On the order of the operators in the Douglas–Rachford algorithm

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Abstract

The Douglas–Rachford algorithm is a popular method for finding zeros of sums of monotone operators. By its definition, the Douglas–Rachford operator is not symmetric with respect to the order of the two operators. In this paper we provide a systematic study of the two possible Douglas–Rachford operators. We show that the reflectors of the underlying operators act as bijections between the fixed points sets of the two Douglas–Rachford operators. Some elegant formulae arise under additional assumptions. Various examples illustrate our results.

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1 Introduction

Throughout this paper we shall assume that X is a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We also assume that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are maxi-

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mally monotone operators[1]. The *resolvent* and the *reflected resolvent* associated with A are $J_A = (\mathrm{Id} + A)^{-1}$ and $R_A = 2J_A - \mathrm{Id}$, respectively[2]. The sum problem for A and B is to *find* $x \in X$ *such that* $x \in (A+B)^{-1}0$. When $(A+B)^{-1}(0) \neq \varnothing$, the Douglas–Rachford splitting algorithm can be used to solve the sum problem. The Douglas–Rachford splitting operator [20] associated with the ordered pair of operators (A, B) is

$$T_{A,B} := \tfrac{1}{2}(\mathrm{Id} + R_B R_A) = \mathrm{Id} - J_A + J_B R_A. \tag{1}$$

By definition, the Douglas–Rachford splitting operator is dependent on the order of the operators A and B, even though the sum problem remains unchanged when interchanging A and B. *The goal of this paper is to investigate the connection between the operators $T_{A,B}$ and $T_{B,A}$.* In passing, we mention the recent related though different work by Yan and Yin on the effect of different orders of the operators in the Alternating Direction Method of Multipliers (ADMM); see [24, Sections 6 and 7]. Our main results can be summarized as follows.

- We show that R_A is an isometric[3] bijection from the fixed points set of $T_{A,B}$ to that of $T_{B,A}$, with inverse $R_B : \mathrm{Fix}\, T_{B,A} \to \mathrm{Fix}\, T_{A,B}$ (see Theorem 2.2).
- When A is an affine relation, we have $(\forall n \in \mathbb{N})\ R_A T^n_{A,B} = T^n_{B,A} R_A$. In particular[4], when $A = N_U$ where U is a closed affine subspace of X, we have $(\forall n \in \mathbb{N})\ T^n_{A,B} = R_A T^n_{B,A} R_A$ and $T^n_{B,A} = R_A T^n_{A,B} R_A$ (see Proposition 2.5(i) and Theorem 2.7(i)).
- Our results connect to recent linear and finite convergence results (see [1], [2], [7], [9], [11], [17], and [18]) for the Douglas–Rachford algorithm (see Remark 2.10).

In Section 2, we present the main results and various examples. The notation we adopt is standard and follows, e.g., [6] and [22].

2 Results

We recall that the Attouch–Théra dual pair of (A, B) (see [3]) is the pair [5] $(A^{-1}, B^{-\circledcirc})$. Following [5], we set $Z := Z_{(A,B)} = (A+B)^{-1}(0)$ and $K := K_{(A,B)} = (A^{-1} + B^{-\circledcirc})^{-1}(0)$,

[1]Recall that $A : X \rightrightarrows X$ is *monotone* if whenever the pairs (x,u) and (y,v) lie in gra A we have $\langle x-y, u-v\rangle \geq 0$, and is *maximally monotone* if it is monotone and any proper enlargement of the graph of A (in terms of set inclusion) does not preserve the monotonicity of A.

[2]The identity operator on X is denoted by Id. It is well-known that, when A is maximally monotone, J_A is single-valued, maximally monotone and firmly nonexpansive and R_A is nonexpansive.

[3]Suppose that C and D are two nonempty subsets of X. We recall that $Q : C \to D$ is an isometry if $(\forall x \in C)(\forall y \in C)\ \|Qx - Qy\| = \|x - y\|$. The set of fixed points of T is $\mathrm{Fix}\, T := \{x \in X \mid x = Tx\}$.

