On a result of Pazy concerning the asymptotic behaviour of nonexpansive mappings

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Abstract

In 1971, Pazy presented a beautiful trichotomy result concerning the asymptotic behaviour of the iterates of a nonexpansive mapping. In this note, we analyze the fixed-point free case in more detail. Our results and examples give credence to the conjecture that the iterates always converge cosmically. The relationship to recent work by Lins is also discussed.

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1 Introduction

Throughout, X is a finite-dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and $T\colon X \to X$ is *nonexpansive*, i.e., $(\forall x \in X)(\forall y \in X)\ \|Tx - Ty\| \leq$

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$\|x-y\|$. Then, using [14], the vector

$$v := P_{\overline{\operatorname{ran}}\,(\operatorname{Id}-T)}(0) \tag{1}$$

is well defined. The following remarkable result was proved by A. Pazy in 1971. (In fact, Fact 1.1 holds in general Hilbert space. See also [16], [17], [18] and [15] for even more general settings. We thank Simeon Reich for bringing these references to our attention.)

Fact 1.1 (Pazy's trichotomy; [14]). *Let $x \in X$. Then*

$$\lim_{n\to\infty} \frac{T^n x}{n} = -v. \tag{2}$$

Moreover, exactly one of the following holds:

(i) $0 \in \operatorname{ran}(\operatorname{Id}-T)$, *and* $(T^n x)_{n\in\mathbb{N}}$ *is bounded for every* $x \in X$.

(ii) $0 \in \overline{\operatorname{ran}}\,(\operatorname{Id}-T) \smallsetminus \operatorname{ran}(\operatorname{Id}-T)$, $\|T^n x\| \to \infty$ *and* $\frac{1}{n}T^n x \to 0$ *for every* $x \in X$.

(iii) $0 \notin \overline{\operatorname{ran}}\,(\operatorname{Id}-T)$, *and* $\lim_{n\to\infty} \frac{1}{n}\|T^n x\| > 0$ *for every* $x \in X$.

A referee brought to our attention that v is also known as the *cycle time vector* of T and $\|v\| = \lim_{n\to\infty} \|T^n x\|/n$ is also known as the *linear rate of growth/escape* of T; see [7] and [11]. Now consider the case when T does not have a fixed point. Let $x \in X$. In view of Fact 1.1, $\|T^n x\| \to \infty$ and it is natural to ask whether additional asymptotic information is available about the (eventually well defined) sequence

$$(Q_n(x))_{n\in\mathbb{N}} := \left(\frac{T^n x}{\|T^n x\|}\right)_{n\in\mathbb{N}}. \tag{3}$$

Since X is finite-dimensional, for every $x \in X$, $(Q_n(x))_{n\in\mathbb{N}}$ has cluster points. If the sequence $(Q_n(x))_{n\in\mathbb{N}}$ actually converges, then we refer to this also as *cosmic convergence*[1]. Combining (1) and (2), we obtain the following necessary condition for cosmic convergence:

$$0 \notin \overline{\operatorname{ran}}\,(\operatorname{Id}-T) \quad\Rightarrow\quad v \neq 0 \text{ and } (\forall x \in X)\ Q_n(x) \to -v/\|v\|. \tag{4}$$

The aim of this note is to provide conditions sufficient for convergence of $(Q_n(x))_{n\in\mathbb{N}}$ in the case when $0 \in \overline{\operatorname{ran}}\,(\operatorname{Id}-T) \smallsetminus \operatorname{ran}(\operatorname{Id}-T)$.

The results in this note nurture the conjecture that the sequence $(Q_n(x))_{n\in\mathbb{N}}$ actually converges. We test this conjecture on special cases of the method of alternating projections and the proximal point algorithm — these are two basic and powerful optimization algorithms that continue to be an intensely studied topic in optimization. We now turn to a related fact, which was kindly brought to our attention by a referee:

[1] It will become clear in Section 2.4 why we speak of cosmic convergence.

