# Optimal rates of linear convergence of relaxed alternating projections and generalized Douglas-Rachford methods for two subspaces

Heinz H. Bauschke,* J.Y. Bello Cruz,† Tran T.A. Nghia,‡ Hung M. Phan,§ and Xianfu Wang¶

November 28, 2015

**Abstract**

We systematically study the optimal linear convergence rates for several relaxed alternating projection methods and the generalized Douglas-Rachford splitting methods for finding the projection on the intersection of two subspaces. Our analysis is based on a study on the linear convergence rates of the powers of matrices. We show that the optimal linear convergence rate of powers of matrices is attained if and only if all subdominant eigenvalues of the matrix are semisimple. For the convenience of computation, a nonlinear approach to the partially relaxed alternating projection method with at least the same optimal convergence rate is also provided. Numerical experiments validate our convergence analysis.

**2010 Mathematics Subject Classification:** Primary 65F10, 65K05; Secondary 65F15, 65B05, 15A18, 90C25, 41A25

**Keywords:** Convergent and semi-convergent matrix, Friedrichs angle, generalized Douglas-Rachford method, linear convergence, principal angle, relaxed alternating projection method.

## 1 Introduction

Methods of alternative projections and Douglas-Rachford play important roles in convex optimization; see, e.g., [1, 2, 3, 6, 11, 14, 15, 16, 18, 20]. They are also widely used in differential equation and signal processing [32, 29]. In order to study convergence rates, error bounds are given for the method of alternating projections [2, 17, 27], [18, Theorem 9.8] and [22, Section 3.4]; for the Douglas-Rachford method [5, 16], [30] and [32, Proposition 4]. The purpose of this paper is to give a systematic convergence rate analysis of relaxed alternating projections, partial relaxed alternating projection, and the generalized Douglas-Rachford method for two subspaces in finite dimensional spaces. The optimal convergence rates are explicitly given in terms of the relaxation

*Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: heinz.bauschke@ubc.ca.
†IME, Federal University of Goias, Goiania, G.O. 74001-970, Brazil. E-mail: yunier@impa.br and yunier@ufg.br.
‡Mathematics & Statistics, Oakland University, Rochester, MI 48309, USA. E-mail: nttran@oakland.edu.
§Department of Mathematical Sciences, University of Massachusetts Lowell, 265 Riverside St., Olney Hall 428, Lowell, MA 01854, USA. E-mail: Hung_Phan@uml.edu.
¶Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada. E-mail: shawn.wang@ubc.ca.

parameters and sine or cosine of principal angles. Our results extend the work by Demanet and Zhang [16], and the work by Bauschke, Deutsch, Hundal and Park [6]. Our quantification of the optimal convergence rate in terms of relaxation parameters and principal angles will shed light on how to choose parameters in practical applications.

To this end, we need a study on the optimal linear convergence rate of the powers of a real or complex matrix $A$. Necessary and sufficient conditions for such convergence were first established by Hensel [26] and later by Oldenburger [41]. The convergence and its *asymptotic rate* play a central role in many well-known algorithms for solving linear systems such as *Jacobi, Gauss-Seidel, successive over-relaxation* methods; see, e.g., [10, 35, 38, 42]. Furthermore, the convergence of the power $A^k$ *in operator norm* is linear and the rate, which is fundamentally different from the asymptotic one mentioned above, is *dominated* by the second-largest modulus of eigenvalues of $A$, $\gamma(A)$, i.e., modulus of the *subdominant or controlling eigenvalues* [28, 40]. Natural questions thus arising are "What is the optimal (smallest) linear convergence rate?" and "When is $\gamma(A)$ exactly the optimal linear convergence rate?". In general, the optimal linear convergence rate does not exist (see Example 2.11 below). However, many iterative linear methods such as the *method of alternating projections* (also known as von Neumann's method) [2, 17] and the *Douglas-Rachford splitting algorithm* [19, 20, 29, 32] do obtain the optimal linear rates of convergence in operator norm; see also [5, 16, 27].

The rest of the paper is organized as follows. In Section 2 we study optimal linear convergence rates of matrices. We give a necessary and sufficient condition for the powers $A^k$ to converge linearly with the optimal rate $\gamma(A)$ via the semi-simpleness of subdominant eigenvalues. The sufficient part is similar to and can be obtained from [43, Theorem 2.9], in which the norm of power matrix is exactly computed by the power of spectral radius when all the *dominant eigenvalues* are *nondefective*, i.e., semisimple; see also our Remark 2.19 for further details. However, to obtain the necessary part, we develop a systematic analysis on matrix powers, derive in Lemma 2.14 a new bound for their norms, which is sharper than the one in [43, Theorem 2.9], and apply the results to optimal convergence rates of convergent matrices. The main contribution of the paper is developed in Sections 3, 4 and 5. In Section 3, using results of Section 2, we analyze optimal linear convergence rates of the relaxed alternating methods [12, 33] and also the generalized Douglas-Rachford splitting methods [16] with parameters for two linear subspaces. To the best of our knowledge, our optimal rates of the relaxed alternating methods established here via *principal angles* [9] are new in the literature and they significantly improve the one of classical alternating methods. Furthermore, our optimal rate of the generalized Douglas-Rachford splitting methods extends the similar result obtained implicitly in Demanet-Zhang [16, Section 2.6] without assuming the trivial intersection of two subspace as in [16]. In Section 4 we introduce and study a nonlinear map that helps to accelerate significantly the convergence of alternating projection methods. This map also allows us to overcome the difficulty of computing the principal angles used to determine the parameter in the relaxed/partial alternating methods aforementioned. In particular, this generalizes one of the result by Bauschke-Deutsch-Hundal-Park [6, Theorem 3.28]. In Section 5, we provide some numerical results to illustrate our convergence theory developed in earlier sections. The numerical experiments indicate that relaxed alternating projection and partial relaxed alternating projection methods perform better than the method of alternating projection in general. Finally, we present our conclusions in Section 6.

**Notation.** Throughout, we denote by $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ the sets of $n \times n$ complex matrices and real matrices, respectively. Let $A$ be a matrix in $\mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$). The notation $A^*$ stands for the adjoint (complex transposed) matrix of $A$. The matrix norm used in this paper is the *operator norm*, i.e., $\|A\| = \max\{\|Ax\| \mid x \in \mathbb{C}^n, \|x\| \le 1\}$, the induced matrix norm. We write $\ker A$, $\operatorname{ran} A$, and rank $A$ as the kernel, range, rank of $A$, respectively. Moreover, $\operatorname{Fix} A := \ker(A - \operatorname{Id})$ is known as the set of fixed points of $A$, where Id is the identity mapping. We say $A$ is *nonexpansive* if $\|Ax\| \leq \|x\|$ for all $x \in \mathbb{C}^n$; furthermore, $A$ is *firmly nonexpansive* if $\|Ax\|^2 + \|x - Ax\|^2 \leq \|x\|^2$ for all $x \in \mathbb{C}^n$. For any subspace $U$ of $\mathbb{R}^n$, we use $P_U$ for the orthogonal *projection operator* to $U$, $\dim U$ for the dimension of $U$, and $U^\perp$ for the orthogonal complement of $U$. We denote $I_n, 0_n, 0_{m\times n}$ by the $n \times n$ identity matrix, the $n \times n$ zero matrix, and the $m \times n$ zero matrix, respectively. $\mathbb{N}$ is the set of nonnegative integers $\{0, 1, \ldots\}$.

## 2 Preliminary results: the optimal convergence rate of matrices

In this section we establish conditions under which convergent matrices attain the optimal convergent rate. Our analysis in later sections hinges on these results on matrices. Let us begin with:

### 2.1 Some definitions and well-known facts about matrices

**Definition 2.1 (convergent matrices)** *Let $A \in \mathbb{C}^{n\times n}$. We say $A$ is* convergent[1] *to $A^\infty \in \mathbb{C}^{n\times n}$ if and only if*

$$\|A^k - A^\infty\| \to 0 \quad \textit{as} \quad k \to \infty. \tag{1}$$

*We say $A$ is* linearly *convergent to $A^\infty$ with rate $\mu \in [0, 1)$ if there are some $M, N > 0$ such that*

$$\|A^k - A^\infty\| \leq M\mu^k \quad \textit{for all} \quad k > N, k \in \mathbb{N}. \tag{2}$$

*Then $\mu$ is called a* linear convergence rate *of $A$. When the infimum of all the convergence rates is also a convergence rate, we say this minimum is the* optimal linear convergence rate[2].

For any $A \in \mathbb{C}^{n\times n}$ we denote by $\sigma(A)$ the *spectrum* of $A$, the set of all eigenvalues. The spectral radius [37, Example 7.1.4] of $A$ is defined by

$$\rho(A) := \max\{|\lambda| \mid \lambda \in \sigma(A)\}. \tag{3}$$

The next fact is the classical formula of spectral radius.

**Fact 2.2 (spectral radius formula)** ([37, Example 7.10.1]) *Let $A \in \mathbb{C}^{n\times n}$. Then we have*

$$\rho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}. \tag{4}$$

With $\lambda \in \sigma(A)$, recall from [37, page 587] that $\operatorname{index}(\lambda)$ is the smallest positive integer $k$ satisfying $\operatorname{rank}(A - \lambda \operatorname{Id})^k = \operatorname{rank}(A - \lambda \operatorname{Id})^{k+1}$. Furthermore, we say $\lambda \in \sigma(A)$ is *semisimple* if $\operatorname{index}(\lambda) = 1$; see, e.g., [37, Exercise 7.8.4].

**Fact 2.3** *For $A \in \mathbb{C}^{n\times n}$, $\lambda \in \sigma(A)$ is semisimple if and only if* $\ker(A - \lambda \operatorname{Id}) = \ker(A - \lambda \operatorname{Id})^2$.

[1] In the literature, $A$ is called convergent if the power $A^k$ converges to 0; moreover, $A$ is semi-convergent whenever the latter limit $A^k$ exists. To avoid the confusion of these two terminologies, we just say $A$ is convergent in both cases.

[2] This is significantly different from the asymptotic convergence rate [10, p. 199].

*Proof*. Note that $\lambda \in \sigma(A)$ is semisimple if and only if

$$\dim[\ker(A - \lambda\,\mathrm{Id})] = n - \mathrm{rank}\,(A - \lambda\,\mathrm{Id}) = n - \mathrm{rank}\,(A - \lambda\,\mathrm{Id})^2 = \dim[\ker(A - \lambda\,\mathrm{Id})^2].$$

Since $\ker(A - \lambda\,\mathrm{Id}) \subset \ker(A - \lambda\,\mathrm{Id})^2$, the equality $\dim[\ker(A - \lambda\,\mathrm{Id})] = \dim[\ker(A - \lambda\,\mathrm{Id})^2]$ holds if and only if $\ker(A - \lambda\,\mathrm{Id}) = \ker(A - \lambda\,\mathrm{Id})^2$. This verifies the proof of the fact. ■

The following result taken from [37] gives us a complete characterization of a convergent matrix.

**Fact 2.4 (limits of powers)** ([37, page 617-618 and page 630]) *For $A \in \mathbb{C}^{n\times n}$, $\lim_{k\to\infty} A^k$ exists if and only if*

(5) $\rho(A) < 1$, *or else*

(6) $\rho(A) = 1$ *and* $\lambda = 1$ *is semisimple and it is the only eigenvalue on the unit circle.*

*When this happens, we have*

(7) $$\lim_{k\to\infty} A^k = A^\infty = \text{ the projector onto } \ker(A - \mathrm{Id}) \text{ along } \mathrm{ran}(A - \mathrm{Id}).$$

*In particular, when $\rho(A) < 1$, we have $A^\infty = 0$.*

The proof of the above fact is indeed based on the spectral resolution of $A^k$ stated below.

**Fact 2.5 (spectral resolution of $A^k$)** ([37, page 603 and page 629]) *For $k \in \mathbb{N}$ and $A \in \mathbb{C}^{n\times n}$ with $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ and $k_i = \mathrm{index}\,(\lambda_i)$, we have*

(8) $$A^k = \sum_{i=1}^{s} \lambda_i^k G_i + \sum_{i=1}^{s} \sum_{j=1}^{k_i-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i\,\mathrm{Id})^j G_i,$$

*where the spectral projector $G_i$'s have the following properties:*

**(i)** *$G_i$ is the projector onto* $\ker((A - \lambda_i\,\mathrm{Id})^{k_i})$ *along* $\mathrm{ran}((A - \lambda_i\,\mathrm{Id})^{k_i})$.

**(ii)** $G_1 + G_2 + \cdots + G_s = \mathrm{Id}$.

**(iii)** $G_i G_j = 0$ *when $i \neq j$.*

**(iv)** $N_i = (A - \lambda_i\,\mathrm{Id})G_i = G_i(A - \lambda_i\,\mathrm{Id})$ *is* nilpotent *of index $k_i$, i.e., $N_i^{k_i} = 0$ and $N_i^{k_i-1} \neq 0$.*

*Furthermore, the second sum in* (8) *disappears when* $\mathrm{index}\,(\lambda_i) = 1$ *for all $i = 1, \ldots, s$.*

**Remark 2.6** Note from Fact 2.5(i) and (iv) that

$$0 \neq N_i^{k_i-1} = (A - \lambda_i\,\mathrm{Id})^{k_i-1} G_i^{k_i-1} = (A - \lambda_i\,\mathrm{Id})^{k_i-1} G_i \quad \text{if} \quad k_i > 1.$$

The limit $A^\infty$ in Fact 2.4 is an oblique projector, not necessarily an orthogonal projector. However, we have:

**Corollary 2.7** *Suppose that $A \in \mathbb{C}^{n\times n}$ is convergent to $A^\infty \in \mathbb{C}^{n\times n}$. Then the following hold:*

**(i)** $A^\infty = P_{\mathrm{Fix}\,A}$ *if and only if* $\mathrm{Fix}\,A = \mathrm{Fix}\,A^*$.

**(ii)** *If $A$ is nonexpansive or normal, then $A^\infty = P_{\mathrm{Fix}\,A}$.*

*Proof*. It follows from (7) that $A^\infty$ is equal to the projector onto $\ker(A - \mathrm{Id})$ along $\mathrm{ran}(A - \mathrm{Id})$. Thanks to the equality [37, (5.9.11)], we have $\mathrm{ran}(A^\infty - \mathrm{Id}) = \mathrm{ran}(A - \mathrm{Id})$. If $A^\infty = P_{\mathrm{Fix}\,A}$, we obtain

$$\mathrm{ran}(A - \mathrm{Id}) = \mathrm{ran}(A^\infty - \mathrm{Id}) = \mathrm{ran}(P_{\mathrm{Fix}\,A} - \mathrm{Id}) = \mathrm{ran}(P_{(\mathrm{Fix}\,A)^\perp}) = (\mathrm{Fix}\,A)^\perp.$$

It follows that

$$\mathrm{Fix}\,A = \left[(\mathrm{Fix}\,A)^\perp\right]^\perp = \mathrm{ran}(A - \mathrm{Id})^\perp = \ker(A^* - \mathrm{Id}) = \mathrm{Fix}\,A^*.$$

Conversely, if $\mathrm{Fix}\,A = \mathrm{Fix}\,A^*$, we have

$$\ker(A - \mathrm{Id}) = \mathrm{Fix}\,A = \mathrm{Fix}\,A^* = \ker(A^* - \mathrm{Id}) = \mathrm{ran}(A - \mathrm{Id})^\perp,$$

which implies in turn that the projector onto $\ker(A - \mathrm{Id})$ along $\mathrm{ran}(A - \mathrm{Id})$ is exactly the orthogonal projection $P_{\mathrm{Fix}\,A}$. The first part **(i)** of the corollary is complete.

