| - M) = -\|a\| \leq 0$.

Altogether, we conclude that

$$F_A(x, y^*) = \sup_{(a,a^*) \in \operatorname{gra} A} \langle x - a, a^* \rangle + \langle a, y^* \rangle \leq \|x^*\|\big(\|x\| + \rho\big) + \rho\|y^*\|.$$

Hence $\operatorname{dom} A \times X^* \subseteq \operatorname{dom} F_A$ and therefore A is rectangular. ■

Example 3.3 Let $A \colon X \rightrightarrows X^*$ be monotone such that $\operatorname{dom} A$ is nonempty and bounded. Then A is rectangular.

Proof. This is immediate from Proposition 3.2 because $\inf \varnothing = +\infty$ and therefore A is strongly coercive. ■

Here is another sufficient condition for rectangularity that was suggested and proved by a referee.

Lemma 3.4 *Suppose that $A \colon X \rightrightarrows X^*$ satisfies $\operatorname{gra} A \neq \varnothing$ and*

$$(9) \qquad \big(\forall (x, x^*) \in \operatorname{gra} A)\big)\big(\forall (y, y^*) \in \operatorname{gra} A)\big) \quad \langle x - y, x^* - y^* \rangle \geq \tfrac{1}{2}\|x^* - y^*\|^2.$$

Then A is monotone, $X \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$, and A is rectangular.

Proof. The monotonicity of A is clear from (9). Now let $(z, x^*) \in X \times \operatorname{ran} A$. Then there exists $x \in A^{-1}x^*$. For every $(y, y^*) \in \operatorname{gra} A$, we obtain with (9) that

$$\begin{aligned} \langle z, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle &= \langle z, x^* \rangle + \langle x - z, x^* - y^* \rangle - \langle x - y, x^* - y^* \rangle \\ &\leq \langle z, x^* \rangle + \|x - z\|\|x^* - y^*\| - \tfrac{1}{2}\|x^* - y^*\|^2 \\ &\leq \langle z, x^* \rangle + \tfrac{1}{2}\|x - z\|^2. \end{aligned}$$

Thus $F_A(z,x^*) \leq \langle z, x^*\rangle + \frac{1}{2}\|x-z\|^2 < +\infty$ and so $X \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$. Consequently, A is rectangular. ■

Let us show how to construct a maximally monotone operator that is rectangular but not paramonotone; before we do so, we recall the following sufficient condition for maximal monotonicity.

Fact 3.5 (See [31, Theorem 3.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $f : X \to \left]-\infty, +\infty\right]$ be convex, lower semicontinuous, and proper with $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$. Then $A + \partial f$ is maximally monotone.*

Proposition 3.6 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let C be a bounded closed convex subset of X such that $0 \in \operatorname{int} C$. Then $A + N_C$ is maximally monotone and rectangular. If A is not paramonotone, then neither is $A + N_C$.*

Proof. Set $B = A + N_C$. By Fact 3.5 or [10, Theorem 3.1], B is maximally monotone. Since $\operatorname{dom} B = \operatorname{dom} A \cap C \subseteq C$ is bounded, we deduce from Example 3.3 that B is rectangular.

Now assume in addition that A is not paramonotone. In view of Lemma 4.3 below, there exists $(a, a^*) \in \operatorname{gra} A$ such that

$$\langle a, a^*\rangle = 0 \quad \text{but} \quad a^* \notin A0. \tag{10}$$

Since $0 \in \operatorname{int} C$, there exists $\delta > 0$ such that $\delta a \in \operatorname{int} C$. The linearity of $\operatorname{gra} A$ yields $(\delta a, \delta a^*) \in \operatorname{gra} A$. Hence $\{(\delta a, \delta a^*), (0,0)\} \subseteq \operatorname{gra} B$, and $\langle \delta a - 0, \delta a^* - 0\rangle = 0$ by (10). However, (10) implies that $\delta a^* \notin A0 = B0$. Therefore, B is not paramonotone. ■

We conclude this section with our first counterexample.

Example 3.7 (rectangular $\not\Rightarrow$ paramonotone in the general case) Suppose that $X = \mathbb{R}^2$ and set

$$A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $A + N_{B_X}$ is maximally monotone and rectangular, but not paramonotone.

