On Fejér monotone sequences and nonexpansive mappings

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Abstract

The notion of Fejér monotonicity has proven to be a fruitful concept in fixed point theory and optimization. In this paper, we present new conditions sufficient for convergence of Fejér monotone sequences and we also provide applications to the study of nonexpansive mappings. Various examples illustrate our results.

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1 Introduction

We assume throughout the paper that

$$X \text{ is a real Hilbert space} \tag{1}$$

with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of X. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called *Fejér monotone* (see, e.g., [10], [11] and [3]) with respect to C if

$$(\forall c \in C)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - c\| \leq \|x_n - c\|\,. \tag{2}$$

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In other words, each point in a Fejér monotone sequence is not further from any point in C than its predecessor. This property has known to be an efficient tool to analyze various iterative algorithms in convex optimization.

The goal of this paper is to present some new conditions sufficient for convergence of Fejér monotone sequences. We also provide applications to the study of nonexpansive mappings.

The paper is organized as follows. In Section 2, we deal with Fejér monotonicity. Section 3 is devoted to applications in fixed point theory. Section 4 concludes the paper with a list of open problems.

The notation we employ is standard and follows, e.g., [4].

2 Fejér monotonicity

We start by recalling some pleasant properties of Fejér monotone sequences.

Fact 2.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that is is Fejér monotone with respect to a nonempty closed convex subset C of X. Then the following hold:*

(i) *The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.*

(ii) *For every $c \in C$, the sequence $(\|x_n - c\|)_{n\in\mathbb{N}}$ converges.*

(iii) *The set of strong cluster points of $(x_n)_{n\in\mathbb{N}}$ lies in a sphere of X.*

(iv) *The "shadow sequence" $(P_C x_n)_{n\in\mathbb{N}}$ converges strongly to a point in C.*

(v) *If $\operatorname{int} C \neq \varnothing$, then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a point in X.*

(vi) *If C is a closed affine subspace of X, then $(\forall n \in \mathbb{N})\ P_C x_n = P_C x_0$.*

(vii) *Every weak cluster point of $(x_n)_{n\in\mathbb{N}}$ that belongs to C must be $\lim_{n\to\infty} P_C x_n$.*

(viii) *The sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to some point in C if and only if all weak cluster points of $(x_n)_{n\in\mathbb{N}}$ lie in C.*

(ix) *If all weak cluster points of $(x_n)_{n\in\mathbb{N}}$ lie in C, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $\lim_{n\to\infty} P_C x_n$.*

Proof. (i)&(ii): [4, Proposition 5.4]. (iii): Clear from (ii). (iv): [4, Proposition 5.7]. (v): [4, Proposition 5.10]. (vi): [4, Proposition 5.9(i)]. (vii): This follows from [4, Corollary 5.11]. (viii): This follows from [4, Theorem 5.5]. (ix): Combine (viii) with (vii). ■

The following result was first presented in [2, Theorem 6.2.2(ii)]; for completeness, we include its short proof.

Lemma 2.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that is is Fejér monotone with respect to a nonempty closed convex subset C of X. Let w_1 and w_2 be weak cluster points of $(x_n)_{n\in\mathbb{N}}$. Then $w_1 - w_2 \in (C - C)^\perp$.*

Proof. Let $(c_1, c_2) \in C \times C$. Using Fact 2.1(ii), set $L_i := \lim_{n\to\infty} \|x_n - c_i\|$, for $i \in \{1, 2\}$. Note that

$$\|x_n - c_1\|^2 = \|x_n - c_2\|^2 + \|c_1 - c_2\|^2 + 2\langle x_2 - c_2, c_2 - c_1\rangle. \tag{3}$$

Now suppose that $x_{k_n} \rightharpoonup w_1$ and $x_{l_n} \rightharpoonup w_2$. Taking the limit in (3) along the two subsequences $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ yields $L_1 = L_2 + \|c_2 - c_1\|^2 + 2\langle w_1 - w_2, c_2 - c_1\rangle$ and $L_1 = L_2 + \|c_2 - c_1\|^2 + 2\langle w_2 - w_2, c_2 - c_1\rangle$. Subtracting the last two equations yields $2\langle c_2 - c_1, w_1 - w_2\rangle = 0$. ■

We are now ready for our first result which can be seen as a finite-dimensional variant of [6, Lemma 2.1] (where A is a closed linear subspace) and Fact 2.1(viii) (where $A = X$).