To justify the second part **(ii)**, suppose in addition that $A$ is nonexpansive. Then $\mathrm{Fix}\,A = \mathrm{Fix}\,A^*$ by [6, Lemma 2.1] and thus $A$ is convergent to $P_{\mathrm{Fix}\,A}$. Moreover, if $A$ is normal, then $A - \mathrm{Id}$ is also normal. Hence, for all $x \in \mathbb{C}^n$ we have

$$\|(A - \mathrm{Id})x\|^2 = \langle (A - \mathrm{Id})^*(A - \mathrm{Id})x, x\rangle = \langle (A - \mathrm{Id})(A - \mathrm{Id})^*x, x\rangle = \|(A - \mathrm{Id})^*x\|^2.$$

The latter clearly shows that $\mathrm{Fix}\,A = \mathrm{Fix}\,A^*$ and thus $A^\infty = P_{\mathrm{Fix}\,A}$. The proof is complete. ■

**Remark 2.8 (convergence, firmly nonexpansiveness and nonexpansiveness)** Let $A \in \mathbb{R}^{n\times n}$. When $A$ is firmly nonexpansive, $A$ is convergent; see, e.g., [3, Example 5.17]. However, the converse implication fails. Indeed, consider, for $n \geq 2$,

$$A = \begin{pmatrix} 0 & n^{-2} \\ n & 0 \end{pmatrix}.$$

Then $A$ is not (firmly) nonexpansive because $Ae_1 = ne_2$ where $e_1 = (1,0)^\intercal$ and $e_2 = (0,1)^\intercal$. On the other hand, the characteristic polynomial is $\lambda \mapsto \lambda^2 - n^{-1}$, which has roots $\pm n^{-1/2}$. Thus $A$ is convergent due to Fact 2.4. Moreover, convergence and nonexpansiveness are independent, e.g., $A = -\,\mathrm{Id}$ is nonexpansive but not convergent.

## 2.2 Asymptotic convergence rates of convergent matrices

We will prove later in this section that whenever $A$ is convergent to $A^\infty$, it is linearly convergent with the rate not smaller than $\rho(A - A^\infty)$. To develop this idea, let us now consider the case of diagonalizable matrices.

**Example 2.9 (diagonalizable case)** Suppose that $A \in \mathbb{C}^{n\times n}$ is *diagonalizable* and that $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ with

$$1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_s|.$$

Note that all eigenvalues $\{\lambda_1, \ldots, \lambda_s\}$ are semisimple when $A$ is diagonalizable. By Fact 2.5 and Fact 2.4, we have $A$ is convergent to $A^\infty$ and that

$$A^k = A^\infty + \lambda_2^k G_2 + \cdots + \lambda_s^k G_s,$$

which yields

$$A^k - A^\infty = \lambda_2^k G_2 + \cdots + \lambda_s^k G_s.$$

It follows that

$$\begin{aligned}\|A^k - A^\infty\| &\le |\lambda_2|^k \left[ \Big(\frac{|\lambda_2|}{|\lambda_2|}\Big)^k \|G_2\| + \cdots + \Big(\frac{|\lambda_s|}{|\lambda_2|}\Big)^k \|G_s\| \right] \\ &\le |\lambda_2|^k \big(\|G_2\| + \cdots + \|G_s\|\big).\end{aligned}$$

Hence $A^k \to A^\infty$ with the linear rate $|\lambda_2|$.

In general an eigenvalue having second-largest modulus after 1 is called a subdominant eigenvalue.

**Definition 2.10 (subdominant eigenvalues)** ([28, 40]) *For* $A \in \mathbb{C}^{n\times n}$*, we define*

$$\gamma(A) := \max\big\{|\lambda| \,|\, \lambda \in \{0\} \cup \sigma(A) \setminus \{1\}\big\}. \tag{10}$$

*An eigenvalue* $\lambda \in \sigma(A)$ *satisfying* $|\lambda| = \gamma(A)$ *is referred as a* subdominant *eigenvalue.*

When $A$ is not diagonalizable, $\gamma(A)$ need not be the convergence rate.

**Example 2.11** Let us consider the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix},$$

which gives us that $\gamma(A) = \frac{1}{2}$. Note also that $A$ is not diagonalizable. Moreover, by induction it is easy to check that

$$A^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^k} & \frac{k}{2^{k-1}} \\ 0 & 0 & \frac{1}{2^k} \end{pmatrix} \qquad \text{for all} \quad k \in \mathbb{N}.$$

Hence we have $A^k \to A^\infty := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as $k \to \infty$. However, observe that

$$\frac{\|A^k - A^\infty\|}{\gamma(A)^k} = 2^k \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2^k} & \frac{k}{2^{k-1}} \\ 0 & 0 & \frac{1}{2^k} \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix} \right\| \to \infty \quad \text{as} \quad k \to \infty.$$

Hence $\gamma(A)$ is not a convergence rate. However, observe further that any $\mu \in (\frac{1}{2}, 1)$ is a convergence rate of $A$. Thus $A$ does not obtain the optimal linear convergence rate. ∎

The following result below shows that whenever a matrix $A$ is convergent, it must be linearly convergent with any rate in $(\gamma(A), 1)$. One may also use the spectral decomposition [34, Proposition 1] to prove. Here, our proof is a bit different, but it is necessary for our further study in the paper.

**Theorem 2.12 (rate of convergence I)** *Suppose that $A \in \mathbb{C}^{n\times n}$ is convergent to $A^\infty \in \mathbb{C}^{n\times n}$. Then we have $\gamma(A) = \rho(A - A^\infty) < 1$ and that*

$$(A - A^\infty)^k = A^k - A^\infty \quad \textit{for all} \quad k \in \mathbb{N}. \tag{11}$$

*Moreover, the following two assertions are satisfied:*

**(i)** *$A$ is linearly convergent with any rate $\mu \in (\gamma(A), 1)$.*

**(ii)** *If $A$ is linearly convergent with rate $\mu \in [0, 1)$, then $\mu \in [\gamma(A), 1)$.*

*Proof*. First let us justify that $\gamma(A) = \rho(A - A^\infty) < 1$ and (11) by considering the two following cases taken from Fact 2.4:

**Case 1:** $\rho(A) < 1$. In this case we have $A^\infty = 0$ by (4). It follows that $\gamma(A) = \rho(A) = \rho(A - A^\infty) < 1$. Note also that (11) is trivial, since $A^\infty = 0$.

**Case 2:** $\rho(A) = 1$, and $\lambda = 1$ is semisimple and the only eigenvalue on the unit circle. Suppose that $\sigma(A) \setminus \{1\} = \{\lambda_2, \ldots, \lambda_s\}$ with $1 > |\lambda_2| \geq \ldots \geq |\lambda_s|$. The Jordan decomposition [37, page 590] of $A$ allows us to find an invertible matrix $P \in \mathbb{C}^{n\times n}$ and $r > 0$ such that

$$A = PJP^{-1} \tag{12}$$

with $J$ being the Jordan form of $A$,

$$J = \begin{pmatrix} I_r & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{pmatrix}, \qquad J(\lambda_j) = \begin{pmatrix} J_1(\lambda_j) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_j) & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{t_j}(\lambda_j) \end{pmatrix},$$

and

$$J_k(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}, \qquad \text{index}\,(\lambda_j) = \max\left\{ d_{j_k} \;\middle|\; k = 1, \ldots, t_j \right\},$$

where $d_{j_k}$ is the dimension of the matrix $J_k(\lambda_j)$, $t_j = \dim(\ker(A - \lambda_j \operatorname{Id}))$. Moreover, it follows from [37, p. 629] that

$$A^\infty = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}. \tag{13}$$

This together with the Jordan decomposition above gives us that

$$A - A^\infty = P \begin{pmatrix} 0_r & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{pmatrix} P^{-1}, \tag{14}$$

which readily yields $\rho(A - A^\infty) = \max\{0, |\lambda_2|\} = \gamma(A) < 1$. Observe further from (12) and (13) that $AA^\infty = A^\infty A = (A^\infty)^2 = A^\infty$. For any $k \in \mathbb{N}$ the latter gives us that

$$(A^k - A^\infty)(A - A^\infty) = A^{k+1} - A^k A^\infty - A^\infty A + (A^\infty)^2 = A^{k+1} - A^\infty - A^\infty + A^\infty = A^{k+1} - A^\infty.$$

By using this expression, we may prove by induction (11) and this completes the first part of the theorem.

Now to verify (i), pick any $\mu \in (\gamma(A), 1) = (\rho(A - A^\infty), 1)$. Employing (4) for operator $A - A^\infty$ allows us to find some $N \in \mathbb{N}$ such that

$$\|A^k - A^\infty\| = \|(A - A^\infty)^k\| \le \mu^k \quad \text{for all} \quad k \ge N,$$

which verifies the linear convergence of $A$ with rate $\mu$.

It remains to prove (ii). Suppose that $A$ is convergent to $A^\infty$ with rate $\mu \in [0, 1)$. Hence there are some $M, N > 0$ such that

$$\|A^k - A^\infty\| \le M\mu^k \quad \text{for all} \quad k > N, k \in \mathbb{N}.$$

Combining this with the spectral radius formula (4) and (11) gives us that

$$\gamma(A) = \rho(A - A^\infty) = \lim_{k\to\infty} \|(A - A^\infty)^k\|^{\frac{1}{k}} = \lim_{k\to\infty} \|A^k - A^\infty\|^{\frac{1}{k}} \le \lim_{k\to\infty} M^{\frac{1}{k}}\mu = \mu,$$

which ensures $\gamma(A) \le \mu$ and thus completes the proof of the theorem. ■

**Remark 2.13** We observe from (14) that if $\lambda \in \sigma(A) \setminus \{0, 1\}$, $\lambda \in \sigma(A - A^\infty)$ and its index does not change.

## 2.3 The optimal convergence rate of convergent matrices

A natural question arising from the above theorem is that in which case $\gamma(A)$ is the optimal linear convergence rate of $A$; see our Definition 2.1. By Theorem 2.12, the actual problem is to describe when $\gamma(A)$ is a convergence rate of $A$; see also our Example 2.11. Theorem 2.15 below gives us a complete answer for this question. To prepare, we need the following lemma, which enhances the spectral radius formula in Fact 2.2 and is of its own interest. It is worth noting that one may prove it by using the Jordan decomposition [37, page 590 and page 618] with similar complexity.

**Lemma 2.14** *Let $A \in \mathbb{C}^{n\times n}$ with spectral radius $\rho(A) > 0$. Define $\alpha := \max\{\operatorname{index}(\lambda) \mid \lambda \in \sigma(A), |\lambda| = \rho(A)\}$. We have*

$$0 < \limsup_{k\to\infty} \frac{\|A^k\|}{\binom{k}{\alpha-1}\rho(A)^k} < \infty. \tag{15}$$

*Proof*. For the matrix $A$, denote the set of distinct eigenvalues in $\sigma(A)$ by $\{\lambda_1, \ldots, \lambda_s\}$ with $\rho(A) = |\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_s|$ and $k_i = \operatorname{index}(\lambda_i)$, $i = 1, \ldots, s$. We get from (8) that

$$A^k = \sum_{i=1}^{s} \lambda_i^k G_i + \sum_{i=1}^{s}\sum_{j=1}^{k_i-1} \binom{k}{j} \lambda_i^{k-j}(A - \lambda_i \operatorname{Id})^j G_i. \tag{16}$$

Denote by

$$\begin{aligned} &E := \{1, \ldots, s\}, \qquad F := \{l \in \mathbb{N} \mid |\lambda_l| = \rho(A), 1 \le l \le s\}, \\ &\text{and} \quad S := \{i \in F \mid \operatorname{index}(\lambda_i) = \alpha\}. \end{aligned} \tag{17}$$

It is clear that $F \supset S \neq \emptyset$. By (16) we have

$$A^k = \underbrace{\sum_{i \in S} \sum_{j=0}^{k_i - 1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \operatorname{Id})^j G_i}_{:=H} + \underbrace{\sum_{i \in E \setminus S} \sum_{j=0}^{k_i - 1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \operatorname{Id})^j G_i}_{:=K}. \tag{18}$$

Note that

$$\begin{aligned} \frac{K}{\binom{k}{\alpha-1} |\lambda_1|^k} &= \sum_{i \in E \setminus S} \sum_{j=0}^{k_i-1} \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_1|^k} \frac{1}{\binom{k}{\alpha-1}} (A - \lambda_i \operatorname{Id})^j G_i \\ &= \sum_{i \in (E \setminus S) \cup F} \sum_{j=0}^{k_i-1} \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_1|^k} \frac{1}{\binom{k}{\alpha-1}} (A - \lambda_i \operatorname{Id})^j G_i \\ &\quad + \sum_{i \in E \setminus F} \sum_{j=0}^{k_i-1} \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_1|^k} \frac{1}{\binom{k}{\alpha-1}} (A - \lambda_i \operatorname{Id})^j G_i. \end{aligned} \tag{19}$$