Fact 1.2 (Lins; [12]). *Suppose that* $\operatorname{Fix} T = \varnothing$. *Then there exists* $w \in X$ *such that for every* $x \in X$, $\langle w, T^n x\rangle \to \infty$.

Lins's result, which can be traced back to [5] and [10], holds in the much more general setting of a finite-dimensional normed space! In our present setting, we observe that when $v \neq 0$, then Fact 1.2 is a consequence of Fact 1.1: indeed, set $w = -v$. Then $\langle w, T^n x\rangle = -n\langle v, (1/n)T^n x\rangle \to -\infty \cdot \langle v, -v\rangle = +\infty$. Furthermore, if our conjecture on cosmic convergence is true, then $Q_n(x) \to q$ and hence $\langle q, T^n x\rangle = \|T^n x\| \langle q, Qn(x)\rangle \to \infty \cdot \langle q, q\rangle = \infty$, i.e., Fact 1.2 holds with $w = q$.

Finally, notation and notions not explicitly defined may be found in [3], [19], or [21].

2 Results

2.1 The one-dimensional case

Theorem 2.1. *Suppose that* X *is one-dimensional and that* $\operatorname{Fix} T = \varnothing$. *Then* T *admits cosmic convergence; in fact, exactly one of the following holds:*

(i) $(\forall x \in X)$ $Tx > x$, $T^n x \to +\infty$, *and* $Q_n(x) \to +1$.

(ii) $(\forall x \in X)$ $Tx < x$, $T^n x \to -\infty$, *and* $Q_n(x) \to -1$.

Proof. We can and do assume that $X = \mathbb{R}$. If there existed a and b in $\mathbb{R}$ such that $Ta > a$ and $Tb < b$, then the Intermediate Value Theorem would provide a point z between a and b such that $Tz = z$, which is absurd in view of the hypothesis. It follows that either $\operatorname{ran}(\operatorname{Id} - T) \subseteq \mathbb{R}_{--}$ or $\operatorname{ran}(\operatorname{Id} - T) \subseteq \mathbb{R}_{++}$. Let us first assume that $\operatorname{ran}(\operatorname{Id} - T) \subseteq \mathbb{R}_{--}$, i.e., $(\forall x \in \mathbb{R})$ $x < Tx$. Let $x \in \mathbb{R}$. On the one hand, we have $x < Tx < T(Tx) = T^2 x < T^3 x < \cdots < T^n x < T^{n+1} x < \cdots$. On the other hand, by Fact 1.1(ii)&(iii), $|T^n x| \to +\infty$. Altogether, $T^n x \to +\infty$ and hence $Q_n(x) \to +1$. Finally, the case when $\operatorname{ran}(\operatorname{Id} - T) \subseteq \mathbb{R}_{++}$ is treated similarly. ■

2.2 Composition of two projectors

In this section, we assume that

$$A \text{ and } B \text{ are nonempty closed convex subsets of } X \tag{5}$$

with corresponding projectors (nearest point mappings) P_A and P_B, respectively, and that

$$T = P_B P_A. \tag{6}$$

Iterating T is a classical algorithm in optimization known as the *Method of Alternating Projections*. The MAP can be traced back to work by John von Neumann [22] and continues to this date to be a very active area of research[2].

We begin with a few technical lemmas.

Lemma 2.2. *Let K be a nonempty closed convex cone. Then*[3] $(K^{\ominus})^{\perp} = K \cap (-K)$.