Proposition 2.3. *Suppose that X is finite-dimensional, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that is Fejér monotone with respect to a nonempty closed convex subset C of X, and let A be a closed convex subset of X such that $C \subseteq A$. If all cluster points of $(P_A x_n)_{n\in\mathbb{N}}$ lie in C, then $(P_A x_n)_{n\in\mathbb{N}}$ converges; in fact,*

$$\lim_{n\to\infty} P_A x_n = \lim_{n\to\infty} P_C x_n. \tag{4}$$

Proof. Set $c^* := \lim_{n\to\infty} P_C x_n$ (see Fact 2.1(iv)). By Fact 2.1(i), $(x_n)_{n\in\mathbb{N}}$ is bounded, hence so is $(P_A x_n)_{n\in\mathbb{N}}$ because P_A is nonexpansive. Now assume that all cluster points of $(P_A x_n)_{n\in\mathbb{N}}$ lie in C. Let c be an arbitrary cluster point of $(P_A x_n)_{n\in\mathbb{N}}$. Then there exist a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a point $x \in X$ such that $x_{k_n} \to x$ and $P_A x_{k_n} \to P_A x = c \in C$. It follows that $c^* \leftarrow P_C x_{k_n} \to P_C c = c$. Hence $c = c^*$ and the result follows. ■

Our second result decouples Fejér monotonicity into two properties in the case when the underlying set can be written as the sum of a set and a cone.

Proposition 2.4. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, let E be a nonempty subset of X and let K be a nonempty convex cone of X. Then the following are equivalent:*

(i) *$(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to $E + K$.*

(ii) *$(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to E and $(\forall n \in \mathbb{N})\ x_{n+1} \in x_n + K^{\oplus}$, where $K^{\oplus} := \{u \in X \mid \inf\langle u, K\rangle \geq 0\}$.*

Proof. Set

$$(\forall x \in X)(\forall n \in \mathbb{N}) \quad \Delta_n(x) := \|x_n - x\|^2 - \|x_{n+1} - x\|^2. \tag{5}$$

Then for every $e \in E$ and $k \in K$, we have

$$\begin{aligned}\Delta_n(e + k) &= \|x_n - e\|^2 + \|k\|^2 - 2\langle x_n - e, k\rangle - \big(\|x_{n+1} - e\|^2 + \|k\|^2 - 2\langle x_{n+1} - e, k\rangle\big) &\text{(6a)}\\ &= \Delta_n(e) + 2\langle x_{n+1} - x_n, k\rangle. &\text{(6b)}\end{aligned}$$

Assume first that (i) holds. Then $(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to E because $E \subseteq E + K$. Let $(e, k) \in E \times K$ and $n \in \mathbb{N}$. Using (5),

$$0 \leq \Delta_n(e + k) = \Delta_n(e) + 2\langle x_{n+1} - x_n, k\rangle. \tag{7}$$

Since K is a cone, this shows that $2\inf\langle x_{n+1}-x_n,\mathbb{R}_{++}k\rangle \geq -\Delta_n(e) > -\infty$. Hence $\langle x_{n+1}-x_n,k\rangle \geq 0$. It follows that $x_{n+1}-x_n \in K^{\oplus}$. Conversely, if (ii) holds, then (6) immediately yields (i). ■

The following consequence of Proposition 2.4 shows that Proposition 2.4 is a generalization of Fact 2.1(vi).

Corollary 2.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and let C be a closed affine subspace of X, say $C=c+Y$, where Y is a closed linear subspace of X. Then $(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to Y if and only if $(\forall n\in\mathbb{N})\ \|x_{n+1}-c\|\leq\|x_n-c\|$ and $x_{n+1}\in x_n+Y^{\perp}$, in which case $(P_Cx_n)_{n\in\mathbb{N}}$ is a constant sequence.*

We continue with the following lemma, which is a slight generalization of a theorem of Ostrowski (see [15, Theorem 26.1]) whose proof we follow.