For $i \in (E \setminus S) \cup F$ and $0 \le j \le k_i - 1$, observe from the definition of $\alpha$ in (17) that $j \le \alpha - 2$. It follows that

$$\frac{\binom{k}{j}}{\binom{k}{\alpha-1}} \left| \frac{\lambda_i^{k-j}}{|\lambda_1|^k} \right| = \frac{(\alpha-1)!(k-\alpha+1)!}{j!(k-j)!} \frac{1}{|\lambda_1|^j} \le \frac{(\alpha-1)!}{j!(k-\alpha+2)} \frac{1}{|\lambda_1|^j} := \varepsilon_1(k) \to 0 \tag{20}$$

as $k \to \infty$. For $i \in E \setminus F$ and $0 \le j \le k_i - 1$, we have

$$\left| \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_1|^k} \right| \le k^j \Big( \frac{|\lambda_i|}{|\lambda_1|} \Big)^{k-j} |\lambda_1|^{-j} := \varepsilon_2(k) \to 0, \tag{21}$$

since $k^j$ is polynomial in $k$ and $(|\lambda_i|/|\lambda_1|)^{k-j}$ is exponential with $|\lambda_i|/|\lambda_1| < 1$ for $i \notin F$. It follows from (19), (20), and (21) that

$$\frac{\|K\|}{\binom{k}{\alpha-1} |\lambda_1|^k} \le \sum_{i \in (E \setminus S) \cup F} \sum_{j=0}^{k_i-1} \varepsilon_1(k) \|(A - \lambda_i)^j G_i\| + \sum_{i \in E \setminus F} \sum_{j=0}^{k_i-1} \varepsilon_2(k) \|(A - \lambda_i)^j G_i\| \to 0. \tag{22}$$

Now note from (18) that

$$\frac{H}{\binom{k}{\alpha-1}|\lambda_1|^k} = \sum_{i\in S}\sum_{j=0}^{\alpha-1}\frac{\binom{k}{j}}{\binom{k}{\alpha-1}}\frac{\lambda_i^{k-j}}{|\lambda_1|^k}(A-\lambda_i\,\mathrm{Id})^j G_i$$

$$(23)\qquad = \sum_{i\in S}\frac{\lambda_i^{k-(\alpha-1)}}{|\lambda_1|^k}(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i + \sum_{i\in S}\sum_{j=0}^{\alpha-2}\frac{\binom{k}{j}}{\binom{k}{\alpha-1}}\frac{\lambda_i^{k-j}}{|\lambda_1|^k}(A-\lambda_i\,\mathrm{Id})^j G_i$$

$$= \sum_{i\in S}\frac{\lambda_i^{k}}{|\lambda_1|^k}\lambda_i^{-(\alpha-1)}(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i + \underbrace{\sum_{i\in S}\sum_{j=0}^{\alpha-2}\frac{\binom{k}{j}}{\binom{k}{\alpha-1}}\frac{\lambda_i^{k-j}}{|\lambda_1|^k}(A-\lambda_i\,\mathrm{Id})^j G_i}_{:=H_1}.$$

Furthermore, for $i\in S$ and $j\le\alpha-2$ similarly to (20) we may prove that

$$\frac{\binom{k}{j}}{\binom{k}{\alpha-1}}\left|\frac{\lambda_i^{k-j}}{|\lambda_1|^k}\right| := \varepsilon_3(k)\to 0 \quad\text{when}\quad k\to\infty,$$

which implies in turn that

$$(24)\qquad \|H_1\| \le \sum_{i\in S}\sum_{j=0}^{\alpha-2}\varepsilon_3(k)\|(A-\lambda_i\,\mathrm{Id})^j G_i\| \to 0 \quad\text{as}\quad k\to\infty.$$

By dividing (18) by $|\lambda_1|^k\binom{k}{\alpha-1}$ and taking $k\to\infty$, we get from (18), (22), (23), and (24) that

$$(25)\qquad \limsup_{k\to\infty}\frac{\|A^k\|}{\binom{k}{\alpha-1}\rho(A)^k} = \limsup_{k\to\infty}\Big\|\sum_{i\in S}\frac{\lambda_i^k}{|\lambda_1|^k}\lambda_i^{-(\alpha-1)}(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i\Big\|$$

$$(26)\qquad \le \sum_{i\in S}|\lambda_i|^{-(\alpha-1)}\|(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i\| < \infty,$$

which verify the right-hand inequality in (15). Furthermore, since $\left|\frac{\lambda_i}{|\lambda_1|}\right| = 1$ for all $i\in S$, by passing to subsequences we may assume without loss of generality that for each $i\in S$ the sequence $\left[\frac{\lambda_i}{|\lambda_i|}\right]^k \to x_i$ with $|x_i| = 1$ as $k\to\infty$. Hence, it follows from (25) that

$$(27)\qquad \limsup_{k\to\infty}\frac{\|A^k\|}{\binom{k}{\alpha-1}\rho(A)^k} \ge \|\sum_{i\in S}x_i\lambda_i^{-(\alpha-1)}(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i\|.$$

The left-hand inequality in (15) is justified when $\sum_{i\in S}x_i\lambda_i^{-(\alpha-1)}(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i \ne 0$. By contraction, suppose $\sum_{i\in S}x_i\lambda_i^{-(\alpha-1)}(A-\lambda_i\,\mathrm{Id})^{\alpha-1}G_i = 0$. By multiplying this equality by $G_l$ with $l\in S$, we get from Fact 2.5 (i) and (iii) that

$$x_l\lambda_l^{-(\alpha-1)}(A-\lambda_l\,\mathrm{Id})^{\alpha-1}G_l = 0,$$

which is impossible since $x_l \neq 0$, $|\lambda_l| = |\lambda_1| \neq 0$, and $(A - \lambda_l \operatorname{Id})^{\alpha-1} G_l \neq 0$ by Remark 2.6. The proof is complete. ∎

**Theorem 2.15 (rate of convergence II)** *Let $A \in \mathbb{C}^{n\times n}$ be convergent to $A^\infty \in \mathbb{C}^{n\times n}$. Then $\gamma(A)$ is the optimal linear convergence rate of $A$ if and only if all the subdominant eigenvalues are semisimple.*

*Proof.* By Theorem 2.12, we only need to prove that $\gamma(A)$ is a linear convergence rate of $A$ if and only if $\lambda$ is semisimple for every eigenvalue $\lambda \in \sigma(A)$ satisfying $|\lambda| = \gamma(A)$. Define $\alpha := \max\left\{\operatorname{index}(\lambda) \mid \lambda \in \sigma(A), |\lambda| = \gamma(A)\right\}$.

If $\gamma(A) = 0$, then by (8) we have $A^k = A^\infty$ for all $k \geq 1$. This means that $A = A^\infty$ and $A^2 = A$, which ensures that $\gamma(A)$ is semisimple and $\gamma(A) = 0$ is a convergence rate. Thus the statement of the theorem is trivial in this case. It remains to prove the theorem when $\gamma(A) > 0$.

If $A$ is linearly convergent to $A^\infty$ with the rate $\gamma(A)$, we find $M, N$ such that

$$\|A^k - A^\infty\| \leq M\gamma(A)^k \quad \text{for all} \quad k > N, k \in \mathbb{N}$$

Note from Theorem 2.12 that $\gamma(A) = \rho(A - A^\infty)$. Thanks to Remark 2.13, all subdominant eigenvalues of $A$ are eigenvalues of $A - A^\infty$ and their indices do not change. It follows from (11) and Lemma 2.14 for $A - A^\infty$ that

$$0 < \limsup_{k\to\infty} \frac{\|(A - A^\infty)^k\|}{\binom{k}{\alpha-1}\rho(A - A^\infty)^k} = \limsup_{k\to\infty} \frac{\|A^k - A^\infty\|}{\binom{k}{\alpha-1}\gamma(A)^k} \leq \limsup_{k\to\infty} \frac{M}{\binom{k}{\alpha-1}},$$

which yields $\alpha = 1$ and thus all subdominant eigenvalues are semisimple.

Conversely, if all subdominant eigenvalues are semisimple, we have $\alpha = 1$. Applying Lemma 2.14 again to $A - A^\infty$ gives us that

$$\limsup_{k\to\infty} \frac{\|A^k - A^\infty\|}{\gamma(A)^k} = \limsup_{k\to\infty} \frac{\|(A - A^\infty)^k\|}{\binom{k}{\alpha-1}\rho(A - A^\infty)^k} < \infty,$$

which verifies that $\gamma(A)$ is the convergence rate of $A$. The proof is complete. ∎

**Remark 2.16** It is worth mentioning that Example 2.9 is also a direct consequence of Theorem 2.15, since all the eigenvalues of $A$ are semisimple when $A$ is diagonalizable. Moreover, $\gamma(A)$ is not the convergence rate in Example 2.11, since $\frac{1}{2} = \gamma(A)$ is not semisimple in this case.

Next let us summarize Fact 2.4, Theorem 2.12, and Theorem 2.15 in the following result, which provides a complete characterization for obtaining the optimal convergence rate.

**Theorem 2.17 (optimal convergence rate)** *Let $A \in \mathbb{C}^{n\times n}$. Then $A$ is convergent with the optimal linear convergence rate $\gamma(A)$ if and only if one of the following holds:*

**(i)** *$\rho(A) < 1$ and all $\lambda \in \sigma(A)$ satisfying $|\lambda| = \gamma(A)$ are semisimple.*

**(ii)** *$\rho(A) = 1$, $\lambda = 1$ is the only eigenvalue on the unit circle, $\lambda = 1$ is semisimple, and all $\lambda \in \sigma(A)$ satisfying $|\lambda| = \gamma(A)$ are semisimple.*

*Proof*. If $A$ is convergent with the optimal linear convergence rate, Theorem 2.12 tells us that $\gamma(A)$ is the optimal convergence rate. Moreover, **(i)** and **(ii)** follow from Fact 2.4 and Theorem 2.15. Conversely, if **(i)** and **(ii)** hold, we also get from Fact 2.4 and Theorem 2.15 that $A$ is convergent with the optimal rate $\gamma(A)$. ∎

**Theorem 2.18** *Let $A \in \mathbb{C}^{n\times n}$ be convergent to $A^\infty$. Then we have*

$$\|A^k - A^\infty\| \le \|A - A^\infty\|^k \tag{28}$$

*and thus $\gamma(A) \le \|A - A^\infty\|$. Furthermore, if $A$ is normal then we have*

$$\|A^k - A^\infty\| = \|A - A^\infty\|^k \tag{29}$$

*and $\gamma(A) = \|A - A^\infty\|$ is the optimal convergence rate of $A$.*

*Proof*. First, observe from (11) in Theorem 2.12 that

$$\|A^k - A^\infty\| = \|(A - A^\infty)^k\| \le \|A - A^\infty\|^k,$$

which clearly ensures (28) and thus $\gamma(A) = \rho(A - A^\infty) \le \|A - A^\infty\|$.

To justify the second part, suppose that $A$ is convergent and normal. We claim that $A - A^\infty$ is also normal. This is trivial when $A^\infty = 0$. It remains to take into account the case $A^\infty \neq 0$. Since $A$ is normal, we can find a diagonal matrix $J = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $|\lambda_1| \ge \ldots \ge |\lambda_n|$ and a unitary matrix $P$ such that $A = PJP^*$. Fact 2.4 tells us that $1 \in \sigma(A)$ and $1 = \lambda_1 = \ldots = \lambda_r > |\lambda_{r+1}| \ge \ldots \ge |\lambda_n|$ for some $r \in \mathbb{N}$. It follows that

$$A^\infty = \lim_{k\to\infty} A^k = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^*. \tag{30}$$

Hence we obtain

$$A - A^\infty = P \begin{pmatrix} 0_{r\times r} & 0 & \cdots & 0 \\ 0 & \lambda_{r+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P^*,$$

which is a normal matrix. The latter formula together with (11) also gives us that

$$\|A^k - A^\infty\| = \|(A - A^\infty)^k\| = \|A - A^\infty\|^k = |\lambda_{r+1}|^k = [\rho(A - A^\infty)]^k = \gamma(A)^k,$$

which ensures (29) and completes the proof of the theorem. ∎

**Remark 2.19** (i). Theorems 2.12 and 2.15 can also be deduced from the spectral radius formula and Jordan factorizations; however, we were not be able to find these results explicitly in literature. The asymptotic convergence bound of matrix powers can be found [43, Theorem 2.9, p. 33], i.e., "Let $A$ be of order $n$ and let $\varepsilon > 0$ be given. Then for any norm $\|\cdot\|$ there exists $\sigma$ (depending on the norm) and $\tau > 0$ (depending on $A$, $\varepsilon$ and the norm) such that for all $k \ge 1$:

$$\sigma\rho(A)^k \le \|A^k\| \le \tau_{A,\varepsilon}(\rho(A) + \varepsilon)^k. \tag{31}$$

If the dominant eigenvalues of $A$ is nondefective, we may take $\varepsilon = 0$." Lemma 2.14 gives a better upper bound for $\|A^k\|$ when $\rho(A) > 0$. The distinguished feature of our results given here is the complete characterizations under which the convergence rate of the matrix power $A^k$ is exactly $\gamma(A)$, rather than $\gamma(A)+\varepsilon$ for some $\varepsilon > 0$; or just sufficient conditions.

(ii). Assume that $\lim_{k\to\infty} A^k$ exists and $\rho(A) = 1$. According to Fact 2.4, the spectral resolution, $A = P + Z$ where $P = G_1$, $P^2 = P$, $PZ = ZP = 0$ and $\rho(Z) = \gamma(A) < 1$, see also [36, Theorem 2.1]. We have $A^k = P + Z^k$ so that $A^k - P = Z^k$. Therefore, the rate of convergence of $A^k$ to $P$ is exactly the rate of convergence of $Z^k$ to 0. This observation is exactly Theorem 2.12 with a different proof, where $A^\infty$ there is $P$ in this decomposition. It is well-known that $\rho(Z)$ is the asymptotic convergence rate meaning that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, $M_1, M_2 > 0$ such that

$$(\forall k \geq N)\ M\rho(Z)^k \leq \|Z^k\| \leq M_2(\rho(Z)+\varepsilon)^k.$$

Our result says that there exists $M_3 > 0$ such that $(\forall k \in \mathbb{N})\ \|Z^k\| \leq M_3\rho(Z)^k$ if and only if all eigenvalues of $Z$ with magnitude $\rho(Z)$ are semisimple. In other words, when one of the eigenvalues of $Z$ with magnitude $\rho(Z)$ is not semisimple, the convergence rate of $Z^k$ to 0 can be as close as $\rho(Z)$ as one wishes, but not exactly $\rho(Z)$.