[4]Throughout the paper we use N_C and P_C to denote the *normal cone* and *projector* associated with a nonempty closed convex subset C of X respectively.

[5]We set $A^{\circledcirc} := (-\mathrm{Id}) \circ A \circ (-\mathrm{Id})$ and $A^{-\circledcirc} := (A^{-1})^{\circledcirc} = (A^{\circledcirc})^{-1}$.

to denote, respectively, the primal and dual solutions. One easily verifies that

$$Z_{(B,A)} = (B + A)^{-1}(0) = Z \qquad \text{and} \qquad K_{(B,A)} = (B^{-1} + A^{-\ovee})^{-1}(0) = -K. \tag{2}$$

We further recall (see [14, Lemma 2.6(iii)] and [5, Corollary 4.9]) that

$$Z = J_A(\operatorname{Fix} T_{A,B}) \quad \text{and} \quad K = (\operatorname{Id} - J_A)(\operatorname{Fix} T_{A,B}), \tag{3}$$

and we will make use of the following useful identity which can be verified using (1):

$$R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_A R_B R_A. \tag{4}$$

Let us recall the definition of the extended solution set[6]

$$\mathcal{S}_{(A,B)} := \{(z, -w) \in X \times X \mid -w \in Bz, w \in Az\}. \tag{5}$$

We start with the following useful result.

Lemma 2.1. *The mapping*

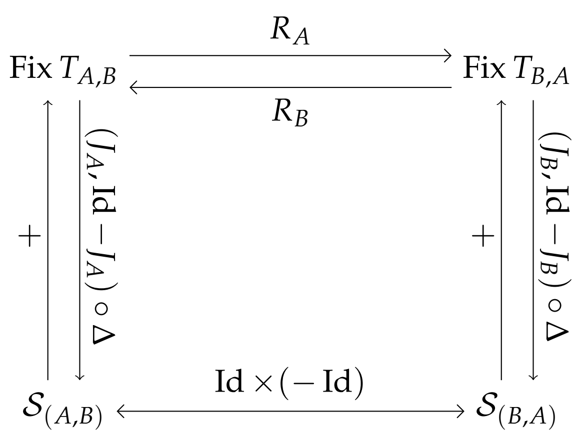
$$\Psi \colon \mathcal{S}_{(A,B)} \to \operatorname{Fix} T_{A,B} \colon (z, k) \mapsto z + k \tag{6}$$

is bijective, with inverse $\Psi^{-1} \colon x \mapsto (J_A x, x - J_A x)$*, and both* Ψ *and* Ψ^{-1} *are continuous.*

Proof. Combine [5, Remark 3.9 and Theorem 4.5]. ■

We are now ready for the first main result.

Theorem 2.2. *R_A is an isometric bijection from* $\operatorname{Fix} T_{A,B}$ *to* $\operatorname{Fix} T_{B,A}$*, with isometric inverse R_B. Moreover, we have the following commutative diagram:*



[6] For further information on the extended solution set, we refer the reader to [15, Section 2.1].

Here $\Delta\colon X \to X \times X\colon x \mapsto (x,x)$. *In particular, we have*

$$R_A\colon \operatorname{Fix} T_{A,B} \to \operatorname{Fix} T_{B,A}\colon z + k \mapsto z - k, \tag{7}$$

where $(z,k) \in \mathcal{S}_{(A,B)}$.