Proof. We will use repeatedly the fact that (see [3, Corollary 6.33]) $(K^{\ominus})^{\ominus} = K$. "$\subseteq$": Indeed, $(K^{\ominus})^{\perp} \subseteq (K^{\ominus})^{\ominus} = K$ and $(K^{\ominus})^{\perp} \subseteq (K^{\ominus})^{\oplus} = -K$; hence, $(K^{\ominus})^{\perp} \subseteq K \cap (-K)$. "$\supseteq$": Let $x \in K \cap (-K)$. Then $\langle x, K^{\ominus}\rangle \leq 0$ and $\langle -x, K^{\ominus}\rangle \leq 0$ and thus $\langle x, K^{\ominus}\rangle = 0$, i.e., $x \in (K^{\ominus})^{\perp}$. ■

Lemma 2.3. *The set of (oriented) functionals separating the sets A and B satisfies*

$$\mathcal{U} := \big\{u \in X \smallsetminus \{0\} \;\big|\; \sup\langle A, u\rangle \leq \inf\langle B, u\rangle\big\} = \big(\overline{\mathrm{cone}}(A - B)\big)^{\ominus} \smallsetminus \{0\}. \tag{7}$$

Moreover[4],

$$(\mathrm{rec}\, A) \cap (\mathrm{rec}\, B) \subseteq \bigcap_{u \in \mathcal{U}} \{u\}^{\perp} = \overline{\mathrm{cone}}(A - B) \cap \overline{\mathrm{cone}}(B - A). \tag{8}$$

Consequently, if $A \cap B = \varnothing$, then $\mathcal{U} \neq \varnothing$ and $(\mathrm{rec}\, A) \cap (\mathrm{rec}\, B)$ is a nonempty closed convex cone that is contained in a proper hyperplane of X.

Proof. Since (7) is easily checked, we turn to (8): Let us first deal with the inclusion. If $\mathcal{U} = \varnothing$, then the intersection is trivially equal to X and we are done. So suppose that $u \in \mathcal{U}$, set $R := \mathrm{rec}\, A$ and $S := \mathrm{rec}\, B$. Then $A + R = A$ and $B + S = B$; consequently, $\sup\langle A + R, u\rangle \leq \inf\langle B + S, u\rangle$. Since R and S are cones, we deduce that $R \subseteq \{u\}^{\ominus}$ and $S \subseteq \{u\}^{\oplus}$. Therefore, $R \cap S \subseteq \{u\}^{\perp}$. This completes the proof of the inclusion. Now $x \in \bigcap_{u \in \mathcal{U}} \{u\}^{\perp} \Leftrightarrow (\forall u \in \mathcal{U})\ \langle x, u\rangle = 0 \Leftrightarrow x \in \mathcal{U}^{\perp} = (\overline{\mathrm{cone}}(A - B))^{\ominus\perp} = \overline{\mathrm{cone}}(A - B) \cap (-\overline{\mathrm{cone}}(A - B)) = \overline{\mathrm{cone}}(A - B) \cap \overline{\mathrm{cone}}(B - A)$ by Lemma 2.2. The "Consequently" part follows from the Separation Theorem (see, e.g., [13, Theorem 2.5]). ■

Lemma 2.4. $0 \in \overline{\mathrm{ran}}\,(\mathrm{Id} - T) \smallsetminus \mathrm{ran}(\mathrm{Id} - T) \Leftrightarrow \mathrm{Fix}\, T = \varnothing$.

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[2]In fact, a Google search for `method of alternating projections` yields more than 400,000 results.

[3]We write $S^{\ominus} := \big\{x \in X \;\big|\; \sup\langle x, S\rangle \leq 0\big\}$ and $S^{\oplus} := -S^{\ominus}$ for a subset S of X.

[4]We use $\mathrm{rec}\, S := \big\{x \in X \;\big|\; x + S \subseteq S\big\}$ to denote the recession cone of a nonempty convex subset of X.

Proof. By [1] (see also [4] for extensions to firmly nonexpansive operators), we always have $0 \in \overline{\operatorname{ran}}\,(\operatorname{Id} - T)$, and this implies the result. ■

We are now ready for the main result of this section.

Theorem 2.5. *Suppose that* $\operatorname{Fix} T = \varnothing$. *Let* $b_0 := x \in X$ *and set* $(\forall n \in \mathbb{N})$ $a_{n+1} := P_A b_n$ *and* $b_{n+1} := P_B a_{n+1} = T b_n$. *Then the following hold:*

(i) $\|a_n\| \to +\infty$, $\|b_n\| \to +\infty$, $b_n - a_n \to g$, *and* $a_{n+1} - b_n \to -g$, *where* $g := P_{\overline{B-A}}(0)$.