(iii). When all subdominant eigenvalues of $A$ are semisimple, all eigenvalues of $Z$ in (ii) with magnitude $\rho(Z)$ are also semisimple and thus *nondefective* in the sense of Stewart [43]. It follows from [43, Theorem 2.9, p. 33] or (31) above that there exists $\tau > 0$ such that $\|Z^k\| \leq \tau\rho(Z)^k$ for all $k > 1$. This together with the spectral resolution aforementioned in (ii) tells us that $A^k$ linearly converges to $P$ with the exact rate $\rho(Z) = \rho(A)$. The latter conclusion is indeed the sufficient part in (2.15) obtained by using a different method.

# 3 Convergence rate analysis of relaxed alternating projection and generalized Douglas-Rachford methods

In this section, using results in Section 2 and principal angles between two subspaces, we will analyze convergence rates of relaxed alternating projections, partial relaxed alternating projection and generalized Douglas-Rachford methods for two subspaces comprehensively. We show that how to choose the relaxation parameter to find the optimal rates of convergence. It turns out that matrices associated with these iteration procedures do have subdominant eigenvalues being semi-simple.

Throughout the section we suppose that $U$ and $V$ are two subspaces of $\mathbb{R}^n$ with $1 \leq p := \dim U \leq \dim V := q \leq n-1$. Note that the whole section will be trivial if $\dim U = 0$ or $\dim V = n$. Let us recall the principal angles and the Friedrichs angles between $U$ and $V$ as follows, which are crucial for our quantitative analysis of convergence rates.

**Definition 3.1 (principal angles)** ([9], [37, page 456]) *The principal angles $\theta_k \in [0, \frac{\pi}{2}]$, $k = 1, \ldots, p$ between U and V are defined by*

$$\begin{aligned}\cos\theta_k &:= \langle u_k, v_k\rangle \\ &= \max\left\{\langle u,v\rangle \,\middle|\, \begin{array}{l} u\in U, v\in V, \|u\|=\|v\|=1, \\ \langle u,u_j\rangle = \langle v,v_j\rangle = 0, j=1,\ldots,k-1 \end{array}\right\} \quad \text{with} \quad u_0 = v_0 := 0.\end{aligned} \tag{32}$$

It is worth mentioning that the vectors $u_k, v_k$ are not uniquely defined, but the principal angles $\theta_k$ are unique with $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_p \leq \frac{\pi}{2}$; see [37, page 456].

**Definition 3.2 (Friedrichs angle)** *The* cosine of the Friedrichs angle $\theta_F \in (0, \frac{\pi}{2}]$ *between $U$ and $V$ is*

$$c_F(U,V) := \max\Big\{\langle u,v\rangle \,\big|\, u \in U \cap (U\cap V)^\perp, v \in V \cap (U\cap V)^\perp, \|u\| = \|v\| = 1\Big\}. \tag{33}$$

In the following proposition we show that the Friedrichs angle is exactly the $(s+1)$-th principal angle $\theta_{s+1}$ where $s := \dim(U \cap V)$.

**Proposition 3.3 (principal angles and Friedrichs angle)** *Let $s := \dim(U\cap V)$. Then we have $\theta_k = 0$ for $k = 1, \dots, s$ and $\theta_{s+1} = \theta_F > 0$.*

*Proof.* Let $x_1, \dots, x_s$ be an orthonormal basis of the subspace $U \cap V$. We may choose $u_k = v_k = x_k$, $k = 1, \dots, s$ from (32). It follows that $\cos\theta_k = \langle x_k, x_k\rangle = 1$ and thus $\theta_k = 0$ for all $k = 1, \dots, s$. Moreover, since $\operatorname{span}\{u_1, \dots, u_s\} = \operatorname{span}\{v_1, \dots, v_s\} = U \cap V$, we obtain from (32) that

$$\cos\theta_{s+1} = \max\big\{\langle u,v\rangle \,\big|\, u \in U, v \in V, \|u\| = \|v\| = 1, u, v \in (U\cap V)^\perp\big\}. \tag{34}$$

This together with (33) tells us that $\theta_{s+1} = \theta_F$. The proof is complete. ∎

The following result follows the idea of [9, 16] to construct the orthogonal projections $P_U$ and $P_V$ with the appearance of the principal angles.

**Proposition 3.4 (principal angles and orthogonal projections)** *Suppose that $p + q < n$. Then we may find an orthogonal matrix $D \in \mathbb{R}^{n\times n}$ such that*

$$P_U = D\begin{pmatrix} I_p & 0 & 0 & 0\\ 0 & 0_p & 0 & 0\\ 0 & 0 & 0_{q-p} & 0\\ 0 & 0 & 0 & 0_{n-p-q}\end{pmatrix} D^* \quad \textit{and} \quad P_V = D\begin{pmatrix} C^2 & CS & 0 & 0\\ CS & S^2 & 0 & 0\\ 0 & 0 & I_{q-p} & 0\\ 0 & 0 & 0 & 0_{n-p-q}\end{pmatrix} D^*, \tag{35}$$

*where $C$ and $S$ are two $p \times p$ diagonal matrices defined by*

$$C := \operatorname{diag}\big(\cos\theta_1, \dots, \cos\theta_p\big) \quad \textit{and} \quad S := \operatorname{diag}\big(\sin\theta_1, \dots, \sin\theta_p\big) \tag{36}$$

*with the principal angles $\theta_1, \dots, \theta_p$ between $U$ and $V$ found in* Definition 3.1. *Consequently, we have*

$$P_U P_V = D\begin{pmatrix} C^2 & CS & 0 & 0\\ 0 & 0_p & 0 & 0\\ 0 & 0 & 0_{q-p} & 0\\ 0 & 0 & 0 & 0_{n-p-q}\end{pmatrix} D^* \textit{ and } P_{U^\perp} P_{V^\perp} = D\begin{pmatrix} 0_p & 0 & 0 & 0\\ -CS & C^2 & 0 & 0\\ 0 & 0 & 0_{q-p} & 0\\ 0 & 0 & 0 & I_{n-p-q}\end{pmatrix} D^*. \tag{37}$$

*Furthermore, the orthogonal projection $P_{U\cap V}$ is computed by*

$$P_{U\cap V} = D\begin{pmatrix} I_s & 0\\ 0 & 0_{n-s}\end{pmatrix} D^* \qquad \textit{with} \qquad s := \dim(U\cap V). \tag{38}$$

*Proof.* Let $Q_U \in \mathbb{R}^{n\times p}$, $Q_{U^\perp} \in \mathbb{R}^{n\times(n-p)}$ and $Q_V \in \mathbb{R}^{n\times q}$ be three matrices such that their columns form three orthonormal bases for $U, U^\perp$ and $V$, respectively. It follows from [37, page 430] that $P_U = Q_U Q_U^*$, $I - P_U = P_{U^\perp} = Q_{U^\perp} Q_{U^\perp}^*$ and $P_V = Q_V Q_V^*$. Furthermore, by [9, Theorem 1] we have that the *Singular Value Decomposition* (SVD) of the $p \times q$ matrix $Q_U^* Q_V$ is

$$Q_U^* Q_V = ACB^* \quad \text{with} \quad C = \operatorname{diag}(\cos\theta_1, \dots, \cos\theta_p) \in \mathbb{R}^{p\times p}, \tag{39}$$

where $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{q\times p}$ satisfy $AA^* = A^*A = B^*B = I_p$. Since all $p$ columns of $B$ are orthonormal and $p \le q$, we may find a $q \times (q-p)$ matrix $B'$ such that $\widetilde{B} := (B, B') \in \mathbb{R}^{q\times q}$ is orthogonal. Define further that $D_1 := Q_U A \in \mathbb{R}^{n\times p}$, we have $D_1^* D_1 = A^* Q_U^* Q_U A = A^* A = I_p$. Note further from (39) that

$$P_U Q_V = Q_U Q_U^* Q_V = Q_U ACB^* = D_1 CB^*. \tag{40}$$

Moreover, we get from (39) that

$$\begin{aligned}
[Q_{U^\perp}^* Q_V]^*[Q_{U^\perp}^* Q_V] &= Q_V^* Q_{U^\perp} Q_{U^\perp}^* Q_V = Q_V^*(\mathrm{Id} - P_U) Q_V = \mathrm{Id} - Q_V^* P_U Q_V \\
&= \mathrm{Id} - Q_V^* Q_U Q_U^* Q_V = \mathrm{Id} - (ACB^*)^*(ACB^*) = \mathrm{Id} - BCA^*ACB^* \\
&= \mathrm{Id} - BC^2B^* = \widetilde{B}\,\widetilde{B}^* - \widetilde{B}\begin{pmatrix} C^2 & 0 \\ 0 & 0_{q-p}\end{pmatrix}\widetilde{B}^* = \widetilde{B}\begin{pmatrix} I_p - C^2 & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^* \\
&= \widetilde{B}\begin{pmatrix} S^2 & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^*.
\end{aligned}$$

Hence the columns of $\widetilde{B}$ are eigenvectors of $[Q_{U^\perp}^* Q_V]^*[Q_{U^\perp}^* Q_V]$. It follows that the SVD of $Q_{U^\perp}^* Q_V$ has the form

$$Q_{U^\perp}^* Q_V = A_1 \begin{pmatrix} S & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^* \tag{42}$$

for some $A_1 \in \mathbb{R}^{(n-p)\times q}$ with $A_1^* A_1 = I_q$. Define $D_2 := Q_{U^\perp} A_1 \in \mathbb{R}^{n\times q}$, we have $D_2^* D_2 = A_1^* Q_{U^\perp}^* Q_{U^\perp} A_1 = A_1^* A_1 = I_q$. Moreover, it follows from (42) that

$$(I - P_U) Q_V = Q_{U^\perp} Q_{U^\perp}^* Q_V = D_2 \begin{pmatrix} S & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^*. \tag{43}$$

Note further that $D_1^* D_2 = A^* Q_U^* Q_{U^\perp} A_1 = 0$, since the columns of $Q_U, Q_{U^\perp}$ are two basis of $U$ and $U^\perp$, respectively. Thus there is an $n \times (n-p-q)$ matrix $D_3$ such that $D := (D_1, D_2, D_3) \in \mathbb{R}^{n\times n}$ is orthogonal. Combining (40) and (43) gives us that

$$Q_V = D_1 CB^* + D_2 \begin{pmatrix} S & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^* = D_1 \begin{pmatrix} C & 0_{p\times(q-p)}\end{pmatrix}\widetilde{B}^* + D_2 \begin{pmatrix} S & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^*.$$

Hence we have

$$\begin{aligned}
P_V = Q_V Q_V^* &= \left[D_1 \begin{pmatrix} C & 0_{p\times(q-p)}\end{pmatrix}\widetilde{B}^* + D_2 \begin{pmatrix} S & 0 \\ 0 & I_{q-p}\end{pmatrix}\widetilde{B}^*\right]\cdot\left[\widetilde{B}\begin{pmatrix} C \\ 0_{(q-p)\times p}\end{pmatrix} D_1^* + \widetilde{B}\begin{pmatrix} S & 0 \\ 0 & I_{q-p}\end{pmatrix} D_2^*\right] \\
&= D_1 C^2 D_1^* + D_1 \begin{pmatrix} CS & 0_{p\times(q-p)}\end{pmatrix} D_2^* + D_2 \begin{pmatrix} SC \\ 0_{(q-p)\times p}\end{pmatrix} D_1^* + D_2 \begin{pmatrix} S^2 & 0 \\ 0 & I_{q-p}\end{pmatrix} D_2^* \\
&= D \begin{pmatrix} C^2 & CS & 0 & 0 \\ CS & S^2 & 0 & 0 \\ 0 & 0 & I_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q}\end{pmatrix} D^*,
\end{aligned}$$

which ensures the second part of (35). Note further that $D_1 D_1^* = Q_U A (Q_U A)^* = Q_U A A^* Q_U^* = Q_U Q_U^* = P_U$. It follows that

$$P_U = D \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q}\end{pmatrix} D^*,$$

which verifies (35). The formulas of $P_U P_V$ and $P_{U^\perp} P_{V^\perp} = (\operatorname{Id} - P_U)(\operatorname{Id} - P_V)$ in (37) can be derived easily from (35). It remains to establish (38). Observe from (37) and Proposition 3.3 that

$$(P_U P_V)^k = D \begin{pmatrix} C^{2k} & C^{2(k-1)}CS & 0 \\ 0 & 0_p & 0 \\ 0 & 0 & 0_{n-2p} \end{pmatrix} D^* \longrightarrow D \begin{pmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{pmatrix} D^* \quad \text{as} \quad k \to \infty.$$

Note further that $\operatorname{Fix}(P_U P_V) = U \cap V = \operatorname{Fix}(P_V P_U)$; see, e.g., [6, Lemma 2.4]. Combining this with (38) and Corollary 2.7 tells us that $P_{U\cap V} = P_{\operatorname{Fix}(P_U P_V)} = D \begin{pmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{pmatrix} D^*$. ■

**Remark 3.5** When $p + q < n$, observe from (37), (33), and Proposition 3.3 that $\gamma(P_U P_V) = \gamma(P_{U^\perp} P_{V^\perp}) = c_F^2(U, V)$. These equalities is also true when $p + q \geq n$ by applying the trick used in Case 2 in the proof of Theorem 3.6. It follows that $c_F(U, V) = c_F(U^\perp, V^\perp)$ by replacing $U, V$ by $U^\perp, V^\perp$, respectively. This equality is known as Solmon's formula; see [17, Theorem 16] and also [39, Theorem 3] for different proofs.

### 3.1 Convergence rate of relaxed alternating projection methods

Throughout this subsection let us denote the classical *alternating projection* mapping by $T := P_U P_V$, which is well-known to be convergent to $P_{U\cap V}$ with the linear rate $c_F^2(U, V) = \cos^2\theta_{s+1}$ with $s = \dim(U \cap V)$; see [17, 27]. We will study some relaxations of this operator and show that a better optimal rate can be obtained. The first kind *relaxed alternating projection mapping* we will study is defined by

$$T_\mu := (1 - \mu)\operatorname{Id} + \mu P_U P_V \quad \text{with} \quad \mu \in \mathbb{R}. \tag{46}$$

It is worth noting that the case $\mu = 0$ is not interesting, since $T_0 = \operatorname{Id}$ is the identity map. Let us analyze the convergence of $T_\mu$ in the following result mainly for the case $\mu \neq 0$. When $\mu = 1$, it recovers the classical result aforementioned.