Proof. Clearly, $0 \in \operatorname{int} B_X$. The lack of paramonotonicity of A is a consequence of Lemma 4.3 below. The conclusion therefore follows from Proposition 3.6. ■

4 Linear relations

In this section, we focus exclusively on linear relations. We start with characterizations of rectangularity and paramonotonicity. These yield information about corresponding properties of the adjoint.

Lemma 4.1 (characterization of rectangularity) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then A is rectangular $\Leftrightarrow$ $\operatorname{dom} A \times \{0\} \subseteq \operatorname{dom} F_A$.*

Proof. "$\Rightarrow$": Clear. "$\Leftarrow$": Let $x \in \operatorname{dom} A$ and $(y, y^*) \in \operatorname{gra} A$. Then

$$\begin{aligned}
F_A(x, y^*) &= F_A\big(\tfrac{1}{2}(2y, 2y^*) + \tfrac{1}{2}(2x - 2y, 0)\big)\\
&\leq \tfrac{1}{2}F_A(2y, 2y^*) + \tfrac{1}{2}F_A(2x - 2y, 0) \quad \text{(since } F_A \text{ is convex by Fact 1.1)}\\
&= \tfrac{1}{2}\langle 2y, 2y^*\rangle + \tfrac{1}{2}F_A(2x - 2y, 0) \quad \text{(by Fact 1.1 and since } \operatorname{gra} A \text{ is linear)}\\
&< +\infty \quad \text{(since } (2x - 2y, 0) \in \operatorname{dom} F_A).
\end{aligned}$$

Thus $(x, y^*) \in \operatorname{dom} F_A$ and hence A is rectangular. ■

Corollary 4.2 (rectangularity of the adjoint) *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation such that $\operatorname{dom} A$ is closed. Then A is rectangular if and only if $A^*|_X$ is rectangular.*

Proof. Observe first that Fact 2.3(ii) yields $\operatorname{dom}(A^*|_X) = \operatorname{dom} A$. "$\Rightarrow$": Let $x \in \operatorname{dom}(A^*|_X)$ and $x^* \in Ax$. Applying Lemma 2.4 and the rectangularity of A, we obtain

$$F_{A^*|_X}(x, 0) = F_A(0, x^*) < +\infty.$$

Combining with Fact 2.3(iv) and Lemma 4.1, we deduce that $A^*|_X$ is rectangular. "$\Leftarrow$": The proof is similar to the implication just established and thus omitted. ■

Lemma 4.3 (characterization of paramonotonicity) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then A is paramonotone if and only if the implication*

$$\left.\begin{array}{c}(a, a^*) \in \operatorname{gra} A\\ \langle a, a^*\rangle = 0\end{array}\right\} \quad \Rightarrow \quad a^* \in A0$$

holds.

Proof. "$\Rightarrow$": Clear. "$\Leftarrow$": Let (a, a^*) and (b, b^*) be in $\operatorname{gra} A$ be such that

$$\langle a - b, a^* - b^*\rangle = 0.$$

Since $\operatorname{gra} A$ is linear, it follows that $(a - b, a^* - b^*) \in \operatorname{gra} A$. The hypothesis now implies that $a^* - b^* \in A0$. Thus, by Fact 2.2(i), we have $Aa = Ab$. Hence $a^* \in Ab$ and $b^* \in Aa$. Therefore, A is paramonotone. ■

Proposition 4.4 (paramonotonicity of the adjoint) *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Then A is paramonotone if and only if $A^*|_X$ is paramonotone.*

Proof. We start by noting that $A^*|_X$ is monotone by Fact 2.3(iv).

"$\Rightarrow$": Let $(a, a^*) \in \operatorname{gra}(A^*|_X)$ be such that

$$\langle a, a^* \rangle = 0. \tag{12}$$

One verifies that $(-a, a^*)$ is monotonically related to $\operatorname{gra} A$. The maximal monotonicity of A yields $(-a, a^*) \in \operatorname{gra} A$. Hence, by (12) and since A is paramonotone, $a^* \in A0$. Thus, by Fact 2.3(i), $a^* \in A^*0 = (A^*|_X)0$. It now follows from Lemma 4.3 that $A^*|_X$ is paramonotone.