Proof. Let $x \in X$ and note that (1) implies that $\operatorname{Fix} T_{A,B} = \operatorname{Fix} R_B R_A$ and $\operatorname{Fix} T_{B,A} = \operatorname{Fix} R_A R_B$. Now $x \in \operatorname{Fix} T_{A,B} \iff x = R_B R_A x \Rightarrow R_A x = R_A R_B R_A x \iff R_A x \in \operatorname{Fix} R_A R_B = \operatorname{Fix} T_{B,A}$, which proves that R_A maps $\operatorname{Fix} T_{A,B}$ into $\operatorname{Fix} T_{B,A}$. By interchanging A and B one sees that R_B maps $\operatorname{Fix} T_{B,A}$ into $\operatorname{Fix} T_{A,B}$. We now show that R_A maps $\operatorname{Fix} T_{A,B}$ onto $\operatorname{Fix} T_{B,A}$. To this end, let $y \in \operatorname{Fix} T_{B,A}$ and note that $R_B y \in \operatorname{Fix} T_{A,B}$ and $R_A R_B y = y$, which proves that R_A maps $\operatorname{Fix} T_{A,B}$ onto $\operatorname{Fix} T_{B,A}$. The same argument holds for R_B. Finally since $(\forall x \in \operatorname{Fix} T_{A,B})\ R_B R_A x = x$, this proves that R_A is a bijection from $\operatorname{Fix} T_{A,B}$ to $\operatorname{Fix} T_{B,A}$ with the desired inverse. To prove that $R_A : \operatorname{Fix} T_{A,B} \to \operatorname{Fix} T_{B,A}$ is an isometry note that $(\forall x \in \operatorname{Fix} T_{A,B})$ $(\forall y \in \operatorname{Fix} T_{A,B})$ we have $\|x - y\| = \|R_B R_A x - R_B R_A y\| \leq \|R_A x - R_A y\| \leq \|x - y\|$.

We now turn to the diagram. The correspondence of $\operatorname{Fix} T_{A,B}$ and $\operatorname{Fix} T_{B,A}$ follows from our earlier argument. The correspondences of $\operatorname{Fix} T_{A,B}$ and $\mathcal{S}_{(A,B)}$, and $\operatorname{Fix} T_{B,A}$ and $\mathcal{S}_{(B,A)}$ follow from applying Lemma 2.1 to $T_{A,B}$ and $T_{B,A}$ respectively. The fourth correspondence is obvious from the definition of $\mathcal{S}_{(A,B)}$ and $\mathcal{S}_{(B,A)}$. To prove (7) we let $y \in \operatorname{Fix} T_{A,B}$ and recall that, in view of Lemma 2.1, we have $y = z + k$ where $(z,k) \in \mathcal{S}_{(A,B)}$ and $R_A(z+k) = (J_A - (\operatorname{Id} - J_A))(z+k) = J_A(z+k) - (\operatorname{Id} - J_A)(z+k) = z - k$. ■

Remark 2.3. *In view of* [5, Remark 3.9, Theorem 4.5 and Corollary 5.5(iii)], *when A and B are paramonotone*[7] *(as is always the case when A and B are subdifferential operators of proper convex lower semicontinuous functions), we can replace $\mathcal{S}_{(A,B)}$ and $\mathcal{S}_{(B,A)}$ by, respectively, $Z \times K$ and $Z \times (-K)$.*

Lemma 2.4. *Suppose that A is an affine relation. Then*

(i) *J_A is affine and $J_A R_A = 2J_A^2 - J_A = R_A J_A$.*

If $A = N_U$, where U is a closed affine subspace of X, then we have additionally:

(ii) $P_U = J_A = J_A R_A = R_A J_A$ *and* $(\operatorname{Id} - J_A) R_A = J_A - \operatorname{Id}$.

(iii) $R_A^2 = \operatorname{Id}$, $R_A = R_A^{-1}$, *and $R_A : X \to X$ is an isometric bijection.*

[7] See [19] for definition and detailed discussion on paramonotone operators.

Proof. (i): The fact that J_A is affine follows from [8, Theorem 2.1(xix)]. Hence $J_A R_A = J_A(2J_A - \mathrm{Id}) = 2J_A^2 - J_A = R_A J_A$.

(ii): It follows from [6, Example 23.4] that $P_U = J_A$. Now using (i) we have $R_A J_A = J_A R_A = 2P_U^2 - P_U = 2P_U - P_U = P_U = J_A$. To prove the last identity note that by (i) we have $(\mathrm{Id} - J_A)R_A = R_A - J_A R_A = 2P_U - \mathrm{Id} - P_U = P_U - \mathrm{Id}$.