**Theorem 3.6 (relaxed alternating projection)** *Let $\theta_{s+1} = \theta_F$ be defined in* Proposition 3.3 *with $s = \dim(U \cap V)$. Then the mapping $T_\mu = (1-\mu)\operatorname{Id} + \mu P_U P_V$, $\mu \in \mathbb{R}$ is convergent if and only if $\mu \in [0, 2)$. Moreover, the following assertions hold:*

**(i)** *If $\mu \in (0, \frac{2}{1+\sin^2\theta_{s+1}}]$, then $T_\mu$ is convergent to $P_{U\cap V}$ with the optimal linear rate $\gamma(T_\mu) = 1 - \mu\sin^2\theta_{s+1}$.*

**(ii)** *If $\mu \in (\frac{2}{1+\sin^2\theta_{s+1}}, 2)$, then $T_\mu$ is convergent to $P_{U\cap V}$ with the optimal linear rate $\gamma(T_\mu) = \mu - 1$.*

*Consequently, when $\mu \neq 0$, $T_\mu$ is convergent to $P_{U\cap V}$ with linear rate smaller than $\cos^2\theta_{s+1}$ if and only if $\mu \in (1, 2 - \sin^2\theta_{s+1})$. Furthermore, $T_\mu$ attains the smallest convergence rate $\frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}}$ at $\mu = \frac{2}{1+\sin^2\theta_{s+1}}$.*

*Proof*. Let us justify the theorem by considering two main cases as follows.

**Case 1:** $p+q<n$, where $1 \le p = \dim U \le q = \dim V \le n-1$. By Proposition 3.4, (35) and (37), we may find some orthogonal matrix $D$ such that

$$T_\mu = (1-\mu)\,\mathrm{Id} + \mu P_U P_V = D \begin{pmatrix} (1-\mu)I_p + \mu C^2 & \mu CS & 0 \\ 0 & (1-\mu)I_p & 0 \\ 0 & 0 & (1-\mu)I_{n-2p} \end{pmatrix} D^* \tag{47}$$
$$= D \begin{pmatrix} I_p - \mu S^2 & \mu CS & 0 \\ 0 & (1-\mu)I_p & 0 \\ 0 & 0 & (1-\mu)I_{n-2p} \end{pmatrix} D^*.$$

It follows that

$$\sigma(T_\mu) = \{1-\mu\sin^2\theta_k |\ k=1,\ldots,p\} \cup \{1-\mu\}. \tag{48}$$

Suppose first that $T_\mu$ is convergent, we get from Fact 2.4 that $\rho(T_\mu) \le 1$ and $-1 \notin \sigma(T_\mu)$. Thus we have $|1-\mu| \le 1$ and $-1 \ne 1-\mu$, which yield $0 \le \mu < 2$. Conversely, suppose that $0 \le \mu < 2$ and observe from Proposition 3.3 that

$$1 = 1-\mu\sin^2\theta_1 = \ldots = 1-\mu\sin^2\theta_s > 1-\mu\sin^2\theta_{s+1} \ge \ldots \ge 1-\mu\sin^2\theta_p \ge 1-\mu > -1.$$

If $\mu = 0$ then $T_\mu = \mathrm{Id}$ is always convergent. If $\mu > 0$ and $s = 0$, it is clear that $1 \notin \sigma(T_\mu)$ by (48). Thus $T_\mu$ is convergent by Fact 2.4. If $\mu > 0$ and $s > 0$, we claim that $1 \in \sigma(T_\mu)$ is semisimple. Indeed, observe from (47) that

$$\ker(T_\mu - \mathrm{Id}) = D \begin{pmatrix} \ker(-\mu S^2) \\ 0_{(n-p)\times 1} \end{pmatrix} = D \begin{pmatrix} \mathbb{R}^s \\ 0_{(n-s)\times 1} \end{pmatrix}. \tag{49}$$

Similarly we also have

$$\ker(T_\mu - \mathrm{Id})^2 = D \begin{pmatrix} \ker(-\mu^2 S^4) \\ 0_{(n-p)\times 1} \end{pmatrix} = D \begin{pmatrix} \mathbb{R}^s \\ 0_{(n-s)\times 1} \end{pmatrix}. \tag{50}$$

It follows from (49) and (50) that 1 is semisimple to $T_\mu$ due to Fact 2.3. This tells that $T_\mu$ is convergent by Fact 2.4. Thus $T_\mu = (1-\mu)\,\mathrm{Id} + \mu P_U P_V$, $\mu \in \mathbb{R}$ is convergent if and only if $\mu \in [0,2)$.

Next let us justify **(i)** and **(ii)** under the assumption that $\mu \in (0,2)$. We claim first that $T_\mu$ is convergent to $P_{U\cap V}$. Indeed, note that

$$\mathrm{Fix}\, T_\mu = \ker[\mu(P_U P_V - \mathrm{Id})] = \ker(P_U P_V - \mathrm{Id}) = \mathrm{Fix}(P_U P_V) = U \cap V.$$

Furthermore, we have

$$\mathrm{Fix}\, T_\mu^* = \ker[\mu(P_V P_U - \mathrm{Id})] = \ker(P_V P_U - \mathrm{Id}) = \mathrm{Fix}(P_V P_U) = V \cap U,$$

which yields in turn the equality $\mathrm{Fix}\, T_\mu = \mathrm{Fix}\, T_\mu^*$. By Corollary 2.7, the mapping $T_\mu$ is convergent to $P_{U\cap V}$.

Now we justify the quantitative characterizations in **(i)** and **(ii)**. Observe from (48) that the subdominant eigenvalue of $T_\mu$ is

$$\gamma(T_\mu) = \max\{|1-\mu\sin^2\theta_{s+1}|, |1-\mu|\}. \tag{51}$$

Note also that

$$(1-\mu\sin^2\theta_{s+1})^2 - (1-\mu)^2 = \mu\cos^2\theta_{s+1}\big[2-\mu(1+\sin^2\theta_{s+1})\big]. \tag{52}$$

*Subcase a:* $\cos^2\theta_{s+1} = 0$. Then we have $\theta_{s+1} = \ldots = \theta_p = \frac{\pi}{2}$ and $\gamma(T_\mu) = |1-\mu|$. In this case it is easy to see that $CS = 0$ and thus $T_\mu$ is diagonalizable by (47). Thanks to Example 2.9 we have $T_\mu$ is convergent with optimal rate $|1-\mu|$. Both **(i)** and **(ii)** are valid in this case.

*Subcase b:* $\cos^2\theta_{s+1} > 0$. Let us consider the following three subsubcases:

*Subsubcase b1:* $\mu \in (0, \frac{2}{\sin^2\theta_{s+1}+1})$. Then we have $|1-\mu\sin^2\theta_{s+1}| > |1-\mu|$ by (52) and thus $\gamma(T_\mu) = |1-\mu\sin^2\theta_{s+1}|$. Observe that

$$1 > a_\mu := 1-\mu\sin^2\theta_{s+1} > 1 - \frac{2\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}} = \frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}} \geq 0. \tag{53}$$

Hence we have $\gamma(T_\mu) = 1-\mu\sin^2\theta_{s+1}$. Suppose further that $\theta_{s+1} = \ldots = \theta_k$ and $\theta_{s+1} \neq \theta_{k+1}$ with some $k \in \{s+1,\ldots,p\}$, we easily check from (47) that

$$\ker(T_\mu - a_\mu \operatorname{Id}) = \ker(T_\mu - a_\mu \operatorname{Id})^2 = D\left(0_{1\times s} \times (\mathbb{R}^{k-s})^* \times 0_{1\times(n-k)}\right)^*,$$

which shows that $a_\mu$ is semisimple by Fact 2.3. Thanks to Theorem 2.15, $T_\mu$ is convergent with the optimal rate $a_\mu$.

*Subsubcase b2:* $\mu = \frac{2}{1+\sin^2\theta_{s+1}} > 1$. Then we obtain from (51) that

$$\gamma(T_\mu) = |1-\mu\sin^2\theta_{s+1}| = |1-\mu| = 1-\mu\sin^2\theta_{s+1} = \mu-1 = \frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}}. \tag{54}$$

It is similar to the above subsubcase that $a_\mu \in \sigma(T_\mu)$ is semisimple. Furthermore, $1-\mu \in \sigma(T_\mu)$ is also semisimple. Indeed, observe that

$$T_\mu - (1-\mu)\operatorname{Id} = D\begin{pmatrix} \mu C^2 & \mu CS & 0 \\ 0 & 0_p & 0 \\ 0 & 0 & 0_{n-2p}\end{pmatrix} D^*, \quad (T_\mu - (1-\mu)\operatorname{Id})^2 = D\begin{pmatrix} \mu^2 C^4 & \mu^2 C^3 S & 0 \\ 0 & 0_p & 0 \\ 0 & 0 & 0_{n-2p}\end{pmatrix} D^*.$$

By using these two expressions, we may check that

$$\ker(T_\mu - (1-\mu)\operatorname{Id}) = \ker(T_\mu - (1-\mu)\operatorname{Id})^2,$$

which yields that $(1-\mu)$ is also semisimple by Fact 2.3. By Theorem 2.15 again, we obtain that $\frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}}$ is the optimal linear convergent rate of $T_\mu$.

*Subsubcase b3:* $\mu > \frac{2}{1+\sin^2\theta_{s+1}} > 1$. It follows from (52) that $|1-\mu\sin^2\theta_{s+1}| < |1-\mu|$. And thus we get from (51) that

$$\gamma(T_\mu) = |1-\mu| = \mu-1 > \frac{2}{1+\sin^2\theta_{s+1}} - 1 = \frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}}. \tag{55}$$

Similarly to the above case, $1-\mu \in \sigma(T_\mu)$ is semisimple. Thus Theorem 2.15 tells us that $\mu-1$ is the convergent rate of $T_\mu$ in this subcase.

Combining Subsubcase b1 and Subsubcase b2 ensures **(i)**, and **(ii)** is exactly the Subsubcase b3. Thus **(i)** and **(ii)** are verified.

Let us complete the proof by verifying the last part of the theorem. When $\mu \in (0, \frac{2}{1+\sin^2\theta_{s+1}}]$, we have $1-\mu\sin^2\theta_{s+1} < \cos^2\theta_{s+1}$ if and only if $\mu > 1$, since $\sin^2\theta_{s+1} > 0$ by Proposition 3.3.

Furthermore, when $\mu \in (\frac{2}{1+\sin^2\theta_{s+1}}, 2)$, we have $\mu - 1 < \cos^2\theta_{s+1}$ if and only if $\mu < 1 + \cos^2\theta_{s+1} = 2 - \sin^2\theta_{s+1}$. Combining these two observations with **(i)** and **(ii)** in the theorem tells us that $T_\mu$ is convergent to $P_{U\cap V}$ with a rate smaller than $\cos^2\theta_{s+1}$ if and only if $\mu \in (1, 2-\sin^2\theta_{s+1})$. Moreover, the optimal rate $\frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}}$ of $T_\mu$ is obtained at $\mu = \frac{2}{1+\sin^2\theta_{s+1}}$ due to (53), (54), and (55).

**Case 2:** $p + q \geq n$. We may find some $k \in \mathbb{N}$ such that $n' := n + k > p + q$. Define $U' := U \times \{0_k\} \subset \mathbb{R}^{n'}$, $V' := V \times \{0_k\} \subset \mathbb{R}^{n'}$, and $T'_\mu = (1-\mu)\operatorname{Id} + \mu P_{U'}P_{V'}$. It is clear that $1 \leq p = \dim U' \leq \dim V' = q$ and $p + q < n'$. Observe from Definition 3.1 that the principal angles between $U'$ and $V'$ are the same with the ones between $U$ and $V$. Moreover, we have $P_{U'} = \begin{pmatrix} P_U & 0 \\ 0 & 0_k \end{pmatrix}$, $P_{V'} = \begin{pmatrix} P_V & 0 \\ 0 & 0_k \end{pmatrix}$, and thus

$$T'_\mu = \begin{pmatrix} T_\mu & 0 \\ 0 & (1-\mu)I_k \end{pmatrix}. \tag{56}$$

Since $q \leq n-1$, there is some $x \in \mathbb{R}^n \setminus \{0\}$ such that $P_V x = 0$. It follows that $Tx = 0$, and thus we have $0 \in \sigma(T)$ and then $1 - \mu \in \sigma(T_\mu)$. If $T_\mu$ is convergent, Fact 2.4 tells us that $-1 < 1-\mu \leq 1$, i.e., $\mu \in [0, 2)$. Conversely, if $\mu \in [0,2)$ we have $T'_\mu$ is convergent due to Case 1. This together with (56) ensures that $T_\mu$ is also convergent. Hence $T_\mu$ is convergent if and only if $\mu \in [0, 2)$.

To verify the convergence rate of $T_\mu$, suppose further that $\mu \in (0,2)$. We note that $\sigma(T_\mu) = \sigma(T'_\mu)$, which implies in turn that $\gamma(T_\mu) = \gamma(T'_\mu)$. It follows from Case 1 that $T'_\mu$ in (56) is convergent to $P_{U'\cap V'} = \begin{pmatrix} P_{U\cap V} & 0 \\ 0 & 0_k \end{pmatrix}$ with the convergence rate $\gamma(T'_\mu)$. This together with (56) yields

$$\|T_\mu^k - P_{U\cap V}\| \leq \|(T'_\mu)^k - P_{U'\cap V'}\|.$$

Thus $\gamma(T'_\mu) = \gamma(T_\mu)$ is the convergence rate of $T_\mu$ by also Theorem 2.12. The analysis of $\gamma(T'_\mu)$ in **(i)** and **(ii)** in Case 1 also guides us to verify **(i)** and **(ii)** for $\gamma(T_\mu)$ in Case 2. Hence the proof is complete. ■

Next we study another kind of relaxation of the the map $T = P_U P_V$, that is

$$S_\mu := P_U((1-\mu)\operatorname{Id} + \mu P_V) = (1-\mu)P_U + \mu P_U P_V; \tag{57}$$

see also [33] for a similar form, which will give us a better optimal rate. Since the proof is similar to the one of Theorem 3.6 above, we only sketch the main steps.