Lemma 2.6. *Let (Y,d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a compact subset C of Y such that $d(x_n,x_{n+1})\to 0$. Then the set of cluster points of $(x_n)_{n\in\mathbb{N}}$ is a compact connected subset of C.*

Proof. Denote the set of cluster points of $(x_n)_{n\in\mathbb{N}}$ by S and assume to the contrary that $S = A\cup B$ where A and B are nonempty closed subsets of X and $A\cap B=\varnothing$. Then

$$\delta := \inf_{(a,b)\in A\times B} d(a,b) > 0. \tag{8}$$

By assumption on $(x_n)_{n\in\mathbb{N}}$, there exists $n_0\in\mathbb{N}$ such that $(\forall n\geq n_0)\ d(x_n,x_{n+1})\leq\delta/3$. Let $a\in A$. Then there exists $m>n_0$ such that $d(x_m,a)<\delta/3$. Because $(x_n)_{n>m}$ has a cluster point in B, there exists a smallest integer $k>m$ such that $d(x_k,B)<2\delta/3$. Then $d(x_{k-1},B)\geq 2\delta/3$ and hence $d(x_k,B)\geq d(x_{k-1},B)-d(x_{k-1},x_k)\geq 2\delta/3-\delta/3=\delta/3$. Thus $\delta/3\leq d(x_k,B)<2\delta/3$. Repeating this argument yields a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(\forall n\in\mathbb{N})\ \delta/3\leq d(x_{k_n},B)<2\delta/3$. Let x be a cluster point of $(x_{k_n})_{n\in\mathbb{N}}$. It follows that

$$\delta/3\leq d(x,B)\leq 2\delta/3 \tag{9}$$

Obviously, $x\notin B$. Hence $x\in A$, and therefore (recall (8)) $\delta\leq\delta(x,B)\leq 2\delta/3<\delta$, which is absurd. ■

An immediate consequence of Lemma 2.6 is the classical Ostrowski result.

Corollary 2.7 (Ostrowski). *Suppose that X is finite-dimensional and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X such that $(x_n)_{n\in\mathbb{N}}$ is* asymptotically regular, *i.e.*, $x_n-x_{n+1}\to 0$. *Then the set of cluster points of $(x_n)_{n\in\mathbb{N}}$ is compact and connected.*

We are now in position to prove the following key result which can be seen as a variant of Fact 2.1(v).

Theorem 2.8 (a new sufficient condition for convergence). *Suppose that X is finite-dimensional and that C is a nonempty closed convex subset of X of co-dimension 1, i.e.,*

$$\operatorname{codim} C := \operatorname{codim}(\operatorname{aff} C - \operatorname{aff} C) = 1. \tag{10}$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence that is Fejér monotone with respect to C and asymptotically regular, i.e., $x_n - x_{n+1} \to 0$. Then $(x_n)_{n\in\mathbb{N}}$ is actually convergent.

Proof. By Fact 2.1(i), $(x_n)_{n\in\mathbb{N}}$ is bounded. Denote by S the set of cluster points of $(x_n)_{n\in\mathbb{N}}$. Since $x_n - x_{n+1} \to 0$, Corollary 2.7 implies that S is connected. Moreover, S lies in a sphere of X due to Fact 2.1(iii). On the other hand, by combining Lemma 2.2 and (10), S lies in a line of X. Altogether S is a connected subset of a sphere that lies on a line. We deduce that S is a singleton. ■

We conclude with two examples illustrating that the assumptions on asymptotic regularity and co-dimension 1 are important.

Example 2.9. Suppose that $X = \mathbb{R}^2$, set $C = \{0\} \times \mathbb{R}$, and $(\forall n \in \mathbb{N})\ x_n = ((-1)^n, 0)$. Then $\operatorname{codim} C = 1$ and $(\forall c \in C)\ (\forall n \in \mathbb{N})\ \|x_n - c\| = \|x_{n+1} - c\|$, hence $(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to C. However, $(x_n)_{n\in\mathbb{N}}$ does not converge. This does not contradict Theorem 2.8 because $\|x_n - x_{n+1}\| = 2 \not\to 0$.