(ii) *All cluster points of* $(b_n/\|b_n\|)_{n\in\mathbb{N}}$ *lie in the set*

$$\big((\operatorname{rec} A) \cap (\operatorname{rec} B)\big) \cap \big((\operatorname{rec} A) \cap (\operatorname{rec} B)\big)^{\oplus}, \tag{9}$$

which is a closed convex cone in X *that properly contains* $\{0\}$.

(iii) *Neither* $(\operatorname{rec} A) \cap (\operatorname{rec} B)$ *nor* $((\operatorname{rec} A) \cap (\operatorname{rec} B))^{\oplus}$ *is a linear subspace of* X.

(iv) **(cosmic convergence)** *The sequence* $(Q_n(x))_{n\in\mathbb{N}} = (b_n/\|b_n\|)_{n\in\mathbb{N}}$ *converges provided one of the following holds:*

  (a) $((\operatorname{rec} A) \cap (\operatorname{rec} B)) \cap ((\operatorname{rec} A) \cap (\operatorname{rec} B))^{\oplus}$ *is a ray.*

  (b) $(\operatorname{rec} A) \cap (\operatorname{rec} B)$ *is a ray.*

  (c) $\dim X = 2$.

Proof. Set $R := (\operatorname{rec} A) \cap (\operatorname{rec} B)$, which is a nonempty closed convex cone.

(i): See [2, Theorem 4.8].

(ii): Note that (i) makes the quotient sequence eventually well defined. Let q be cluster point of $(b_n/\|b_n\|)_{n\in\mathbb{N}}$, say

$$\frac{b_{k_n}}{\|b_{k_n}\|} \to q \tag{10}$$

for some subsequence $(b_{k_n})_{n\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$. Then [3, Proposition 6.50] implies that $q \in \operatorname{rec} B$. Furthermore, since $a_{k_n} - b_{k_n} \to -g$ and $\|b_{k_n}\| \to +\infty$, we deduce that

$$\frac{a_{k_n}}{\|b_{k_n}\|} = \frac{a_{k_n} - b_{k_n}}{\|b_{k_n}\|} + \frac{b_{k_n}}{\|b_{k_n}\|} \to q. \tag{11}$$

As before, this implies that $q \in \operatorname{rec} A$. Thus

$$q \in (\operatorname{rec} A) \cap (\operatorname{rec} B) = R. \tag{12}$$

On the other hand, using [23, Theorem 3.1] and [3, Proposition 6.34], we have

$$
\begin{aligned}
b_{n+1} - b_n &= (b_{n+1} - a_{n+1}) + (a_{n+1} - b_n) \in \operatorname{ran}(P_B - \mathrm{Id}) + \operatorname{ran}(P_A - \mathrm{Id}) && \text{(13a)}\\
&\subseteq \overline{\operatorname{ran}}\,(P_B - \mathrm{Id}) + \overline{\operatorname{ran}}\,(P_A - \mathrm{Id}) = (\operatorname{rec} B)^{\oplus} + (\operatorname{rec} A)^{\oplus} && \text{(13b)}\\
&\subseteq \overline{(\operatorname{rec} A)^{\oplus} + (\operatorname{rec} B)^{\oplus}} = \big((\operatorname{rec} A) \cap (\operatorname{rec} B)\big)^{\oplus} = R^{\oplus}. && \text{(13c)}
\end{aligned}
$$

It follows that $b_n - b_0 = \sum_{k=0}^{n-1}(b_{k+1} - b_k) \in nR^{\oplus} = R^{\oplus}$; hence, $(b_n - b_0)/\|b_n - b_0\| \in R^{\oplus}$ which implies that $q \in R^{\oplus}$. Altogether, $q \in R \cap R^{\oplus}$. Since $\|q\| = 1$, we deduce that $\{0\} \subsetneqq R \cap R^{\oplus}$. Finally, if R was a linear subspace of X, then $R \cap R^{\oplus} = R \cap R^{\perp} = \{0\}$, which is absurd. Hence R is not a linear subspace. If $R^{\oplus}$ were a linear subspace of X, then so would be $R^{\oplus\oplus} = R$, which is absurd.