"$\Leftarrow$": Let $(a, a^*) \in \operatorname{gra} A$ be such that $\langle a, a^* \rangle = 0$. By Lemma 2.5, $(a, -a^*) \in \operatorname{gra}(A^*|_X)$. Since $A^*|_X$ is paramonotone and using Fact 2.3(i), we deduce that $a^* \in (A^*|_X)0 = A^*0 = A0$. Lemma 4.3 now implies that A is paramonotone. ■

The next result shows that rectangularity is a sufficient condition for paramonotonicity in the linear case investigated in this section.

Proposition 4.5 (rectangular $\Rightarrow$ paramonotone in the linear case) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation such that A is rectangular. Then A is paramonotone.*

Proof. Let $(a, a^*) \in \operatorname{gra} A$ be such that $\langle a, a^* \rangle = 0$. Take $b \in \operatorname{dom} A$. Then, since A is linear and rectangular,

$$+\infty > F_A(b, 0) \geq \sup_{t\in\mathbb{R}} \big(\langle b, ta^* \rangle - \langle ta, ta^* \rangle\big) = \sup_{t\in\mathbb{R}} \langle b, ta^* \rangle.$$

Hence $a^* \in (\operatorname{dom} A)^\perp$. In view of Fact 2.3(i), $a^* \in A0$. Therefore, by Lemma 4.3, A is paramonotone. ■

Remark 4.6 Some comments regarding Proposition 4.5 are in order.

(i) The linearity assumption on A in Proposition 4.5 is not superfluous, see Example 3.7 above.

(ii) The maximality assumption on A in Proposition 4.5 is not superfluous, see Example 4.7 below.

(iii) The converse implication in Proposition 4.5 fails even when A is additionally assumed to be single-valued and continuous; see Example 5.7 below.

Example 4.7 Suppose that X contains at least two linearly independent vectors. Then there exist $a \in X \smallsetminus \{0\}$ and $a^* \in X^* \smallsetminus \{0\}$ such that $\langle a, a^*\rangle = 0$. Now define A via $\operatorname{gra} A = \mathbb{R}(a, a^*)$. Then A is a monotone linear relation such that A is rectangular yet A is not paramonotone.

Proof. Let a and b be linearly independent vectors in X and set $Y := \mathbb{R}a$. A separation theorem (see, e.g., [24, Corollary 1.9.7]) yields a vector $a^* \in X^*$ such that $\langle a, a^*\rangle = 0$ and $\langle b, a^*\rangle = \inf_{y\in Y} \|b - y\| > 0$ so that $a^* \neq 0$. It is clear that A is a monotone linear relation.

Let $x \in \operatorname{dom} A$, say $x = ra$ for some $r \in \mathbb{R}$. Then $F_A(x, 0) = \sup_{s\in\mathbb{R}}(\langle ra, sa^*\rangle + \langle sa, 0\rangle - \langle sa, sa^*\rangle) = \sup_{s\in\mathbb{R}} 0 = 0$. Hence $(x, 0) \in \operatorname{dom} F_A$ and thus A is rectangular by Lemma 4.1.

Now $(a, a^*) \in \operatorname{gra} A$ and $\langle a, a^*\rangle = 0$. On the other hand, $A0 = 0$. Altogether, $a^* \notin A0$. By Lemma 4.3, A cannot be paramonotone. ■

Proposition 4.5 raises the question when paramonotonicity implies rectangularity. Our main result (Theorem 4.9) shows that this implication holds under relatively mild assumptions. We shall require the following result concerning kernels and ranges.

Fact 4.8 (See [9, Theorem 3.2(i)&(ii)].) *Suppose that X is reflexive, and let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then* $\ker A = \ker A^*$ *and* $\overline{\operatorname{ran} A} = \overline{\operatorname{ran} A^*}$.

Theorem 4.9 (main result) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Suppose that X is reflexive, and that* $\operatorname{dom} A$ *and* $\operatorname{ran} A_+$ *are closed. Then the following are equivalent:*

(i) *A is rectangular.*

(ii) $\overline{\operatorname{ran} A} = \operatorname{ran} A_+$.