Example 2.10. Suppose that $X = \mathbb{R}^2$, set $C = \{(0,0)\} \subseteq X$, and $(\forall n \in \mathbb{N})\ \theta_n = \sum_{k=1}^{n}(1/k)$ and $x_n = \cos(\theta_n)(1,0) + \sin(\theta_n)(0,1)$. Then $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular and Fejér monotone with respect to C. However, the set of cluster points of $(x_n)_{n\in\mathbb{N}}$ is the unit sphere because the harmonic series diverges. Again, this does not contradict Theorem 2.8 because $\operatorname{codim} C = 2 \neq 1$.

3 Asymptotic behaviour of nonexpansive mappings

From now on, we assume that

$$\boxed{T\colon X \to X \ \text{ is nonexpansive.}} \tag{11}$$

Let x and y be in X. It is clear that $(\forall n \in \mathbb{N})\ \|T^{n+1}x - T^{n+1}y\| \leq \|T^n x - T^n y\|$ is bounded. The following question is thus extremely natural:

$$\text{Under which conditions on } T \text{ must } (T^n x - T^n y)_{n\in\mathbb{N}} \text{ always converge weakly?} \tag{12}$$

We first note that (12) will impose some restriction on T:

Example 3.1. Suppose that $X = \mathbb{R}$, that $T = -\operatorname{Id}$, that $x \neq 0$ and that $y = 0$. Then the sequence $(T^n x - T^n y)_{n\in\mathbb{N}} = ((-1)^n x)_{n\in\mathbb{N}}$ is not convergent.

The following two results are well known.

Fact 3.2. (See, e.g., [16, Corollary 6].) *Exactly one of the following holds:*

(i) $\operatorname{Fix}(T) = \varnothing$ *and* $(\forall x \in X)\ \|T^n x\| \to \infty$.

(ii) $\operatorname{Fix}(T) \neq \varnothing$ *and* $(\forall x \in X)\ (T^n x)_{n\in\mathbb{N}}$ *is bounded.*

Fact 3.3. (See, e.g., [1, Theorem 1.2].) *Suppose that* $\operatorname{Fix}(T) \neq \varnothing$ *and let* $x \in X$*. Then* $(T^n x)_{n\in\mathbb{N}}$ *is weakly convergent if and only if* $T^n x - T^{n+1}x \rightharpoonup 0$*; if this is the case, then* $(T^n x)_{n\in\mathbb{N}}$ *converges weakly to a point in* $\operatorname{Fix}(T)$*.*

To make further progress, let us recall that $\overline{\operatorname{ran}}\,(\operatorname{Id} - T)$ is a nonempty closed convex set, which makes the vector

$$v := P_{\overline{\operatorname{ran}}(\operatorname{Id} - T)} 0 \tag{13}$$

well defined (see [9], [1] and [16]), and which gives rise to the generalized (possibly empty) fixed point set

$$\operatorname{Fix}(v + T) = \big\{ x \in X \;\big|\; x = v + Tx \big\}. \tag{14}$$

We now recall the following helpful fact.

Fact 3.4. (See [8, Proposition 2.4].) *Suppose that* $\operatorname{Fix}(v+T) \neq \varnothing$*. Then the following hold:*

(i) $\operatorname{Fix}(v+T) - \mathbb{R}_+ v \subseteq \operatorname{Fix}(v+T)$.

(ii) $(\forall y \in \operatorname{Fix}(v+T))(\forall n \in \mathbb{N})\ T^n y = y - nv$.

(iii) *For every* $x \in X$*, the sequence* $(T^n x + nv)_{n\in\mathbb{N}}$ *is Fejér monotone with respect to* $\operatorname{Fix}(v+T)$*.*

Remark 3.5. Suppose that $\operatorname{Fix}(v+T) \neq \varnothing$. Then

$$(\forall x \in X)(\forall y \in X) \quad (T^n x - T^n y)_{n\in\mathbb{N}} \ \text{ is weakly convergent} \tag{15a}$$

if and only if

$$(\forall x \in X) \quad (T^n x + nv)_{n\in\mathbb{N}} \ \text{ is weakly convergent.} \tag{15b}$$

Indeed, if (15a) holds, then (15b) follows by choosing $y \in \operatorname{Fix}(v+T)$ and recalling Fact 3.4(ii). Conversely, assume that (15b) holds. Then $(T^n x + nv)_{n\in\mathbb{N}}$ and $(T^n y + nv)_{n\in\mathbb{N}}$ are weakly convergent, and so is their difference which yields (15a).