(iv): In view of (ii), $R \cap R^{\oplus}$ contains a ray and it suffices to show that $R \cap R^{\oplus}$ is precisely a ray. Indeed, each of the listed conditions guarantees that — for (iv)(c) use (iii). ■

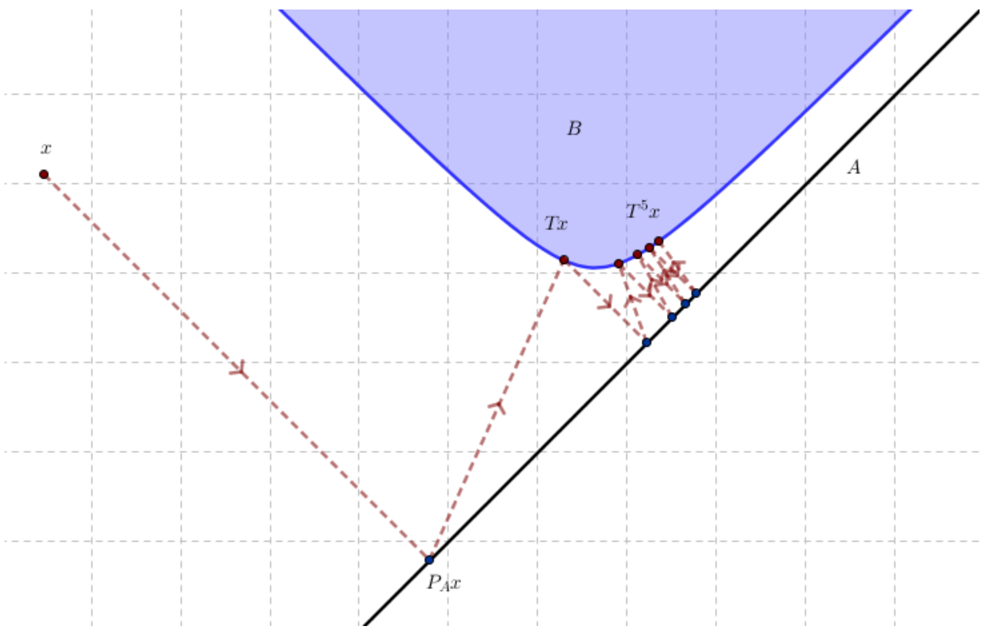

Figure 1: A `GeoGebra` [8] snapshot in $\mathbb{R}^2$ for two sets A (the black line) and B (the blue region) illustrating Theorem 2.5(iv)(c). Shown are the first few iterates of the sequence $(T^n x)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}$ (red points) and of the sequence $(a_n)_{n\in\mathbb{N}}$ (blue points). We visually confirm cosmic convergence: the sequence $(Q_n(x))_{n\in\mathbb{N}}$ converges to $(1/\sqrt{2})(1,1)$.

In Figure 1, we visualize Theorem 2.5(iv)(c) for the case when A and B are nonintersecting unbounded closed convex subsets in the Euclidean plane.

2.3 Firmly nonexpansive operators

Recall that $x \in X$ belongs to the *horizon cone* of a nonempty subset C of X, written $x \in C^\infty$ if there exist sequences $(c_n)_{n\in\mathbb{N}}$ in C and $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{++}$ such that $\lambda_n \to 0$ and $\lambda_n c_n \to x$. Note that $C^\infty = \overline{C}^\infty$; furthermore, if C is closed and convex, then $C^\infty = \operatorname{rec} C$ (see [21, Section 6.G]). The notion of the horizon cone allows us to present a superset of cluster points of the iterates of T.