(iii) $\ker A_+ = \ker A$.

(iv) *A is paramonotone.*

(v) *A^* is paramonotone.*

(vi) *A^* is rectangular.*

Proof. Note that Fact 2.3(ii) yields $\operatorname{dom} A = \operatorname{dom} A^*$. Thus, by Fact 2.2(ii), we have

$$(\forall x \in \operatorname{dom} A) \quad \langle x, Ax\rangle = \langle x, A_+x\rangle. \tag{13}$$

Now take $y \in \operatorname{dom} A$ such that $\langle y, Ay\rangle = \{0\}$. By (13), $\langle y, A_+y\rangle = \{0\}$. Hence, by Fact 2.3(v) and Fact 1.4, $0 \in A_+y$ and thus $y \in \ker A_+$. We have established the implication

$$y \in \operatorname{dom} A \text{ and } \langle y, Ay\rangle = \{0\} \quad \Rightarrow \quad y \in \ker A_+\,. \tag{14}$$

"(i)⇔(ii)": Let $x \in \operatorname{dom} A$. Then $x \in \operatorname{dom} A^*$ by Fact 2.3(ii). Note that for every $b \in \operatorname{dom} A$ $\langle x, Ab\rangle = \langle b, A^*x\rangle$ by Fact 2.2(ii). Hence, using also the fact that $\operatorname{gra} A$ is a linear subspace, we compute

$$
\begin{cases}
F_A(x,0) & = \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x, a^*\rangle + \langle a, 0\rangle - \langle a, a^*\rangle\big) \\
& = \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x, a^*\rangle - \langle a, a^*\rangle\big) \\
& = \tfrac{1}{2}\sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x, 2a^*\rangle - \tfrac{1}{2}\langle 2a, 2a^*\rangle\big) \\
& = \tfrac{1}{2}\sup_{(b,b^*)\in \operatorname{gra} A} \big(\langle x, b^*\rangle - \tfrac{1}{2}\langle b, b^*\rangle\big) \\
& = \tfrac{1}{2}\sup_{(b,b^*)\in \operatorname{gra} A} \big(\langle A^*x, b\rangle - \tfrac{1}{2}\langle b, b^*\rangle\big) \\
& = \tfrac{1}{2}\sup_{b\in X} \big(\langle A^*x, b\rangle - q_A(b)\big) \\
& = \tfrac{1}{2} q_A^*(A^*x).
\end{cases} \tag{15}
$$

From the Brøndsted-Rockafellar Theorem (see, e.g., [32, Theorem 3.1.2]) and Fact 2.3(v) it follows that $\operatorname{ran} A_+ \subseteq \operatorname{dom} \overline{q_A}^* \subseteq \overline{\operatorname{ran} A_+} = \operatorname{ran} A_+$. Thus $\operatorname{ran} A_+ = \operatorname{dom} \overline{q_A}^* = \operatorname{dom} q_A^*$. Hence, using (15), Lemma 4.1, the closedness of $\operatorname{ran} A_+$, and Fact 4.8, we deduce that

$$
\begin{cases}
A \text{ is rectangular} & \Leftrightarrow \operatorname{ran} A^* \subseteq \operatorname{ran} A_+ \\
& \Leftrightarrow \overline{\operatorname{ran} A^*} \subseteq \operatorname{ran} A_+ \\
& \Leftrightarrow \overline{\operatorname{ran} A} \subseteq \operatorname{ran} A_+.
\end{cases} \tag{16}
$$

On the other hand, by Fact 4.8, $\operatorname{ran} A_+ \subseteq \operatorname{ran} A + \operatorname{ran} A^* \subseteq \operatorname{ran} A + \overline{\operatorname{ran} A} = \overline{\operatorname{ran} A}$. Thus recalling (16), we see that A is rectangular $\Leftrightarrow \overline{\operatorname{ran} A} = \operatorname{ran} A_+$.