(iii): Because R_A is affine, it follows from (ii) that $R_A^2 = R_A(2J_A - \mathrm{Id}) = 2R_A J_A - R_A = 2P_U - (2P_U - \mathrm{Id}) = \mathrm{Id}$. Finally let $x, y \in X$. Since R_A is nonexpansive we have $\|x - y\| = \|R_A^2 x - R_A^2 y\| \leq \|R_A x - R_A y\| \leq \|x - y\|$, hence all the inequalities become equalities which completes the proof. ■

We now turn to the iterates of the Douglas–Rachford algorithm.

Proposition 2.5. *Suppose that A is an affine relation. Then the following hold:*

(i) $(\forall n \in \mathbb{N})$ *we have* $R_A T_{A,B}^n = T_{B,A}^n R_A$.

(ii) $R_A Z = J_A \operatorname{Fix} T_{B,A}$ *and* $R_A K = (J_A - \mathrm{Id})(-\operatorname{Fix} T_{B,A})$.

If B is an affine relation, then we additionally have:

(iii) $T_{A,B} R_B R_A = R_B R_A T_{A,B}$.

(iv) $4(T_{A,B} T_{B,A} - T_{B,A} T_{A,B}) = R_B R_A^2 R_B - R_A R_B^2 R_A$. *Consequently,* $T_{A,B} T_{B,A} = T_{B,A} T_{A,B} \iff R_B R_A^2 R_B = R_A R_B^2 R_A$.

(v) *If* $R_A^2 = R_B^2 = \mathrm{Id}$, *then*[8] $T_{A,B} T_{B,A} = T_{B,A} T_{A,B}$.

Proof. (i): It follows from (4), Lemma 2.4(i) and (1) that $R_A T_{A,B} - T_{B,A} R_A = 2J_A T_{A,B} - J_A - J_A R_B R_A = J_A(2T_{A,B} - \mathrm{Id}) - J_A R_B R_A = J_A(2(\frac{1}{2}(\mathrm{Id} + R_B R_A)) - \mathrm{Id}) - J_A R_B R_A = J_A R_B R_A - J_A R_B R_A = 0$, which proves the claim when $n = 1$. The general proof follows by induction.

(ii): Using (3), Lemma 2.4(i) and Theorem 2.2, we have $R_A Z = R_A J_A(\operatorname{Fix} T_{A,B}) = J_A R_A(\operatorname{Fix} T_{A,B}) = J_A(\operatorname{Fix} T_{B,A})$. Now using that the inverse resolvent identity[9],

[8] In passing, we point out that this is equivalent to saying that $A = N_U$ and $B = N_V$ where U and V are closed affine subspaces of X. Indeed, $R_A^2 = \mathrm{Id} \iff J_A = J_A^2$ and therefore we conclude that $\operatorname{ran} J_A = \operatorname{Fix} J_A$. Combining with [25, Theorem 1.2] yields that J_A is a projection, hence A is an affine normal cone operator using [6, Example 23.4].

[9] Recall the when A is maximally monotone the inverse resolvent identity states that $J_A + J_{A^{-1}} = \mathrm{Id}$. Consequently, $R_{A^{-1}} = -R_A$.

Lemma 2.4(i) applied to A^{-1} and Theorem 2.2, we obtain $R_A K = -R_{A^{-1}} J_{A^{-1}}(\operatorname{Fix} T_{A,B}) = -J_{A^{-1}} R_{A^{-1}}(\operatorname{Fix} T_{A,B}) = -J_{A^{-1}}(-R_A \operatorname{Fix} T_{A,B}) = (J_A - \mathrm{Id})(-\operatorname{Fix} T_{B,A})$.

(iii): Note that $T_{A,B}$ and $T_{B,A}$ are affine. It follows from (1) that $T_{A,B} R_B R_A = T_{A,B}(2T_{A,B} - \mathrm{Id}) = 2T_{A,B}^2 - T_{A,B} = (2T_{A,B} - \mathrm{Id})T_{A,B} = R_B R_A T_{A,B}$.