Theorem 2.6. *Suppose that* $\operatorname{Fix} T = \varnothing$, *let* $x_0 := x \in X$, *and set* $(\forall n \in \mathbb{N})\ x_{n+1} := Tx_n$. *Then the following hold:*

(i) *All cluster points of* $(x_n/\|x_n\|)_{n\in\mathbb{N}}$ *lie in the cone*

$$R := \big(\operatorname{ran} T\big)^\infty \cap \big(\overline{\operatorname{cone}}\operatorname{ran}(T-\operatorname{Id})\big). \tag{14}$$

(ii) *If* T *is firmly nonexpansive, then* $R = \operatorname{rec}(\overline{\operatorname{ran}}\, T) \cap (\overline{\operatorname{cone}}\operatorname{ran}(T-\operatorname{Id}))$.

(iii) **(cosmic convergence)** *If* R *is a ray, then* $(Q_n(x))_{n\in\mathbb{N}} = (x_n/\|x_n\|)_{n\in\mathbb{N}}$ *converges.*

Proof. By Fact 1.1, $\|x_n\| \to \infty$; thus, the quotient sequence is eventually well defined. (i): Let q be a cluster point of $(x_n/\|x_n\|)_{n\in\mathbb{N}}$. It is clear that $q \in (\operatorname{ran} T)^\infty$. For every $n \in \mathbb{N}$, we have $x_{n+1} - x_0 = \sum_{k=0}^{n}(x_{k+1} - x_k) \in (n+1)\operatorname{ran}(T-\operatorname{Id}) \subseteq \operatorname{cone}\operatorname{ran}(T-\operatorname{Id})$; hence, $(x_{n+1}-x_0)/\|x_{n+1}\| \in \overline{\operatorname{cone}}\operatorname{ran}(T-\operatorname{Id})$ and thus $q \in \overline{\operatorname{cone}}\operatorname{ran}(T-\operatorname{Id})$. Note that since T is nonexpansive, $\overline{\operatorname{ran}}\,(\operatorname{Id}-T)$ is convex (see [14, Lemma 4]). (ii): Since T is firmly nonexpansive, so is $\operatorname{Id}-T$ which implies that $\overline{\operatorname{ran}}\,(\operatorname{Id}-(\operatorname{Id}-T)) = \overline{\operatorname{ran}}\, T$ is convex (again by [14, Lemma 4]). The conclusion now follows because the horizon cone and recession cone coincide for closed convex sets. (iii): This is clear. ■

The following result allows a reduction to lower-dimensional cases.

Theorem 2.7. *Let* Y *be a linear subspace of* X, *and let* $B\colon Y \rightrightarrows Y$ *be maximally monotone. Set* $A := BP_Y$ *and suppose that* $T = J_A := (\operatorname{Id}+A)^{-1}$. *Let* $x \in X$. *Then the following hold:*

(i) $A\colon X \rightrightarrows X$ *is maximally monotone and* $T = P_{Y^\perp} + J_B P_Y$, *where* $J_B := (\operatorname{Id}+B)^{-1}$.

(ii) $(\forall n \in \mathbb{N})\ T^n x = P_{Y^\perp}x + J_B^n(P_Y x)$.

Proof. (i): This follow from [3, Proposition 23.23]. (ii): Clear from (i) and induction. ■

We are now in a position to obtain a positive result for proximity operators of certain convex functions. Proximity operators are at the heart of many optimization algorithms such as the *Proximal Point Algorithm* (see [20] and references therein), which continues to of interest in optimization[5].

[5] In fact, a Google search for `proximal point algorithm` yields more than 500,000 results!