"(ii)⇔(iii)": By Fact 2.2(iii) and Fact 4.8, we have $(\operatorname{ran} A)^\perp = \ker A^* = \ker A$. By Fact 2.3(iii)&(v), A_+ is maximally monotone and so $\operatorname{gra} A_+$ is closed. By [11, Proposition 2.8], $(A_+)^* = A_+$. Hence, using Fact 2.2(iii) again, we obtain $(\operatorname{ran} A_+)^\perp = \ker A_+$. This establishes the equivalence (ii)⇔(iii).

"(iii)⇒(iv)": Let $(a, a^*) \in \operatorname{gra} A$ be such that $\langle a, a^*\rangle = 0$. Then, by (14) and the assumption, $a \in \ker A_+ = \ker A$. Thus, $0 \in Aa$. By Fact 2.2(i), $Aa = 0 + A0 = A0$ and hence $a^* \in A0$. Lemma 4.3 now implies that A is paramonotone.

"(iv)⇒(iii)": Let $x \in \ker A_+$ and $x^* \in Ax$. Then by (13), $\langle x, x^*\rangle = \langle x, Ax\rangle = \langle x, A_+x\rangle = 0$. Since A is paramonotone, $x^* \in A0$ and hence $Ax = x^* + A0 = A0$. Thus $x \in \ker A$ and so

$$
\ker A_+ \subseteq \ker A. \tag{17}
$$

On the other hand, take $x \in \ker A$. Then $\langle x, Ax\rangle = \langle x, 0\rangle = 0$. Thus, by (14), $x \in \ker A_+$. We have shown that $\ker A \subseteq \ker A_+$. Combining with (17), we obtain $\ker A = \ker A_+$.

"(iv)⇔(v)": Proposition 4.4. "(i)⇔(vi)": Corollary 4.2. ■

Remark 4.10 In Theorem 4.9, the assumption that $\operatorname{ran} A_+$ is closed is not superfluous: indeed, let A and B be defined as in Example 5.7 below, where A and B are even continuous linear monotone operators defined on a Hilbert space, and set $C = A + B$. Then

$$\operatorname{ran} C_+ = \operatorname{ran} \frac{A + B + (A + B)^*}{2} = \operatorname{ran} \frac{A + B + A - B}{2} = \operatorname{ran} A,$$

is a proper dense subspace of X, and C is paramonotone but not rectangular. Note that $\operatorname{dom} C = X$. We do not know whether the assumption on the closure of the domain in Theorem 4.9 is superfluous.

The statement of Theorem 4.9 simplifies significantly in the finite-dimensional setting that we state next. (See also [23, Proposition 3.2] and [2, Remark 4.11] for related results pertaining to the case when A is single-valued and thus identified with a matrix.)

Corollary 4.11 (finite-dimensional setting) *Suppose that X is finite-dimensional, and let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Then the following are equivalent:*

(i) *A is rectangular.*

(ii) $\operatorname{ran} A_+ = \operatorname{ran} A$.

(iii) $\ker A_+ = \ker A$.

(iv) *A is paramonotone.*

(v) *A^* is paramonotone.*

(vi) *A^* is rectangular.*

5 Linear (single-valued) operators

Proposition 5.2 below, which is complementary to Corollary 4.2 and provides characterizations of rectangularity for continuous linear monotone operators, was first proved by Brezis and Haraux in [14, Proposition 2] in a Hilbert space setting. A different proof was provided in [2, Theorem 4.12]. Let us now generalize to Banach spaces. (We mention that most of the proof of (i)⇔(ii) follows along the lines of [2, Theorem 4.12(i)⇔(ii)].) We will need the following fact.

Fact 5.1 (See [25, Proposition 3.3 and Proposition 1.11].) *Let $f : X \to \left]-\infty, +\infty\right]$ be lower semicontinuous and convex, with* $\operatorname{int}\operatorname{dom} f \neq \varnothing$. *Then f is continuous on* $\operatorname{int}\operatorname{dom} f$ *and* $\partial f(x) \neq \varnothing$ *for every* $x \in \operatorname{int}\operatorname{dom} f$.