(iv) We have

$$\begin{aligned} 4(T_{A,B}T_{B,A} - T_{B,A}T_{A,B}) &= 4\left(\tfrac{1}{2}(\mathrm{Id} + R_B R_A)\tfrac{1}{2}(\mathrm{Id} + R_A R_B) - \tfrac{1}{2}(\mathrm{Id} + R_A R_B)\tfrac{1}{2}(\mathrm{Id} + R_B R_A)\right) \\ &= \mathrm{Id} + R_B R_A + R_A R_B + R_B R_A^2 R_B - (\mathrm{Id} + R_A R_B + R_B R_A \\ &\quad + R_A R_B^2 R_A) = R_B R_A^2 R_B - R_A R_B^2 R_A. \end{aligned} \tag{8}$$

(v): This is a direct consequence of (iii). ■

With regards to Proposition 2.5(i), one may inquire whether the conclusion still holds when R_A is replaced by R_B. We now give an example illustrating that the answer to this question is negative.

Example 2.6. *Suppose that $X = \mathbb{R}^2$, that $U = \mathbb{R} \times \{0\}$, that $V = \{0\} \times \mathbb{R}_+$, that $A = N_U$ and that $B = N_V$. Then A is linear, hence $R_A T_{A,B} = T_{B,A} R_A$, however $R_B T_{A,B} \neq T_{B,A} R_B$ and $R_B T_{B,A} \neq T_{A,B} R_B$.*

Proof. The identity $R_A T_{A,B} = T_{B,A} R_A$ follows from applying Proposition 2.5(i) with $n = 1$. Now let $(x,y) \in \mathbb{R}^2$. Elementary calculations show that $R_A(x,y) = (x,-y)$. and $R_B(x,y) = (-x,|y|)$. Consequently, (1) implies that $T_{A,B}(x,y) = (0,y^+)$ and $T_{B,A}(x,y) = (0,y^-)$ [10]. Therefore, $R_B T_{A,B}(x,y) = (0,y^+)$, $T_{B,A} R_B(x,y) = (0,0)$, $R_B T_{B,A}(x,y) = (0,-y^+)$, and $T_{A,B} R_B(x,y) = (0,|y|)$. The conclusion then follows from comparing the last four equations. ■

We are now ready for our second main result.

Theorem 2.7 (When A is normal cone of closed affine subspace). *Suppose that U is a closed affine subspace and that $A = N_U$. Then the following hold:*

(i) $(\forall n \in \mathbb{N})$ $R_A T_{B,A}^n = T_{A,B}^n R_A$, $T_{B,A}^n = R_A T_{A,B}^n R_A$ *and* $T_{A,B}^n = R_A T_{B,A}^n R_A$.

(ii) $R_A \colon \operatorname{Fix} T_{B,A} \to \operatorname{Fix} T_{A,B}$, $Z = J_A(\operatorname{Fix} T_{B,A})$, *and* $K = (J_A - \mathrm{Id})(\operatorname{Fix} T_{B,A})$.

(iii) *Suppose that V is a closed affine subspace of X and that $B = N_V$. Then $T_{A,B} R_A R_B = R_A R_B T_{A,B}$ and $T_{A,B} T_{B,A} = T_{B,A} T_{A,B}$.*

[10] For every $x \in \mathbb{R}$, we set $x^+ := \max\{x,0\}$ and $x^- := \min\{x,0\}$

Proof. (i): Let $n \in \mathbb{N}$. It follows from Proposition 2.5(i) and Lemma 2.4(iii) that $T_{A,B}^n = R_A R_A T_{A,B}^n = R_A T_{B,A}^n R_A$. Hence $T_{A,B}^n R_A = R_A T_{B,A}^n R_A R_A = R_A T_{B,A}^n$.