**Theorem 3.7 (partial relaxed alternating projection)** *The map $S_\mu := P_U((1-\mu)\operatorname{Id} + \mu P_V) = (1-\mu)P_U + \mu P_U P_V$ is convergent if and only if $\mu \in [0, \frac{2}{\sin^2\theta_p})$ with the convention $\frac{1}{0} = \infty$. Moreover, the following assertions hold:*

**(i)** *If $\mu \in (0, \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}]$, then $S_\mu$ is convergent to $P_{U\cap V}$ with the optimal linear convergence rate $\gamma(S_\mu) = 1 - \mu\sin^2\theta_{s+1}$.*

**(ii)** *If $\mu \in (\frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}, \frac{2}{\sin^2\theta_p})$, then $S_\mu$ is convergent to $P_{U\cap V}$ with the optimal linear convergence rate $\gamma(S_\mu) = \mu\sin^2\theta_p - 1$.*

*Consequently, when $\mu \neq 0$, $S_\mu$ is convergent to $P_{U\cap V}$ with the optimal linear convergence rate smaller than $\cos^2\theta_{s+1} = c_F^2(U,V)$ if and only if $\mu \in (1, \frac{2-\sin^2\theta_{s+1}}{\sin^2\theta_p})$. Furthermore, $S_\mu$ attains the smallest linear convergence rate $\frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ at $\mu = \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$.*

*Proof*. We separate the proof into two main cases as below:

**Case 1:** $p+q<n$ with $1 \le p = \dim U \le q = \dim V \le n-1$. It follows from (35) and (37) that there is some orthogonal matrix $D \in \mathbb{R}^{n\times n}$ such that

$$S_\mu = D\begin{pmatrix}(1-\mu)I_p+\mu C^2 & \mu CS & 0\\ 0 & 0_p & 0\\ 0 & 0 & 0_{n-2p}\end{pmatrix}D^* = D\begin{pmatrix}I_p-\mu S^2 & \mu CS & 0\\ 0 & 0_p & 0\\ 0 & 0 & 0_{n-2p}\end{pmatrix}D^*. \tag{58}$$

Hence we have

$$\sigma(S_\mu) = \{1-\mu\sin^2\theta_k |\ k=1,\ldots,p\} \cup \{0\}. \tag{59}$$

Suppose that $S_\mu$ is convergent, we get from Fact 2.4 that

$$-1 < 1-\mu\sin^2\theta_p \quad \text{and} \quad 1-\mu\sin^2\theta_{s+1} \le 1. \tag{60}$$

Since $\theta_{s+1} = \theta_F \neq 0$ by Proposition 3.3, the latter gives us that $\mu \in [0, \frac{2}{\sin^2\theta_p})$. Conversely, suppose that $\mu \in [0, \frac{2}{\sin^2\theta_p})$, we have

$$1 = 1-\mu\sin^2\theta_1 = \cdots = 1-\mu\sin^2\theta_s \ge 1-\mu\sin^2\theta_{s+1} \ge \cdots \ge 1-\mu\sin^2\theta_p > -1. \tag{61}$$

If $\mu = 0$ then $S_\mu = P_U$ is always convergent. If $\mu > 0$ and $s = 0$, it is clear that $1 \notin \sigma(S_\mu)$ by (59). Thanks to Fact 2.4, we have $S_\mu$ is convergent. If $\mu > 0$ and $s > 0$, it is similar to the corresponding part of Theorem 3.6 that $1 \in \sigma(S_\mu)$ is semisimple. Combining (61) with Fact 2.4 gives us that $S_\mu$ is convergent. Thus $S_\mu$ is convergent if and only if $\mu \in [0, \frac{2}{\sin^2\theta_p})$.

To verify **(i)** and **(ii)**, assume further that $\mu \in (0, \frac{2}{\sin^2\theta_p})$. Let us claim that $S_\mu$ is convergent to $P_{U\cap V}$. Via the explicit form of $S_\mu$ in (58), we can easily check that

$$\operatorname{Fix} S_\mu = \ker(S_\mu - \mathrm{Id}) = D((\mathbb{R}^s)^* \times 0_{1\times(n-s)})^* = \ker(S_\mu^* - \mathrm{Id}) = \operatorname{Fix} S_\mu^*.$$

Note also from (38) that

$$U \cap V = \operatorname{Fix} P_{U\cap V} = D((\mathbb{R}^s)^* \times 0_{1\times(n-s)})^*.$$

It follows that $\operatorname{Fix} S_\mu = \operatorname{Fix} S_\mu^* = U\cap V$. Thanks to Corollary 2.7, we have $S_\mu$ is convergent to $P_{U\cap V}$.

Next we justify the qualitative characterizations in **(i)** and **(ii)**. Observe from (59) and (61) that

$$\gamma(S_\mu) = \max\{|1-\mu\sin^2\theta_{s+1}|, |1-\mu\sin^2\theta_p|\}. \tag{62}$$

Note also that

$$(1-\mu\sin^2\theta_{s+1})^2 - (1-\mu\sin^2\theta_p)^2 = \mu(\sin^2\theta_p - \sin^2\theta_{s+1})[2-\mu(\sin^2\theta_{s+1}+\sin^2\theta_p)]. \tag{63}$$

*Subcase a:* $\sin\theta_p = \sin\theta_{s+1}$, i.e., $\theta_{s+1} = \theta_{s+2} = \cdots = \theta_p$. Hence we have $\sigma(S_\mu) = \{1-\mu\sin^2\theta_s, 1-\mu\sin^2\theta_{s+1}, 0\}$ and $\gamma(S_\mu) = |1-\mu\sin^2\theta_{s+1}|$. Moreover, it is easy to check that $c_\mu := 1-\mu\sin^2\theta_{s+1}$ is semisimple by showing that $\ker(S_\mu - c_\mu\,\mathrm{Id}) = \ker(S_\mu - c_\mu\,\mathrm{Id})^2$.

*Subcase b:* $\sin\theta_p \neq \sin\theta_{s+1}$, i.e., $\sin\theta_p > \sin\theta_{s+1}$. We continue the proof by taking into account three different cases as follows.

*Subsubcase b1:* $\mu \in (0, \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p})$. Then we have from (63) that $|1-\mu\sin^2\theta_{s+1}| > |1-\mu\sin^2\theta_p|$, which gives us that $\gamma(S_\mu) = |1-\mu\sin^2\theta_{s+1}|$ by (62). Moreover, note that

$$c_\mu = 1-\mu\sin^2\theta_{s+1} > 1-\frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}\sin^2\theta_{s+1} = \frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p} > 0. \tag{64}$$

Thanks to the structure of $S_\mu$ in (58), we may check that $c_\mu$ is semisimple. Thus $c_\mu = \gamma(S_\mu)$ is the optimal linear convergence rate of $S_\mu$ by Theorem 2.15.

*Subsubcase b2:* $\mu = \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$. Thus

$$\gamma(S_\mu) = |1-\mu\sin^2\theta_{s+1}| = |1-\mu\sin^2\theta_p| = \frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p} > 0. \tag{65}$$

We can check that $c_\mu = 1-\mu\sin^2\theta_{s+1}$ and $d_\mu := 1-\mu\sin^2\theta_p$ are semisimple in this case via Fact 2.3. This together with Theorem 2.15 tells us that $\gamma(S_\mu) = 1-\mu\sin^2\theta_{s+1} = \frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ is the optimal linear rate of $S_\mu$.

*Subsubcase b3:* $\mu \in (\frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}, \frac{2}{\sin^2\theta_p})$. It follows from (63) that $|1-\mu\sin^2\theta_{s+1}| < |1-\mu\sin^2\theta_p|$, which yields $\gamma(S_\mu) = |1-\mu\sin^2\theta_p|$ by (62). Moreover, observe that

$$\mu\sin^2\theta_p - 1 > \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}\sin^2\theta_p - 1 = \frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p} > 0. \tag{66}$$

We also have $d_\mu = 1-\mu\sin^2\theta_p$ is semisimple via Fact 2.3. Thanks to Theorem 2.15, $\gamma(S_\mu) = \mu\sin^2\theta_p - 1$ is the optimal linear convergence rate of $S_\mu$.

Combining Subsubcase b1 and Subsubcase b2 gives us **(i)**. Furthermore, Subsubcase b3 exactly verifies **(ii)**. The last part of the theorem is indeed a direct consequence of **(i)** and **(ii)**. The proof of the theorem for Case 1 is complete.

**Case 2:** $p+q \geq n$. Then we find some $k \in \mathbb{N}$ such that $n' := n+k > p+q$ and define $U' := U \times \{0_k\} \subset \mathbb{R}^{n'}$, $V' := V \times \{0_k\} \subset \mathbb{R}^{n'}$, and $S'_\mu = (1-\mu)P_{U'} + \mu P_{U'}P_{V'}$. It is clear that $1 \leq p = \dim U' \leq \dim V' = q$ and $p+q < n'$. Moreover, we also have

$$S'_\mu = \begin{pmatrix} S_\mu & 0 \\ 0 & 0_k \end{pmatrix},$$

which shows that $S'_\mu$ is convergent if and only if $S_\mu$ is convergent. The rest of the proof is quite similar to the corresponding one in Theorem 3.6. ■

**Remark 3.8** It is clear that the optimal linear rate $\frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ of $S_\mu$ is smaller than the one $\frac{1-\sin^2\theta_{s+1}}{1+\sin^2\theta_{s+1}}$ of $T_\mu$ in Theorem 3.6. Note further from the above theorem that $S_2 = P_U R_V$ with $R_V := 2P_V - \mathrm{Id}$, which is known as the *reflection-projection* method [7, 11] is convergent to $P_{U\cap V}$ if and only if $2 < \frac{2}{\sin^2\theta_p}$, i.e., $\theta_p < \frac{\pi}{2}$. When this case is fulfill, the optimal linear rate of the reflection-projection method is $\max\{|1-2\sin^2\theta_{s+1}|, |1-2\sin^2\theta_p|\}$ by (62). Besides the definition of $\theta_{s+1}, \theta_p$ in Definition 3.1 and Definition 3.2, we may also obtain $\theta_{s+1}, \theta_p$ in following formulas

$$\cos^2\theta_{s+1} = \|P_U P_V - P_{U\cap V}\| \quad \text{and} \quad \sin^2\theta_p = \|P_U - P_U P_V\|^2 = \|P_U - P_U P_V P_U\| \tag{67}$$

from (35), (37), and (38).

**Remark 3.9 (finite termination)** From Theorem 3.6, observe that the map $T_\mu$ has the linear convergence rate 0, i.e., it will always terminate after finite powers if and only if $\theta_{s+1} = \frac{\pi}{2}$ and $\mu = 1$. Similarly, we get from Theorem 3.7 that $S_\mu$ has the linear convergence rate 0 if and only if $\mu = \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ and $\theta_{s+1} = \theta_p$. The latter condition is clearly satisfied when $\dim(U\cap V) = p-1$ and $\mu = \frac{1}{\sin^2\theta_{s+1}}$; e.g., $U$ and $V$ are two different lines passing the origin in $\mathbb{R}^2$, or $U$ is a line in $\mathbb{R}^3$ and $V$ is a hyperplane in $\mathbb{R}^3$ with $U \not\subset V$, or $U$ and $V$ are two different hyperplanes in $\mathbb{R}^3$, etc.

## 3.2 Convergence rate of the generalized Douglas-Rachford method

Convergence rates of many specific matrices relating to the Douglas-Rachford operator

$$R := P_U P_V + P_{U^\perp} P_{V^\perp} = \frac{R_U R_V + \mathrm{Id}}{2} = \frac{R_{U^\perp} R_{V^\perp} + \mathrm{Id}}{2} \tag{68}$$

have been discussed in [16]. One of the particular cases there is the so-called *generalized Douglas-Rachford operator* $R_\mu$ defined by

$$R_\mu := (1-\mu)\,\mathrm{Id} + \mu R.$$

The convergence rate of this mapping has been obtained in Demanet-Zhang [16] under an additional condition $U\cap V = \{0\}$. In the following result we give a complete characterization of the convergence of this map and also show that the condition $U\cap V=\{0\}$ can be relaxed.

**Theorem 3.10 (generalized Douglas-Rachford method)** *The map $R_\mu$ is convergent if and only if $\mu \in [0,2)$. Moreover, the following assertions hold:*

**(i)** *$R_\mu$ is normal.*

**(ii)** *If $\mu \in (0,2)$ then $R_\mu$ is convergent to $P_{\mathrm{Fix}\,R} = P_{(U\cap V)\oplus(U^\perp\cap V^\perp)}$ with the optimal linear convergence rate $\gamma(R_\mu) = \sqrt{\mu(2-\mu)\cos^2\theta_{s+1} + (1-\mu)^2}$, where $s := \dim(U\cap V)$.*

*Proof.* As proceeded in the proof of Theorem 3.6 and Theorem 3.7, we consider two major cases as below.

**Case 1:** $p+q<n$. By using the expressions of (37), we easily establish that

$$\begin{aligned} R_\mu &= D\begin{pmatrix} C^2+(1-\mu)S^2 & \mu CS & 0 & 0\\ -\mu CS & C^2+(1-\mu)S^2 & 0 & 0\\ 0 & 0 & (1-\mu)I_{q-p} & 0\\ 0 & 0 & 0 & I_{n-p-q}\end{pmatrix} D^*\\ &= D\begin{pmatrix} I_p-\mu S^2 & \mu CS & 0 & 0\\ -\mu CS & I_p-\mu S^2 & 0 & 0\\ 0 & 0 & (1-\mu)I_{q-p} & 0\\ 0 & 0 & 0 & I_{n-p-q}\end{pmatrix} D^*;\end{aligned} \tag{69}$$

see also a similar form on [16, page 14]. It is easy to check that $R_\mu^* R_\mu = R_\mu R_\mu^*$, i.e., $R_\mu$ is normal. Thus **(i)** is satisfied. We may get from the above format and the block determinant formula, c.f., [37, page 475] that

$$\sigma(R_\mu) = \begin{cases} \{\cos^2\theta_k + (1-\mu)\sin^2\theta_k \pm \mathrm{i}\mu\cos\theta_k\sin\theta_k \,|\, k=1,\ldots,p\}\cup\{1\} & \text{if } \; q=p,\\ \{\cos^2\theta_k + (1-\mu)\sin^2\theta_k \pm \mathrm{i}\mu\cos\theta_k\sin\theta_k \,|\, k=1,\ldots,p\}\cup\{1\}\cup\{1-\mu\} & \text{if } \; q>p,\end{cases}$$

where $\mathrm{i} := \sqrt{-1}$. For any $k = 1, \dots, p$, we have

$$\begin{aligned}\left|1-\mu\sin^2\theta_k \pm \mathrm{i}\mu\cos\theta_k\sin\theta_k\right| &= \sqrt{(1-\mu\sin^2\theta_k)^2 + \left[\mu\cos\theta_k\sin\theta_k\right]^2} \\ &= \sqrt{[\mu\cos^2\theta_k + (1-\mu)]^2 + \mu^2\cos^2\theta_k(1-\cos^2\theta_k)} \\ &= \sqrt{\mu(2-\mu)\cos^2\theta_k + (1-\mu)^2}.\end{aligned}$$

Suppose further that $R_\mu$ is convergent. Then we get from Fact 2.4 that

$$\mu(2-\mu)\cos^2\theta_{s+1} + (1-\mu)^2 \le 1,$$

which yields $\mu(2-\mu)(1-\cos^2\theta_{s+1}) \ge 0$ and thus $\mu \in [0,2]$, since $\cos^2\theta_{s+1} < 1$. Next let us consider three particular subcases of $\mu$.