Corollary 2.8. *Let $f\colon \mathbb{R} \to \left]-\infty,+\infty\right]$ be convex, lower semicontinuous, and proper, let $a \in X$ such that $\|a\| = 1$, set $F\colon X \to \left]-\infty,+\infty\right] : x \mapsto f(\langle a, x\rangle)$, and suppose that $T = P_F := (\operatorname{Id} + \partial F)^{-1}$ is the associated proximity operator. Let $x \in X$. Then*

$$(\forall n \in \mathbb{N}) \quad T^n x = P_{\{a\}^\perp}(x) + P_f^n\big(\,\langle a, x\rangle\,\big)a, \tag{15}$$

where $P_f := (\operatorname{Id} + \partial f)$ is the proximity operator of f. Consequently, if f is bounded below but without minimizers, then T admits cosmic convergence and $(Q_n(x))_{n\in\mathbb{N}} = (T^n x/\|T^n x\|)_{n\in\mathbb{N}}$ converges either to $+a$ or to $-a$.

Proof. Set $Y := \mathbb{R}a$ and $\varphi\colon Y \to \left]-\infty,+\infty\right] : \xi a \mapsto f(\xi)$. Then $F = \varphi \circ P_Y$ and $\partial F = (\partial\varphi) \circ P_Y$. By Theorem 2.7, $Tx = P_{\{a\}^\perp}(x) + P_\varphi(\langle a, x\rangle\, a) = P_{\{a\}^\perp}(x) + P_f(\langle a, x\rangle)a$. Concerning the "Consequently" part, observe that if f is bounded below but without minimizers, then $0 \in \overline{\operatorname{ran}}\,(\operatorname{Id} - P_f) \smallsetminus \overline{\operatorname{ran}}\,(\operatorname{Id} - P_f)$ and the result follows from Theorem 2.1. ■

Let us consider two examples: the first one is covered by our analysis but the second one is not.

Example 2.9. *Suppose that $X = \mathbb{R}^2$ and set*

$$F\colon \mathbb{R}^2 \to \left]-\infty,+\infty\right] : (\xi_1, \xi_2) \mapsto \begin{cases} \frac{1}{\xi_1+\xi_2}, & \text{if } \xi_1 + \xi_2 > 0; \\ +\infty, & \text{otherwise,} \end{cases} \tag{16}$$

and suppose that $T = P_F$. Set $a := (1,1)/\sqrt{2}$, and $f(\xi) = 1/(\sqrt{2}\xi)$, if $\xi > 0$ and $f(\xi) = +\infty$ otherwise. Then Corollary 2.8 applies and we obtain cosmic convergence; indeed,

$$Q_n(x) = \frac{T^n(x)}{\|T^n(x)\|} \to a. \tag{17}$$

Example 2.10. *Suppose that $X = \mathbb{R}^2$, set*

$$F\colon \mathbb{R}^2 \to \left]-\infty,+\infty\right] : (\xi_1, \xi_2) \mapsto \begin{cases} \frac{\exp(\xi_1)}{\xi_2}, & \text{if } \xi_2 > 0; \\ +\infty, & \text{otherwise,} \end{cases} \tag{18}$$

and suppose that $T = P_F$. Then F is not *of a form that makes Corollary 2.8 applicable. Interestingly, numerical experiments suggest that*

$$Q_n(x) = \frac{T^n x}{\|T^n x\|} \to (-1, 0); \tag{19}$$

however, we do not have a proof for this conjecture.

We conclude this section with a comment on Fact 1.1.

Remark 2.11. *Since T is nonexpansive, the operator $A := \operatorname{Id} - T$ is maximally monotone. Let $t \in \mathbb{R}_{++}$. Then tA is maximally monotone as well, and therefore its resolvent, $J_{tA} = (\operatorname{Id} + tA)^{-1}$, has full domain. By studying these resolvents carefully, Plant and Reich (see [15, Section 4]) noted that*

$$\lim_{t \to \infty} \frac{d\big(0, \operatorname{ran}(\operatorname{Id} + t(\operatorname{Id} - T))\big)}{t} = 0, \tag{20}$$

which is actually equivalent *to* (2). *In fact, the authors' work is placed even in the setting of Banach spaces and contains further results on the dynamics of iterating T.*