(ii): The statement for R_A follows from combining Theorem 2.2 and Lemma 2.4(iii). In view of (3), Lemma 2.4(ii) and Theorem 2.2 one learns that $Z = J_A(\operatorname{Fix} T_{A,B}) = J_A R_A(\operatorname{Fix} T_{A,B}) = J_A(\operatorname{Fix} T_{B,A})$. Finally, (3), Lemma 2.4(iii) and (ii), and Theorem 2.2 imply that $K = (\operatorname{Id} - J_A)(\operatorname{Fix} T_{A,B}) = (\operatorname{Id} - J_A) R_A (R_A \operatorname{Fix} T_{A,B}) = (J_A - \operatorname{Id}) \operatorname{Fix} T_{B,A}$.

(iii): In view of (i) applied to A and B we have $T_{A,B} R_A R_B = R_A T_{B,A} R_B = R_A R_B T_{A,B}$. The second identity follows from combining Proposition 2.5(v) and Lemma 2.4(iii) applied to both A and B. ■

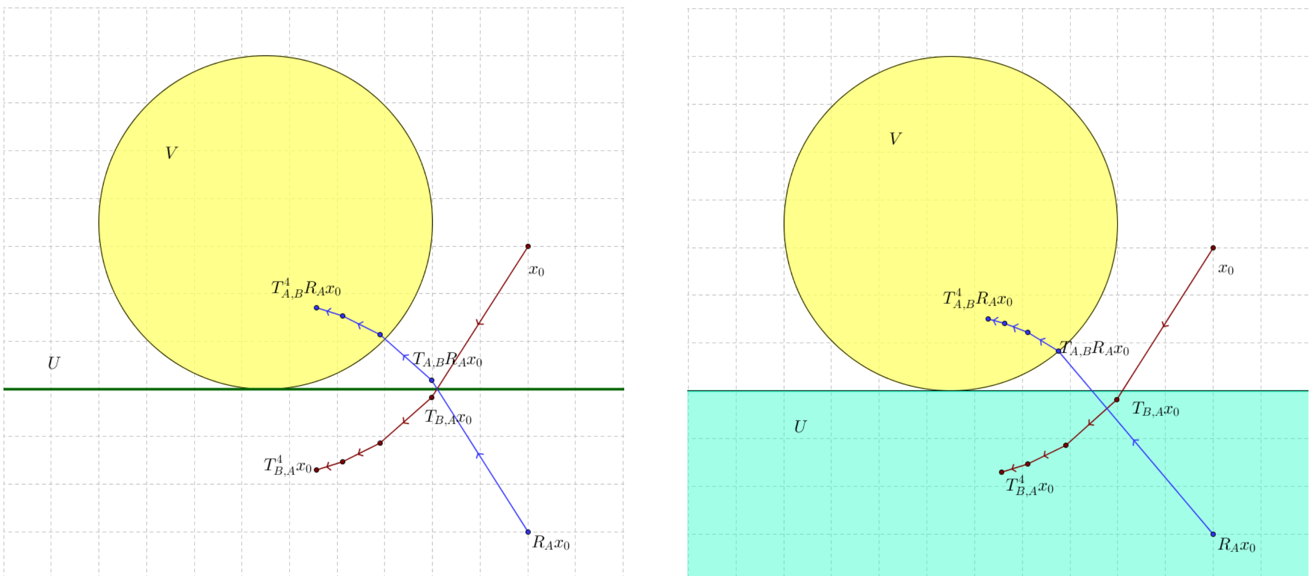


Figure 1: A `GeoGebra` [16] snapshot. Left: Two closed convex sets in $\mathbb{R}^2$, U is a linear subspace (green line) and V (the ball). Right: Two closed convex sets in $\mathbb{R}^2$, U is the half-space (cyan region) and V (the ball). Shown are also the first five terms of the sequences $(T_{A,B}^n R_A x_0)_{n \in \mathbb{N}}$ (blue points) and $(T_{B,A}^n x_0)_{n \in \mathbb{N}}$ (red points) in each case. The left figure illustrates Theorem 2.7(i) while the right figure illustrates the failure of this result when the subspace is replaced by a cone.

The conclusion of Theorem 2.7(iii) may fail when we assume that A or B is an affine, but not a normal cone, operator as we illustrate next.