We can now give a mild sufficient condition for (12):

Theorem 3.6. *Suppose that* $X = \mathbb{R}$*, that* $v \neq 0$*, and that* $\operatorname{Fix}(v+T) \neq \varnothing$*. Then the sequence* $(T^n x + nv)_{n\in\mathbb{N}}$ *is convergent.*

Proof. By Fact 3.4(i)&(iii), the sequence $(T^n x + nv)_{n\in\mathbb{N}}$ is Fejér monotone with respect to $C := \operatorname{Fix}(v+T)$, and C contains a ray. Therefore, $\operatorname{int} C \neq \varnothing$ and Fact 2.1(v) yields the convergence of $(T^n x + nv)_{n\in\mathbb{N}}$. ■

Remark 3.7. Example 3.1 shows that the assumption that $v \neq 0$ in Theorem 3.6 is important.

Theorem 3.8. *Suppose that T is affine, say $T\colon x \to Lx + b$, where L is linear and nonexpansive, and $b \in X$. Suppose furthermore that L is asymptotically regular*[1]*, and let x and y be points in X. Then*

$$T^n x - T^n y = L^n(x-y) \to P_{\mathrm{Fix}(L)}(x-y). \tag{16}$$

Proof. Using [8, Theorem 3.2(ii)], we have $(\forall n \in \mathbb{N})\ T^n x - T^n y = L^n x - L^n y = L^n(x - y)$. The asymptotic regularity assumption yields $L^n(x-y) - L^{n+1}(x-y) \to 0$. Using [4, Proposition 5.27], we see that altogether $T^n x - T^n y = L^n(x-y) \to P_{\mathrm{Fix}(L)}(x-y)$ ■

To make further progress we impose now additional assumptions on T. Recall that our nonexpansive T is *averaged* if there exist a nonexpansive operator $R : X \to X$ and a constant $\alpha \in \,]0,1[$ such that $T = (1-\alpha)\,\mathrm{Id} + \alpha R$; equivalently, (see, e.g., [4, Proposition 4.25])

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \tfrac{1-\alpha}{\alpha}\,\|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \le \|x-y\|^2. \tag{17}$$

If $\alpha = 1/2$, then T is said to be *firmly nonexpansive*. Averaged operators have proven to be a useful class in fixed point theory and optimization; see [1] and [12].

The following result yields a generalized asymptotic regularity for averaged nonexpansive operators.

Lemma 3.9. *Suppose that T is averaged and that $\mathrm{Fix}(v+T) \neq \varnothing$. Then for every $x \in X$, $T^n x - T^{n+1}x \to v$; equivalently, $(T^n x + nv)_{n\in\mathbb{N}}$ is asymptotically regular.*

Proof. Let $x \in X$ and $y \in \mathrm{Fix}(v+T)$. Since T is averaged, it follows from (17) and Fact 3.4(ii) that there exists $\alpha \in \,]0,1[$ such that

$$(\forall n \in \mathbb{N}) \quad \|T^{n+1}x - T^{n+1}y\|^2 \le \|T^n x - T^n y\|^2 - \tfrac{1-\alpha}{\alpha}\|T^n x - T^{n+1}x - v\|^2. \tag{18}$$

Telescoping yields $\sum_{n=0}^{\infty} \left\|T^n x - T^{n+1}x - v\right\|^2 < +\infty$ and consequently $T^n x - T^{n+1}x \to v$. ■

Amazingly, on the real line, averagedness is a sufficient condition for (12):

Theorem 3.10. *Suppose that $X = \mathbb{R}$ and that T is averaged. Let x and y be in $\mathbb{R}$. Then the sequence $(T^n x - T^n y)_{n\in\mathbb{N}}$ is convergent.*

Proof. Set $(\forall n \in \mathbb{N})\ a_n := T^n x - T^n y$. We must show that $(a_n)_{n\in\mathbb{N}}$ is convergent. From (17), there exists $\alpha \in \,]0,1[$ such that

$$(\forall n \in \mathbb{N}) \quad a_{n+1}^2 + \tfrac{1-\alpha}{\alpha}(a_n - a_{n+1})^2 \le a_n^2. \tag{19}$$

Set $\beta := 1 - 2\alpha$ and note that $0 \le |\beta| < 1$. By viewing (19) as a quadratic inequality in a_{n+1}, we learn that

$$(\forall n \in \mathbb{N}) \quad |a_{n+1}| \le |a_n| \ \text{ and } a_{n+1} \text{ lies between } a_n \text{ and } \beta a_n. \tag{20}$$

[1] Recall that T is asymptotically regular at x if $T^n x - T^{n+1}x \to 0$ and that T is asymptotically regular if it is asymptotically regular at every point.