*Subcase a.* $\mu = 2$. Then all eigenvalues of $R_\mu$ have magnitude 1. By Fact 2.4, we have

$$1-\mu\sin^2\theta_k \pm \mathrm{i}\mu\cos\theta_k\sin\theta_k = 1 \quad \text{for all} \quad k = 1,\dots,p, \tag{70}$$

which implies in turn that $\sin\theta_{s+1}\cos\theta_{s+1} = 0$ and thus $\theta_{s+1} = \frac{\pi}{2}$, since $\sin\theta_{s+1} > 0$ by Proposition 3.3. It follows that

$$1-\mu\sin^2\theta_{s+1} \pm \mathrm{i}\mu\cos\theta_{s+1}\sin\theta_{s+1} = -1,$$

which contradicts (70). Hence when $\mu = 2$, $R_\mu$ is not convergent.

*Subcase b:* $\mu = 0$. It is obvious that $R_\mu = \mathrm{Id}$ is convergent to Id with rate 0.

*Subcase c:* $0 < \mu < 2$. By Propodition 3.3 we have

$$\begin{aligned}1 &= \sqrt{\mu(2-\mu)\cos^2\theta_1 + (1-\mu)^2} = \cdots = \sqrt{\mu(2-\mu)\cos^2\theta_s + (1-\mu)^2} \\ &> \sqrt{\mu(2-\mu)\cos^2\theta_{s+1} + (1-\mu)^2} \ge \sqrt{\mu(2-\mu)\cos^2\theta_{s+2} + (1-\lambda)^2} \\ &\ge \cdots \ge \sqrt{\mu(2-\mu)\cos^2\theta_p + (1-\mu)^2} \ge |1-\mu|.\end{aligned} \tag{71}$$

Since $R_\mu$ is normal, it follows from Fact 2.4 and Corollary 2.7 that $R_\mu$ is convergent. Hence we have $R_\mu$ is convergent if and only if $\mu \in [0,2)$.

It remains to verify **(ii)** in this case. Suppose that $\mu \in (0,2)$, we get from the normality of $R_\mu$ and Theorem 2.18 that $\gamma(R_\mu) = \sqrt{\mu(2-\mu)\cos^2\theta_{s+1} + (1-\mu)^2}$ (by (71)) is the optimal linear convergence rate of $R_\mu$ and that $R_\mu$ is convergent to $P_{\mathrm{Fix}\, R_\mu} = P_{\mathrm{Fix}\, R}$. Moreover, we have $\mathrm{Fix}\, R = (U \cap V) \oplus (U^\perp \cap V^\perp)$ by [5, Proposition 3.6]. This ensures **(ii)** and thus completes the proof of the theorem for Case 1.

**Case 2:** $p + q \ge n$. Similarly to the proof of Theorem 3.6 and Theorem 3.7, we find $k > 0$ such that $n + k := n' > p + q$. Define further that $U' := U \times \{0_k\} \subset \mathbb{R}^{n'}$, $V' := V \times \{0_k\} \subset \mathbb{R}^{n'}$, and $R'_\mu = (1-\mu)\,\mathrm{Id} + \mu[P_{U'}P_{V'} + P_{(U')^\perp}P_{(V')^\perp}]$. It is easy to verify that

$$R'_\mu = \begin{pmatrix} R_\mu & 0 \\ 0 & I_k \end{pmatrix}. \tag{72}$$

Note from Case 1 that $R'_\mu$ is normal, and so is $R_\mu$. Morever, we get from (72) that $R_\mu$ is convergent if and only if $R'_\mu$ is convergent with the same rate. The analysis of the convergence of $R'_\mu$ in Case 1 justifies all the statement of the theorem in this case. The proof is complete. ■

**Remark 3.11** (1). Unlike the relaxed alternating projection methods studied in Theorem 3.6 and 3.7, convergence rate of the (over and under) relaxation of the Douglas-Rachford algorithm is always bigger than the original one due to

$$\gamma(R_1) = \cos\theta_{s+1} \le \sqrt{\mu(2-\mu)\cos^2\theta_{s+1} + (1-\mu)^2} = \gamma(R_\mu) \quad \text{for all} \quad \mu \in [0,2).$$

Moreover, it is worth mentioning here that Theorem 3.10 also tells us that $R_2 = R_U R_V$, which is known as *reflection-reflection* method will never be convergent in the case of two nontrivial subspaces with $1 \le \dim U, \dim V \le n-1$.

(2). For the linear convergence rate of the Douglas-Rachford method on a general Hilbert space, see [5].

## 4 A nonlinear approach to the alternating projection method

Throughout this section, we also suppose that $U$ and $V$ are two subspaces of $\mathbb{R}^n$ with $1 \le p = \dim U \le \dim V = q \le n-1$. From Theorem 3.7, we know that the map $S_\mu$ (57) obtains its smallest rate $\frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ at $\mu = \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$. This rate is smaller than the optimal rate of $T_\mu$ and $T$. However, it is not trivial to determine $\theta_{s+1}$ and $\theta_p$ to construct $\mu = \frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ for $S_\mu$ especially with big dimensions of $U$ and $V$; see Definition 3.1, Definition 3.2, and (67). In this section we introduce a simple *nonlinear* mapping, by using the idea of a line search [6, 23, 25] for the map $S_\mu$, so that the iterative sequence given by this nonlinear mapping is linearly convergent to the projection on $U \cap V$ with the same optimal rate mentioned above, so at least as fast convergence rate as the one using the optimal relaxation parameter. One may think of this mapping as the partial relaxed alternating projection with an adaptive parameter $\mu(x)$ depending on each iteration period. This is a technique employed for other iterative methods; see, e.g., [3, 4, 11, 12, 13, 21].

**Definition 4.1** *Define the map $B_T$ with $T = P_U P_V$ by*

(73) $$B_T(x) := P_U((1-\mu_x)x + \mu_x P_V x) = (1-\mu_x)P_U x + \mu_x P_U P_V x,$$

*where*

(74) $$\mu_x := \begin{cases} \dfrac{\langle P_U x - P_U P_V x, x\rangle}{\|P_U x - P_U P_V x\|^2} & \text{if} \quad P_U x - P_U P_V x \neq 0 \\ 1 & \text{if} \quad P_U x - P_U P_V x = 0. \end{cases}$$

**Remark 4.2** In [4, 6, 23], an *accelerated* mapping of $T$ is introduced by using the line-search [25] as

(75) $$A_T(x) := (1-\lambda_x)x + \lambda_x P_U P_V x,$$

where

(76) $$\lambda_x = \begin{cases} \dfrac{\langle x - P_U P_V x, x\rangle}{\|x - P_U P_V x\|^2} & \text{if} \quad x - P_U P_V x \neq 0 \\ 1 & \text{if} \quad x - P_U P_V x = 0. \end{cases}$$

It is worth noting that $\mu_x = \lambda_x$ and $B_T x = A_T x$ when $x \in U$.

Set $M := U \cap V$. The proof of the following convenient fact can be found in [18, Lemma 9.2]

$$P_U P_M = P_M P_U = P_V P_M = P_M P_V = P_M. \tag{77}$$

The main result in this section is Theorem 4.5, before proving it we provide two useful lemmas.

**Lemma 4.3** *For each $x \in \mathbb{R}^n$ and $y \in U \cap V$ we have*

$$\min_{\mu \in \mathbb{R}} \|(1-\mu)P_U x + \mu P_U P_V x - y\| = \|B_T x - y\|. \tag{78}$$

*Moreover, $\mu = \mu_x$ as given in* (74) *is the unique minimizer when $P_U x - P_U P_V x \neq 0$.*

*Proof*. When $P_U x = P_U P_V x$, inequality (78) is trivial. Now suppose that $P_U x \neq P_U P_V x$ and note that

$$\begin{aligned}\|(1-\mu)P_U x + \mu P_U P_V x - y\|^2 &= \|(1-\mu)(P_U x - y) + \mu(P_U P_V x - y)\|^2 \\ &= (1-\mu)\|P_U x - y\|^2 + \mu\|P_U P_V x - y\|^2 - \mu(1-\mu)\|P_U x - P_U P_V x\|^2. \end{aligned}\tag{79}$$

This is a quadratic in $\mu$ and thus attains its minimum at the following unique minimizer

$$\begin{aligned}\mu &= \frac{1}{2}\frac{\|P_U x - y\|^2 - \|P_U P_V x - y\|^2 + \|P_U x - P_U P_V x\|^2}{\|P_U x - P_U P_V x\|^2} \\ &= \frac{\langle P_U x - y, P_U x - P_U P_V x\rangle}{\|P_U x - P_U P_V x\|^2}.\end{aligned}\tag{80}$$

Since $y \in U \cap V$, we derive that

$$\langle y, P_U x - P_U P_V x\rangle = \langle y, P_U P_{V^\perp} x\rangle = \langle P_U y, P_{V^\perp} x\rangle = \langle y, P_{V^\perp} x\rangle = 0. \tag{81}$$

Moreover, note that $\langle P_U x, P_U x - P_U P_V x\rangle = \langle x, P_U x - P_U P_V x\rangle$. This together with (79), (80) and (81) tells us that the left-hand side of (78) attains its minimum at $\mu_x$ in (74). We have verified (78), thus completed the proof of the lemma. ■

**Lemma 4.4** *For any $\mu \in \mathbb{R}$ and $x \in U$ we have*

$$S_\mu x - P_{U\cap V} x = ((1-\mu)P_U + \mu P_U P_V P_U - P_{U\cap V})(x - P_{U\cap V} x), \tag{82}$$

*where $S_\mu = (1-\mu)P_U + \mu P_U P_V$ defined in* Theorem 3.7. *Moreover, when $\mu = \frac{2}{\sin^2\theta_{s+1} + \sin^2\theta_p}$ with $s = \dim(U \cap V)$ and $\theta_{s+1}, \theta_p$ found in* Definition 3.1, *we have*

$$\|(1-\mu)P_U + \mu P_U P_V P_U - P_{U\cap V}\| = \frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_{s+1} + \sin^2\theta_p}. \tag{83}$$

*Proof*. Set $M := U \cap V$. For any $x \in U$ we get from (77) that

$$\begin{aligned}((1-\mu)P_U + \mu P_U P_V P_U - P_M)(x - P_M x) &= ((1-\mu)P_U + \mu P_U P_V P_U)x - P_M x - P_M x + P_M^2 x \\ &= (1-\mu)P_U x + \mu P_U P_V x - P_M x \\ &= S_\mu x - P_M x,\end{aligned}$$

which verifies (82). To justify (83), without loss of generality, suppose that $p+q<n$ (otherwise, we follow the trick used in the proof of Case 2 of Theorem 3.7). It is easy to check from (35) and (38) that

$$
\begin{aligned}
(1-\mu)P_U+\mu P_UP_VP_U-P_M &= D\begin{pmatrix} (1-\mu)I_p+\mu C^2-\begin{pmatrix} I_s & 0\\ 0 & 0_{p-s}\end{pmatrix} & 0\\ 0 & 0_{n-p}\end{pmatrix}D^*\\
&= D\begin{pmatrix} 0_s & & & & 0\\ & 1-\mu\sin^2\theta_{s+1} & & & \\ & & \ddots & & \\ & & & 1-\mu\sin^2\theta_p & \\ 0 & & & & 0_{n-p}\end{pmatrix}D^*
\end{aligned}
$$

for some orthogonal matrix $D\in\mathbb{R}^{n\times n}$. When $\mu=\frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$, we get that

$$
\begin{aligned}
\|(1-\mu)P_U+\mu P_UP_VP_U-P_M\| &= \max\left\{|1-\mu\sin^2\theta_p|,|1-\mu\sin^2\theta_{s+1}|\right\}\\
&= \frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}.
\end{aligned}
$$

This ensures (83) and completes the proof of the lemma. ■

We are ready to establish the main result of this section as follows.