Example 2.8. *Suppose that $X = \mathbb{R}^2$, that $U = \mathbb{R} \times \{0\}$, that $A = N_U$ and that $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.*

Then B is linear and maximally monotone but not a normal cone *operator and* $\frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} =$

$T_{A,B}T_{B,A} \neq T_{B,A}T_{A,B} = \frac{1}{9}\left(\begin{smallmatrix} 5 & 1 \\ 1 & 2 \end{smallmatrix}\right)$

Corollary 2.9. *Suppose that U is an affine subspace and that $A = N_U$. Let x and y be in X. Then the following hold:*

(i) $(\forall n \in \mathbb{N})$ $J_A T^n_{B,A} x = J_A T^n_{A,B} R_A x$. *Consequently, if $Z \neq \varnothing$ then $(J_A T^n_{B,A} x)_{n\in\mathbb{N}}$ converges weakly to a point in Z.*

(ii) $\|T_{A,B}x - T_{A,B}y\| = \|T_{B,A}R_A x - T_{B,A}R_A y\| \leq \|R_A x - R_A y\|$.

Proof. (i): It follows from Lemma 2.4(iii), Proposition 2.5(i) and Lemma 2.4(ii) that $J_A T^n_{B,A} x = J_A T^n_{B,A} R_A R_A x = J_A R_A T^n_{A,B} R_A x = J_A T^n_{A,B} R_A x$, as claimed. The convergence of the sequence $(J_A T^n_{B,A} x)_{n\in\mathbb{N}}$ follows from e.g., [6, Theorem 25.6].

(ii): Apply Lemma 2.4(iii) with x and y replaced with $T_{A,B}x$ and $T_{A,B}y$, Proposition 2.5(i) with $n = 1$, and use nonexpansiveness of $T_{B,A}$. ■

Remark 2.10.

(i) *The results of Theorem 2.7 and Corollary 2.9 are of interest when the Douglas–Rachford algorithm is applied to find the zero of the sum of more than two operators in which case one can use a parallel splitting method (see, e.g., [13, Section 2.2] or [6, Proposition 25.7]), where one operator is the normal cone operator of the diagonal subspace in a product space.*

(ii) *A second glance at the proof of Theorem 2.7(i) reveals that the result remains true if J_B is replaced by any operator $Q_B \colon X \to X$ (and R_B is replaced by $2Q_B - \mathrm{Id}$, of course). This is interesting because in* [1], *[2], [17] and [18], Q_B is chosen to be a selection of the (set-valued) projector onto a set V that is* not convex. *Hence the generalized variant of Theorem 2.7(i) then guarantees that the orbits of the two Douglas–Rachford operators are related via*

$$(\forall n \in \mathbb{N}) \quad T^n_{B,A} = R_A T^n_{A,B} R_A. \tag{9}$$

(iii) *As a consequence of (ii) and Lemma 2.4(iii), we see that if* linear convergence *is guaranteed for the iterates of $T_{A,B}$ then the same holds true for the iterates of $T_{B,A}$ provided that U is a closed affine subspace, V is a nonempty closed set, $A = N_U$ and J_B is a selection of the projection onto V. This is not particularly striking when we compare to sufficient conditions that are already symmetric in A and B (such as, e.g., $\mathrm{ri}\, U \cap \mathrm{ri}\, V \neq \varnothing$ in [9] and [21]); however, this is a new insight when the sufficient conditions are* not symmetric *(as in, e.g., [1], [10] [17] and [18]).*

(iv) *A comment similar to (iii) can be made for* finite convergence *results. In [7] the authors, generalizing results of [23], prove the finite convergence of the Douglas–Rachford algorithm for the case when $A = N_U$, $B = N_V$, U is a closed affine subspace and V is a polyhedral* *set such that Slater's condition $U \cap \operatorname{int} V \neq \varnothing$ holds. Observe that Slater's condition is* nonsymmetric *and therefore, in view of Theorem 2.7(i), we obtain a novel sufficient condition for finite convergence.*

We now turn to the Borwein–Tam method [12].