If some $a_{n_0} = 0$, then $a_n \to 0$ and we are done. So assume that $a_n \neq 0$ for every $n \in \mathbb{N}$. If $(a_n)_{n\in\mathbb{N}}$ changes sign only finitely many times, then $(a_n)_{n\in\mathbb{N}}$ is eventually always positive or negative. Since $(|a_n|)_{n\in\mathbb{N}}$ is decreasing, we deduce that $(a_n)_{n\in\mathbb{N}}$ is convergent. Finally, we assume that $(a_n)_{n\in\mathbb{N}}$ changes signs frequently. If $n \in \mathbb{N}$ and $\operatorname{sgn}(a_{n+1}) = -\operatorname{sgn}(a_n)$, then $|a_{n+1}| \leq |\beta||a_n|$; since this occurs infinitely many times, it follows that $a_n \to 0$. ■

Theorem 3.11. *Suppose that X is finite-dimensional, that T is averaged, that $\operatorname{Fix}(v+T) \neq \varnothing$, and that that $\operatorname{codim}\operatorname{Fix}(v+T) \leq 1$. Then for every $(x,y) \in X \times X$, the sequence $(T^n x - T^n y)_{n\in\mathbb{N}}$ is convergent.*

Proof. In view of Remark 3.5, we let $x \in X$ and must show that $(T^n x + nv)_{n\in\mathbb{N}}$ is convergent. Set $C := \operatorname{Fix}(v+T)$ and $(\forall n \in \mathbb{N})\ x_n := T^n x + nv$. By Fact 3.4(iii), $(x_n)_{n\in\mathbb{N}}$ is Fejér monotone with respect to C. Suppose first that $\operatorname{codim} C = 0$. Then $\operatorname{int} C \neq \varnothing$ and we are done by Fact 2.1(v). Now assume that $\operatorname{codim} C = 1$. By Lemma 3.9, $(x_n)_{n\in\mathbb{N}}$ is asymptotically regular. Altogether, by Theorem 2.8, $(x_n)_{n\in\mathbb{N}}$ is convergent. ■

Corollary 3.12. *Let $x \in X$. Suppose that $X = \mathbb{R}^2$, that T is averaged, that $v \neq 0$, and that $\operatorname{Fix}(v+T) \neq \varnothing$. Then for every $(x,y) \in X \times X$, the sequence $(T^n x - T^n y)_{n\in\mathbb{N}}$ is convergent.*

Proof. Because $v \neq 0$, Fact 3.4(i) implies that $\dim \operatorname{Fix}(v+T) \geq 1$, i.e., $\operatorname{codim}\operatorname{Fix}(v+T) \leq \dim(X) - 1 = 1$. The result now follows from Theorem 3.11. ■

4 Open problems

We now present a list of open problems that may be easier than the general question (12). Let x and y be in X.

P1: Suppose that $X = \mathbb{R}$, $v = 0$ but $\operatorname{Fix}(T) = \varnothing$. Is $(T^n x - T^n y)_{n\in\mathbb{N}}$ convergent?

P2: Suppose that $X = \mathbb{R}$, $v \neq 0$ but $\operatorname{Fix}(v+T) = \varnothing$. Is $(T^n x - T^n y)_{n\in\mathbb{N}}$ convergent?

P3: Does Corollary 3.12 remain true if $\dim(X) \geq 3$?

P4: What can be said for (12) if we replace "weakly" by "strongly"?

Let us conclude with an example which numerically illustrates that the answer to **P3** may be positive.