Proposition 5.2 (characterizations of rectangularity) *Let $A : X \to X^*$ be continuous, linear, and monotone. Then the following are equivalent:*

(i) *A is rectangular.*

(ii) $(\exists\,\beta > 0)(\forall x \in X)$ $\langle x, Ax\rangle \geq \beta\|Ax\|^2$, *i.e., A is β-cocoercive.*

(iii) *$A^*|_X$ is rectangular.*

If X is a real Hilbert space, then (i)–(iii) *are also equivalent to either of the following:*

(iv) $(\exists\,\gamma > 0)$ $\|\gamma A - \operatorname{Id}\| \leq 1$.

(v) $(\exists\,\beta > 0)$ $A^{-1} - \beta\operatorname{Id}$ *is monotone.*

Proof. "(i)⇒(ii)": Set

$$f\colon X \to \left]-\infty, +\infty\right] : x \mapsto F_A(x, 0).$$

Note that $X \times \{0\} \subseteq \operatorname{dom} F_A$ by Lemma 4.1. Thus, by Fact 5.1, f is continuous. Since $(0, 0) \in \operatorname{gra} A$, Fact 1.1 now yields $f(0) = F_A(0, 0) = \langle 0, 0\rangle = 0$. In view of the continuity of f at 0, there exist $\alpha > 0$ and $\delta > 0$ such that

$$\big(\forall y \in \delta B_X\big) \quad \sup_{a\in X}\big(\langle y, Aa\rangle - \langle a, Aa\rangle\big) = F_A(y, 0) = f(y) \leq \alpha.$$

Hence $(\forall y \in \delta B_X)(\forall a \in X)$ $\langle y, Aa\rangle \leq \alpha + \langle a, Aa\rangle$. Thus,

$$(\forall a \in X) \quad \delta\|Aa\| = \sup\langle \delta B_X, Aa\rangle \leq \alpha + \langle a, Aa\rangle.$$

Replacing a by ρa and invoking the linearity of A, we obtain

$$(\forall a \in X)(\forall \rho \in \mathbb{R}) \quad \begin{cases} 0 & \leq -|\rho|\delta\|Aa\| + \alpha + \rho^2\langle a, Aa\rangle \\ & \leq -\rho\delta\|Aa\| + \alpha + \rho^2\langle a, Aa\rangle. \end{cases} \tag{18}$$

Then, by Fact 2.1 and (18), it follows that

$$(\forall a \in X) \quad \delta^2\|Aa\|^2 \leq 4\alpha\langle a, Aa\rangle.$$

The desired conclusion holds with $\beta := \delta^2/(4\alpha)$.

"(ii)⇒(i)": Let $x \in X$. Then

$$\begin{aligned}
F_A(x,0) &= \sup_{a\in X}\big(\langle x, Aa\rangle - \langle a, Aa\rangle\big)\\
&\le \sup_{a\in X}\big(\|x\|\cdot\|Aa\| - \beta\|Aa\|^2\big)\\
&\le \sup_{t\in\mathbb{R}}\big(t\|x\| - \beta t^2\big)\\
&= \frac{\|x\|^2}{4\beta}.
\end{aligned}$$

Hence $X \times \{0\} \subseteq \operatorname{dom} F_A$ and the conclusion follows from Lemma 4.1.

"(iii)⇔(i)": Corollary 4.2. Now assume that X is a real Hilbert space. The proof concludes as follows. "(i)⇔(iv)": [2, Theorem 4.12]. "(ii)⇔(v)": [6, Example 22.6]. ■

Remark 5.3 Some comments regarding Proposition 5.2 are in order.

(i) When X is a real Hilbert space and $A^* = A \neq 0$, then (ii) holds with $\beta = 1/\|A\|$; see [6, Corollary 18.17] or [5, Corollary 3.4].

(ii) The condition in (v) is also known as *strong monotonicity* of A^{-1} with constant β. The constant β in item (ii) is indeed the same as the one in item (v).

Let us now provide some "bad" operators (see Section 6 for some "good" operators).

Example 5.4 (The Volterra operator is neither rectangular nor paramonotone) (See also [2, Example 3.3] and [22, Problem 148].) Suppose that $X = L^2[0,1]$. Recall that the *Volterra integration operator* is defined by

$$V\colon X \to X\colon x \mapsto Vx, \quad \text{where} \quad Vx\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_0^t x,$$

and that its adjoint is given by

$$V^*\colon X \to X\colon x \mapsto V^*x, \quad \text{where} \quad V^*x\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_t^1 x.$$

Then V is neither paramonotone nor rectangular; consequently, V^* is neither paramonotone nor rectangular.