Proposition 2.11. *Suppose that U is an affine subspace of X, that $A = N_U$, and set*

$$T_{[A,B]} := T_{A,B} T_{B,A}. \tag{10}$$

Then the following holds:

(i) $T_{[A,B]} = R_A T_{[B,A]} R_A = (T_{A,B} R_A)^2 = (R_A T_{B,A})^2$.

(ii) *Suppose that V is an affine subspace and that $B = N_V$. Then*[11] $T_{[A,B]} = T_{[B,A]}$. *Consequently* $T_{[A,B]} = (R_B T_{A,B})^2 = (T_{B,A} R_B)^2 = \frac{1}{2}(T_{A,B} + T_{B,A})$, *and $T_{[A,B]}$ is firmly nonexpansive.*

Proof. (i): Using (10) and Theorem 2.7(i) with $n = 1$ we obtain $T_{[A,B]} = R_A T_{B,A} R_A T_{B,A} = R_A T_{B,A} T_{A,B} R_A = R_A T_{[B\ A]} R_A = T_{A,B} R_A T_{A,B} R_A = (T_{A,B} R_A)^2 = (R_A T_{B,A})^2$.

(ii): The identity $T_{[A,B]} = T_{[B,A]}$ follows from Theorem 2.7(iii) and (10). Now combine with (i) with A and B switched, and use [12, Remark 4.1]. That $T_{[A,B]}$ (hence $T_{[B,A]}$) is firmly nonexpansive follows from the firm nonexpansiveness of $T_{A,B}$ and $T_{B,A}$ and the fact that the class of firmly nonexpansive operators is closed under convex combinations (see, e.g., [6, Example 4.31]). ■

Following [12], the Borwein–Tam method specialized to two nonempty closed convex subsets U and V of X, iterates the operator $T_{[A,B]}$ of (10), where $A = N_U$ and $B = N_V$. We conclude with an example that shows that if A or B is not an affine normal cone operator then $T_{[A,B]}$ and $T_{[B,A]}$ need not be firmly nonexpansive.

Example 2.12. *Suppose that $X = \mathbb{R}^2$, that $U = \mathbb{R}_+ \cdot (1,1)$, that $V = \mathbb{R} \times \{0\}$, that $A = N_U$ and that $B = N_V$. Then neither $T_{[A,B]}$ nor $T_{[B,A]}$ is firmly nonexpansive.*

Proof. Let $(x, y) \in \mathbb{R}^2$. Using (1) we verify that $T_{A,B}(x,y) = (\frac{1}{2}(x+y)^+, y - \frac{1}{2}(x+y)^+)$ and $T_{B,A}(x,y) = (\frac{1}{2}(x-y)^+, y + \frac{1}{2}(x-y)^+)$. Now let $\alpha > 0$, let $x = (-2\alpha, 2\alpha)$ and let $y = (0,0)$. A routine calculation shows that $T_{[A,B]}x = T_{A,B}T_{B,A}(-2\alpha, 2\alpha) = (\alpha, \alpha)$ and $T_{[A,B]}y = T_{A,B}T_{B,A}(0,0) = (0,0)$, hence $\langle T_{[A,B]}x - T_{[A,B]}y, (\mathrm{Id} - T_{[A,B]})x - (\mathrm{Id} - T_{[A,B]})y\rangle = \langle(\alpha,\alpha), (-3\alpha, \alpha)\rangle = -2\alpha^2 < 0$. Applying similar argument to $T_{[B,A]}$ with $x = (-2\alpha, -2\alpha)$ and $y = (0,0)$ shows that $\langle T_{[B,A]}x - T_{[B,A]}y, (\mathrm{Id} - T_{[B,A]})x - (\mathrm{Id} - T_{[B,A]})y\rangle = \langle(-3\alpha, -\alpha), (\alpha, -\alpha)\rangle = -2\alpha^2 < 0$. It then follows from e.g., [6, Proposition 4.2] that neither $T_{[A,B]}$ nor $T_{[B,A]}$ is firmly nonexpansive. ■

[11] See [4, Proposition 3.5] for the case when U and V are linear subspaces.

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