Example 4.1. Suppose that $X = \mathbb{R}^3$ and let A and B be two closed balls in X. Set[2] $T = \frac{1}{2}(\operatorname{Id} + R_B R_A)$. Then T is firmly nonexpansive and hence averaged. (In fact, T is the *Douglas–Rachford operator* [14] associated with the sets A and B.) It follows from [5, Theorem 3.5] that $A \cap (B+v) + N_{\overline{A-B}}v \subseteq \operatorname{Fix}(v+T) \subseteq v + A \cap (B+v) + N_{\overline{A-B}}v$. Furthermore, [7, Example 5.7] implies that $N_{\overline{A-B}}v$ is a ray, hence $\operatorname{Fix}(v+T)$ is ray and therefore $\dim \operatorname{Fix}(v+$

[2]For a nonempty closed convex subset C of X we use N_C, P_C and $R_C := 2P_C - \operatorname{Id}$ to denote the normal cone operator , the projector and the reflector associated with C, respectively.

$T) = 1$ and so $\operatorname{codim} \operatorname{Fix}(v + T) = 2$. Even though Theorem 3.11 is not applicable here, we still conjecture that $(T^n x + nv)_{n\in\mathbb{N}}$ converges (see Figure 1 below).

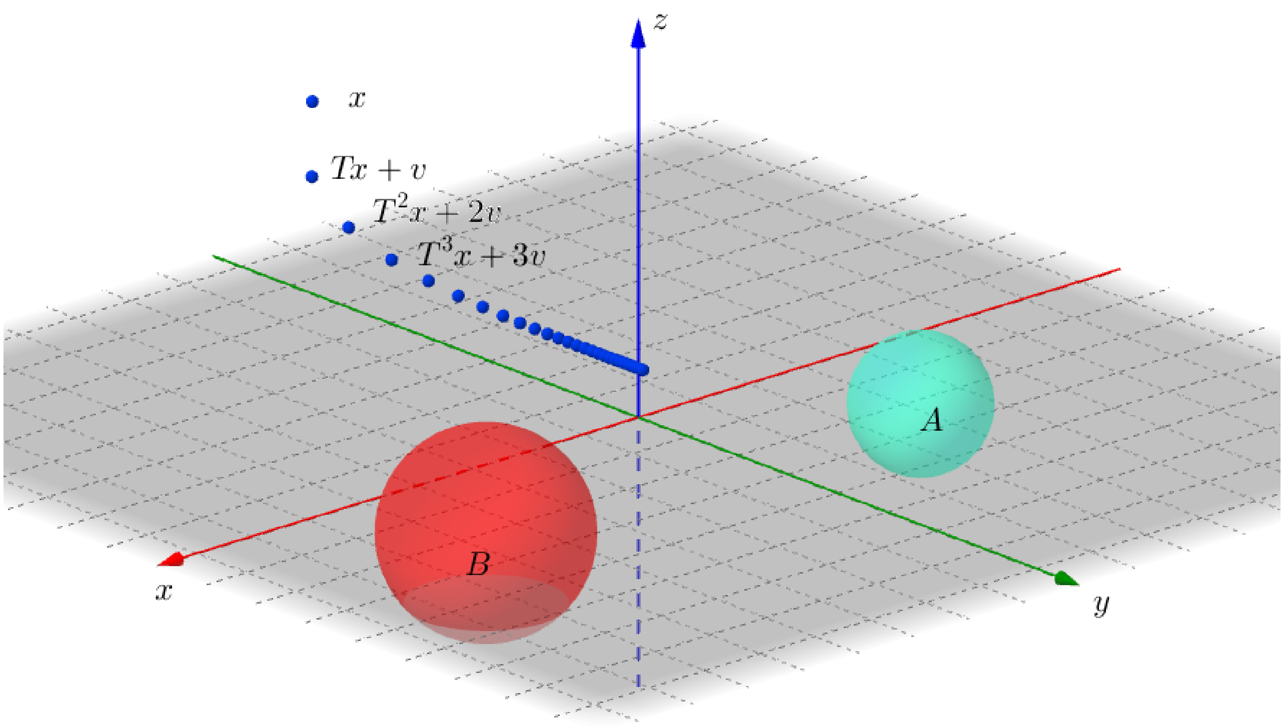

Figure 1: A `GeoGebra` [13] snapshot that illustrates Example 4.1. The first few terms of the sequence $(T^n x + nv)_{n\in\mathbb{N}}$ (blue points) are depicted.

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