2.4 Poincaré metric and cosmic interpretation

In this section, we provide a different interpretation of our convergence results which also motivates the terminology "cosmic convergence" used above. We first observe that X can be equipped with the *Poincaré metric*, which is defined by

$$\Delta\colon X \to X \to \mathbb{R}\colon (x, y) \mapsto \left\| \frac{x}{1 + \|x\|} - \frac{y}{1 + \|y\|} \right\|. \tag{21}$$

Note that Δ is just the standard Euclidean metric after the bijection $x \mapsto x/(1 + \|x\|)$ between X and the *open* unit ball was applied. The metric space (X, Δ) is not complete; however, regular convergence of sequences in the Euclidean space X is preserved. To complete (X, Δ), define the equivalence relation

$$x \equiv y \quad :\Leftrightarrow \quad x \in \mathbb{R}_{++} y \tag{22}$$

on $X \smallsetminus \{0\}$, with equivalence class

$$\operatorname{dir} x := \mathbb{R}_{++} x \tag{23}$$

for $x \in X \smallsetminus \{0\}$. Following [21], we write

$$\operatorname{hzn} X := \big\{ \operatorname{dir} x \;\big|\; x \in X \smallsetminus \{0\} \big\} \quad \text{and} \quad \operatorname{csm} X := X \cup \operatorname{hzn} X. \tag{24}$$

Here hzn is the *horizon* of X while $\operatorname{csm} X$ denotes the *cosmic closure* of X. A convenient representer of $\operatorname{dir} x$ is $x/\|x\|$. These particular representers form the unit *sphere* which we can think of adjoining to the open unit ball. More precisely, we extend Δ from $X \times X$ to $\operatorname{csm} X \times \operatorname{csm} X$ as follows:

$$\left.\begin{array}{r} x \in X \\ \operatorname{dir} y \in \operatorname{hzn} X \end{array}\right\} \;\Rightarrow\; \Delta(x, \operatorname{dir} y) := \Delta(\operatorname{dir} y, x) := \left\| \frac{x}{1 + \|x\|} - \frac{y}{\|y\|} \right\|. \tag{25}$$

and

$$\left.\begin{array}{r} \operatorname{dir} x \in \operatorname{hzn} X \\ \operatorname{dir} y \in \operatorname{hzn} X \end{array}\right\} \;\Rightarrow\; \Delta(\operatorname{dir} x, \operatorname{dir} y) := \Delta(\operatorname{dir} y, \operatorname{dir} x) := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|. \tag{26}$$

Equipped with Δ, the Bolzano-Weierstrass theorem implies that the cosmic closure csm X is a (sequentially) *compact* metric space; in particular, any sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\|x_n\| \to +\infty$ has a convergent subsequence in $(\operatorname{csm} X, \Delta)$. In the previous sections, we concentrated on the case when $(x_n)_{n\in\mathbb{N}} = (T^n x)_{n\in\mathbb{N}}$ and Fix $T = \varnothing$; then, of course, it may or may not be true that the entire sequence converges in $(\operatorname{csm} X, \Delta)$. This provides an *a posteriori* motivation for our terminology. Finally, we are grateful to a referee for pointing out that if T was fixed-point free and nonexpansive with respect to the Poincaré metric (rather than the Euclidean norm), then results of Beardon [6] imply that all iterates converge to a single point in the horizon.

2.5 Conclusion

We have taken a closer look at Pazy's trichotomy theorem for nonexpansive operators. The question whether or not $(T^n x)_{n\in\mathbb{N}}$ always cosmically converges when T has no fixed points remains open; however, we have related our conjecture to work by Lins, and presented various partial results indicating that the answer may be affirmative. Future work may focus on analyzing larger classes of nonexpansive operators, e.g., general proximity operators or averaged operators. Another promising avenue may be to use tools from non-euclidean geometry (see [9]). Furthermore, it is presently unclear how the presented results extend to infinite-dimensional settings. Last but not least, we hope that this work builds another bridge between researchers in fixed point theory and in optimization; as we have learnt, the former community has powerful results to offer while the latter can provide interesting classes of algorithmic mappings motivated by applications.

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