Proof. Set $e \equiv 1 \in X$. Then by [2, Example 3.3], V is a continuous, injective, linear, and maximally monotone, with symmetric part $V_+\colon x \mapsto \frac{1}{2}\langle x, e\rangle e$. Now pick $x \in \{e\}^\perp \smallsetminus \{0\}$ (for instance, consider $t \mapsto t - \frac{1}{2}$). Then

$$\langle x, Vx\rangle = \langle x, V_+x\rangle = \big\langle x, \tfrac{1}{2}\langle e, x\rangle e\big\rangle = \tfrac{1}{2}\langle e, x\rangle^2 = 0.$$

However, since $x \neq 0$ and V is injective, we have $Vx \neq 0 = V0$. Thus V cannot be paramonotone. By Proposition 4.5, V is not rectangular. It follows from Proposition 5.2 and Proposition 4.4 that V^* is neither paramonotone nor rectangular. ■

Proposition 5.6 below provides a mechanism for constructing paramonotone linear operators that are not rectangular. We shall employ the following very useful sufficient condition for maximal monotonicity and continuity.

Fact 5.5 (See [34, Proposition 26.4.(a)&(b)] and [26, Corollary 2.6 and Proposition 3.2(h)].) *Let $A\colon X \to X^*$ be monotone and linear. Then A is maximally monotone and continuous.*

Proposition 5.6 *Suppose that X is reflexive. Let A and B be continuous, linear, and monotone on X. Suppose that A is paramonotone and* $\operatorname{ran} A$ *is a proper dense subspace of X, and that B is bijective and skew. Then $A+B$ is maximally monotone, strictly monotone and paramonotone, but not rectangular.*

Proof. Set $C := A + B$. By Fact 5.5, A, B, and C are maximally monotone, linear, and continuous. Since $\operatorname{ran} A$ is proper dense, Fact 2.2(iii) and Fact 4.8 imply that

$$\ker A = (\operatorname{ran} A^*)^{\perp} = (\operatorname{ran} A)^{\perp} = \{0\}. \tag{20}$$

Since B is skew, for every $x \in X$,

$$\langle x, Cx\rangle = \langle x, (A+B)x\rangle = \langle x, Ax\rangle.$$

Using (20) and the paramonotonicity of A, we obtain

$$(\forall x \in X) \quad \langle x, Cx\rangle = 0 \Leftrightarrow Ax = 0 \Leftrightarrow x = 0.$$

Therefore, C is strictly monotone and thus paramonotone.

By the Bounded Inverse Theorem, B^{-1} is a bounded linear operator; thus, there exists $\gamma > 0$ such that

$$(\forall x \in X) \quad \|Bx\| \geq \gamma\|x\|. \tag{21}$$

We now claim that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$(\forall n \in \mathbb{N}) \quad \|x_n\| = 1, \quad \text{while} \quad Ax_n \to 0. \tag{22}$$

Since $\operatorname{ran} A$ is not closed, we take $z^* \in \overline{\operatorname{ran} A} \smallsetminus \operatorname{ran} A$. Thus there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that

$$Ay_n \to z^*. \tag{23}$$

Let us show that $\|y_n\| \to +\infty$. Otherwise, $(y_n)_{n\in\mathbb{N}}$ possesses a a weakly convergent subsequence—which for convenience we still denote by $(y_n)_{n\in\mathbb{N}}$— say, $y_n \rightharpoonup y \in X$. By [24, Theorem 3.1.11], $Ay_n \rightharpoonup Ay$. Combining with (23), we deduce that $z^* = Ay$, which contradicts our assumption that $z^* \notin \operatorname{ran} A$. Hence $\|y_n\| \to +\infty$. We assume that $(\forall n \in \mathbb{N})$ $y_n \neq 0$ and we set $x_n := y_n/\|y_n\|$. By (23), $Ax_n \to 0$. This completes the proof of (22).

Now suppose that $\beta \geq 0$ satisfies

$$(\forall x \in X) \quad \langle x, Cx\rangle \geq \beta\|Cx\|^2. \tag{24}$$

Using (24), we estimate that for every $n \in \mathbb{N}$

$$\begin{aligned}\|Ax_n\| = \|Ax_n\| \cdot \|x_n\| &\geq \langle x_n, Ax_n\rangle = \langle x_n, Cx_n\rangle \\ &\geq \beta\|Cx_n\|^2 = \beta\|(A+B)x_n\|^2 \\ &\geq \beta\big(\|Bx_n\| - \|Ax_n\|\big)^2.\end{aligned}$$

Taking $\underline{\lim}$ and recalling (22)&(21), we deduce that $0 \geq \beta\gamma^2$ and hence $\beta = 0$. Therefore, by Proposition 5.2, C cannot be rectangular. ■

The following is a realization of Proposition 5.6 in the Hilbert space of real square-summable sequences. It provides a counterexample complementary to Example 3.7.

Example 5.7 (paramonotone $\not\Rightarrow$ rectangular) Suppose that $X = \ell^2$, and define two continuous linear maximally monotone operators on X via

$$A\colon X \to X\colon (x_n)_{n\in\mathbb{N}} \mapsto \big(\tfrac{1}{n}x_n\big)_{n\in\mathbb{N}}$$

and

$$B\colon X \to X\colon (x_n)_{n\in\mathbb{N}} \mapsto (-x_2, x_1, -x_4, x_3, \ldots).$$

Then $A + B$ is maximally monotone, strictly monotone and paramonotone, but not rectangular.

6 Displacement mappings

For certain maximally monotone operators in Hilbert space, we are able to obtain sharper conclusions as the following result illustrates. Let us assume that X is a Hilbert space, and let $T\colon X \to X$. Recall that T is *nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\| \leq \|x - y\|;$$

T is *firmly nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle.$$

It is straightforward to show that T is firmly nonexpansive if and only if $2T - \operatorname{Id}$ is nonexpansive.

We now turn to our last result and thank a referee for suggesting a simpler proof.

Theorem 6.1 (The displacement mapping) *Suppose that X is a Hilbert space, let $T\colon X \to X$ be nonexpansive, and define the corresponding* displacement mapping *by*

$$A = \operatorname{Id} - T.$$

Then the following hold:

(i) *A is maximally monotone.*

(ii) *A is $\frac{1}{2}$-cocoercive, i.e., $\frac{1}{2}A$ is firmly nonexpansive.*

(iii) *A is rectangular.*

(iv) *A^{-1} is strongly monotone with constant $\frac{1}{2}$, i.e., $A^{-1} - \frac{1}{2}\operatorname{Id}$ is monotone.*

(v) *A^{-1} is strictly monotone.*

(vi) *A is paramonotone.*

Proof. (i): The maximal monotonicity of A is well known; see, e.g., [6, Example 20.26].

(ii): Since $2(\frac{1}{2}A) - \operatorname{Id} = -T$ is nonexpansive, it follows that $\frac{1}{2}A$ is firmly nonexpansive.

(iii): Combine (ii) with Lemma 3.4.

(iv): Combine (ii) with [6, Example 22.6].

(v): Immediate from (iv).

(vi): A^{-1} is strictly monotone by (v) and hence paramonotone. Thus, as the inverse of a paramonotone operator, $(A^{-1})^{-1} = A$ is paramonotone as well. ■

The displacement mapping in the following example is generally not a subdifferential operator (unless $m = 2$); however, it is rectangular and paramonotone. Moreover, the rectangularity of the displacment mapping played a crucial role in [1] and [7].

Example 6.2 (cyclic right-shift operator) Let Y be a real Hilbert space, let $m \in \mathbb{N}$, and suppose that X is the Hilbert product space Y^m. Define the *cyclic right-shift operator* R by

$$R\colon X \to X\colon (x_1, x_2, \ldots, x_m) \mapsto (x_m, x_1, \ldots, x_{m-1}).$$

Then $\operatorname{Id} - R$ is rectangular and paramonotone.

Proof. Since $(\forall x \in X)\ \|Rx\| = \|x\|$, we deduce that R is nonexpansive. Now apply Theorem 6.1. ■

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