**Theorem 4.5** *For any $k\in\mathbb{N}$ and $x\in\mathbb{R}^n$, we have*

$$\|B_T^{k+1}(x)-P_{U\cap V}x\|\le\left[\frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_p+\sin^2\theta_{s+1}}\right]^k\cos^2\theta_{s+1}\|x-P_{U\cap V}x\|, \tag{84}$$

*where $\theta_{s+1}$ and $\theta_p$ are the principal angles found in* Definition 3.1. *Hence the algorithm $B_T^k(x)\to P_{U\cap V}(x)$ is at least as fast as the partial relaxed projection* (57). *Furthermore, when $x\in U$ we obtain a shaper inequality*

$$\|B_T^{k+1}(x)-P_{U\cap V}x\|\le\left[\frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_p+\sin^2\theta_{s+1}}\right]^{k+1}\|x-P_{U\cap V}x\|. \tag{85}$$

*Proof*. For any $x\in\mathbb{R}^n$, define $y=P_{U\cap V}x$ and $M=U\cap V$, note that $y=P_MB_Tx$. Fix $\mu:=\frac{2}{\sin^2\theta_{s+1}+\sin^2\theta_p}$ and $\gamma:=\frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}$. We obtain

$$
\begin{aligned}
\|B_T^{k+1}(x)-P_{U\cap V}x\| &= \|B_T(B_T^k(x))-y\|\le\|S_\mu(B_T^k(x))-y\| \qquad \text{(by (78))}\\
&= \|((1-\mu)P_U+\mu P_UP_VP_U-P_M)(B_T^k(x)-y)\| \qquad \text{(by (82) and } B_T^kx\in U)\\
&\le \|(1-\mu)P_U+\mu P_UP_VP_U-P_M\|\cdot\|B_T^k(x)-y\|\\
&= \frac{\sin^2\theta_p-\sin^2\theta_{s+1}}{\sin^2\theta_{s+1}+\sin^2\theta_p}\|B_T^k(x)-y\| \qquad \text{(by (83))}\\
&\le\cdots\\
&\le \gamma^k\|B_T(x)-y\|\le\gamma^k\|S_1(x)-y\| \qquad \text{(by (78) again)}
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma^k \|(P_U P_V - P_M)(x - P_M x)\| \qquad \text{(by (77))} \\
&\le \gamma^k \|P_U P_V - P_M\| \cdot \|x - P_M x\| = \gamma^k \cos^2 \theta_{s+1} \|x - P_M x\| \qquad \text{(by (67))}.
\end{aligned}
$$

This verifies (84). To justify (85), suppose further that $x \in U$, note that

$$
B_T x - P_M x = [(1 - \mu_x) P_U + \mu_x P_U P_V P_U](x - P_M x).
$$

With $y = P_M x$, following the above inequalities gives us that

$$
\begin{aligned}
\|B_T^{k+1}(x) - P_{U\cap V} x\| &\le \gamma^k \|B_T(x) - y\| \le \gamma^k \|S_\mu x - y\| \qquad \text{(by (78))} \\
&= \gamma^k \|((1-\mu)P_U + \mu P_U P_V P_U - P_M)(x - y)\| \qquad \text{(by (82))} \\
&\le \gamma^{k+1} \|x - y\| \qquad \text{(by (83))},
\end{aligned}
$$

which ensures (85) and completes the proof of the theorem. ■

**Remark 4.6** As discussed at the beginning of this section, though the map $S_\mu$ obtains the optimal linear convergence rate at $\mu_0 := \frac{2}{\sin^2\theta_{s+1} + \sin^2\theta_p}$, computing $\theta_{s+1}$ and $\theta_p$ may be expensive when the dimensions of $U$ and $V$ are big. Our nonlinear map $B_T$ indeed has a similar form to $S_\mu$ and also obtains the same rate with $S_{\mu_0}$, but it is easier to compute $\mu_x$ in (74) and hence $B_T(x)$ for any $x \in \mathbb{R}^n$.

The following corollary suggests a convergence rate for the accelerated map $A_T$ in (75). This is actually a counterpart of Bauschke-Deutsch-Hundal-Park [6, Theorem 3.28] when $T = P_U P_V$, which is not *selfadjoint* as required in [6, Theorem 3.28].

**Corollary 4.7** *Let $T = P_U P_V$. Then for any $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we have*

$$
\|A_T^k(Tx) - P_{U\cap V} x\| \le \left[\frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_p + \sin^2\theta_{s+1}}\right]^k \cos^2\theta_{s+1} \|x - P_{U\cap V} x\|, \tag{86}
$$

*where $A_T$ is defined in* (75) *and where $\theta_{s+1}$ and $\theta_p$ are the principal angles as in* Definition 3.1.

*Proof*. For any $x \in \mathbb{R}^n$, note that $Tx \in U$, $P_{U\cap V} Tx = T P_{U\cap V} x = P_{U\cap V} x$ by (77), and that $A_T^k(Tx) = B_T^k(Tx)$. Thus we get from (85) that

$$
\begin{aligned}
\|A_T^k(Tx) - P_{U\cap V} x\| &\le \left[\frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_p + \sin^2\theta_{s+1}}\right]^k \|Tx - P_{U\cap V} x\| \\
&= \left[\frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_p + \sin^2\theta_{s+1}}\right]^k \|(T - P_{U\cap V})(x - P_{U\cap V} x)\| \\
&\le \left[\frac{\sin^2\theta_p - \sin^2\theta_{s+1}}{\sin^2\theta_p + \sin^2\theta_{s+1}}\right]^k \cos^2\theta_{s+1} \|x - P_{U\cap V} x\| \qquad \text{(by (67))},
\end{aligned}
$$

which completes the proof of the corollary. ■

## 5 Numerical experiments

In this section, we present numerical computations comparing several algorithms developed in previous sections with some classic methods for finding $P_{U\cap V}x_0$. Our test algorithms are the following

- $B_T$ defined by (73) and (74);
- $S_\mu$ with $\mu_1 = \frac{2}{\sin^2\theta_F+\sin^2\theta_p}$ (the best parameter); $\mu_2 = \frac{1}{\sin^2\theta_p} \in [0, \frac{2}{\sin^2\theta_p})$; and $\mu_3 = \frac{1}{2} + \frac{1}{\sin^2\theta_p} \in [1, \frac{2}{\sin^2\theta_p})$ (see Theorem 3.7);
- $T_\mu$ with $\mu_1 = \frac{2}{1+\sin^2\theta_F}$ (the best parameter) and $\mu_2 = 1.5 \in [0,2)$ (see Theorem 3.6);
- the classic method of alternating projections (MAP);
- the classic Douglas-Rachford method (DR).

There are (at least) two angles that might affect the convergence: $\theta_F$ (the Friedrichs angle, see Proposition 3.3) and $\theta_p$, thus we will use them to categorize the pairs of subspaces. Our numerical set up is as follows. We assume that $X = \mathbb{R}^{100}$ and define $\mathcal{X}$ to be the set of all subspaces of $X$. First, we define our *primary* categories based on the Friedrichs angle (in radians) as follows

(87a) $$W_1 := \big\{(U,V) \in \mathcal{X}^2 \bigm| 0 < \theta_F < 0.05\big\};$$

(87b) $$W_2 := \big\{(U,V) \in \mathcal{X}^2 \bigm| 0.05 \le \theta_F < 0.1\big\};$$

(87c) $$W_3 := \big\{(U,V) \in \mathcal{X}^2 \bigm| 0.1 \le \theta_F < 0.5\big\};$$

(87d) $$\text{and} \quad W_4 := \big\{(U,V) \in \mathcal{X}^2 \bigm| 0.5 \le \theta_F < 1\big\}.$$

Since we always have $\theta_F \le \theta_p \le \pi/2$, we define our *secondary* categories as follows[3]

(88) $$Z_j := \big\{(U,V) \in \mathcal{X}^2 \bigm| \theta_p > \theta_F \quad \text{and} \quad \tfrac{\theta_p-\theta_F}{\frac{\pi}{2}-\theta_F} \in [\tfrac{j-1}{5}, \tfrac{j}{5})\big\}, \quad j = 1,\ldots,5.$$

Thus, there are 20 induced categories $W_i \cap Z_j$ for $i = 1,\ldots,4$ and $j = 1,\ldots,5$. In each $W_i \cap Z_j$, we randomly generated 5 pairs of subspaces $U$ and $V$ of $X$ such that $\dim U \le \dim V$ and $U \cap V \ne \{0\}$. So there are 100 pairs of subspaces. For each pair of subspaces, we choose randomly 10 starting points, each with Euclidean norm 10. This results in a total of $1,000$ instances for each algorithm. Note that the sequences to monitor are as follows

| Algorithm | sequence $(z_k)$ to monitor |
|---|---|
| $B_T$ (see (73)) | $(B_T)^k(x_0)$ |
| $S_\mu$ (see Theorem 3.7) | $(S_\mu)^k(x_0)$ |
| $T_\mu$ (see Theorem 3.6) | $(T_\mu)^k(x_0)$ |
| MAP | $(P_UP_V)^k(x_0)$ |
| DR (see 68) | $P_V(\frac{\mathrm{Id}+R_UR_V}{2})^k(x_0)$ |

[3]We do not test the case $\theta_F = \theta_{s+1} = \theta_p$: in such case, it is proved in Theorems 3.7 and 4.5 that $S_{\mu_1}$ and $B_T$ converge after a single step, i.e., they are the clear winners!

We terminate the algorithm when the current iterate of the monitored sequence $(z_n)_{n\in\mathbb{N}}$ satisfies

$$d_{U\cap V}(z_k) \leq 0.01 \tag{89}$$

for the first time or when the number of iterations reaches 100,000 (i.e., problem unsolved). In applications, we in general would not have access to this information but here we use it to see the true performance of these algorithms.

The data and figures in this section were computed with the help of `Julia` (see [31]) and `Gnuplot` (see [24]) running on a (CPU) Intel Core i7 (2.5GHz) with 16Gb RAM DDR3 (1600MHz). We describe the experimental results from two perspectives: number of iterations, and computation time and memory.

### 5.1 Comparison of number of iterations

In Figures 1, 2, and 3, the horizontal axis represents the Friedrichs angle between two subspaces; and the vertical axis represents the (median) number of iterations, more specifically, the median is computed over 10 instances of one pair of subspaces.

In Figure 1, we compare $B_T$, the "best" versions $S_{\mu_1}$ and $T_{\gamma_1}$, MAP, and DR. We see that $B_T$ is generally the fastest when $\theta_F > 0.02$. This can be interpreted by the fact that $B_T$ optimizes its parameter $\mu_x$ at each iteration. While when $\theta_F \leq 0.02$, DR seems to be the fastest, which was also previously observed in [5]. To some extent, the case $\theta_F \leq 0.02$ is close to "singularity". In fact, [8] studies a special case of singularity and shows that DR has some advantages over other projections methods. Nevertheless, the complete theory remains open. In Figure 2, we compare $S_{\mu_i}$, $i = 1, 2, 3$. The results suggest that the "best" version $S_{\mu_1}$ is somewhat faster than $S_{\mu_2}$ and $S_{\mu_3}$. In Figure 3, we compare $T_{\gamma_i}$, $i = 1, 2$. On the contrary, it is not clear that the "best" version $T_{\gamma_1}$ is more favorable than $T_{\gamma_2}$.

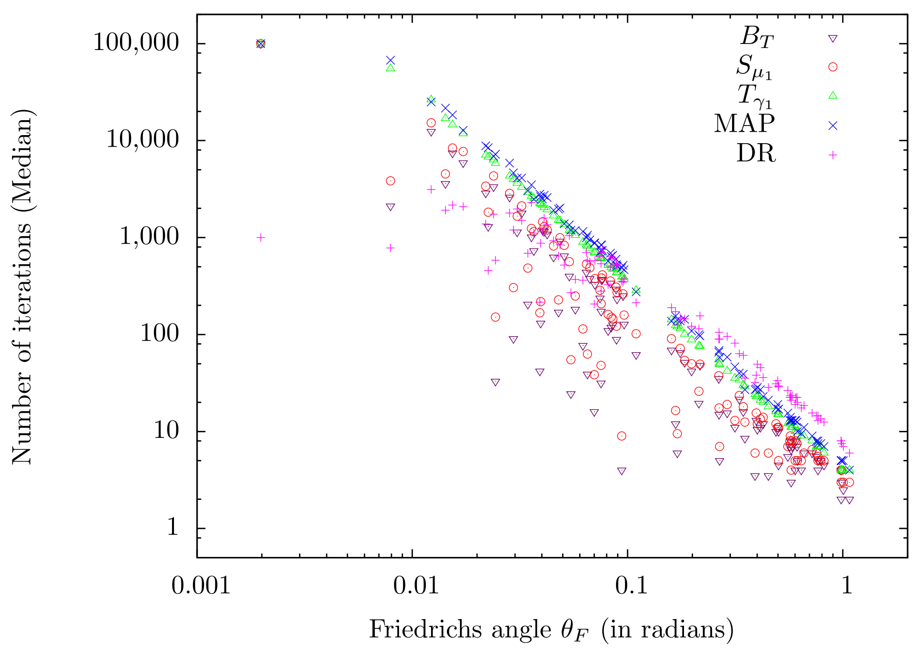


Figure 1: $B_T$ is the fastest for large $\theta_F$, while DR is the fastest for small $\theta_F$.

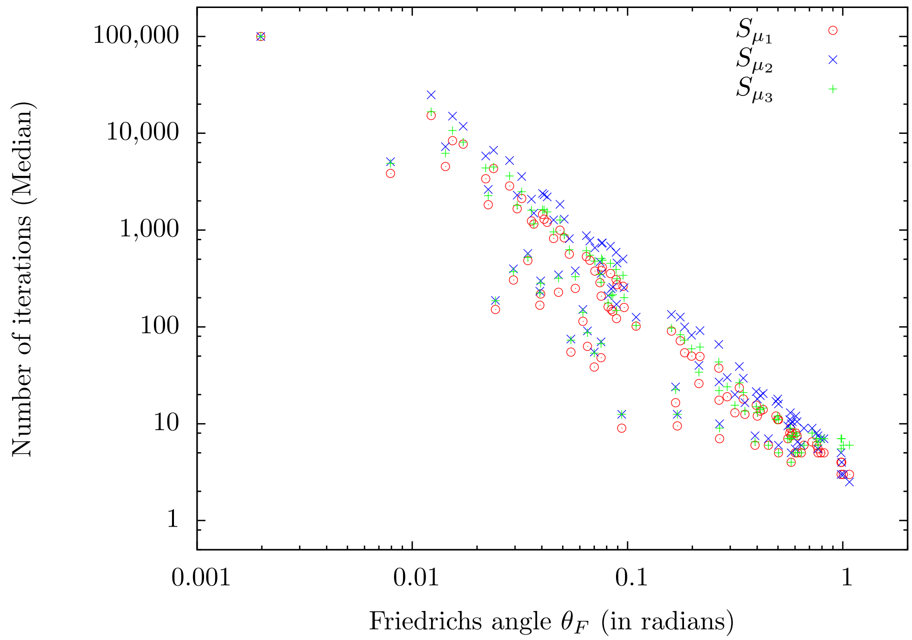


Figure 2: $S_\mu$ with $\mu_1 = \frac{2}{\sin^2\theta_F + \sin^2\theta_p}$ ("best"); $\mu_2 = \frac{1}{\sin^2\theta_p}$; and $\mu_3 = \frac{1}{2} + \frac{1}{\sin^2\theta_p}$.

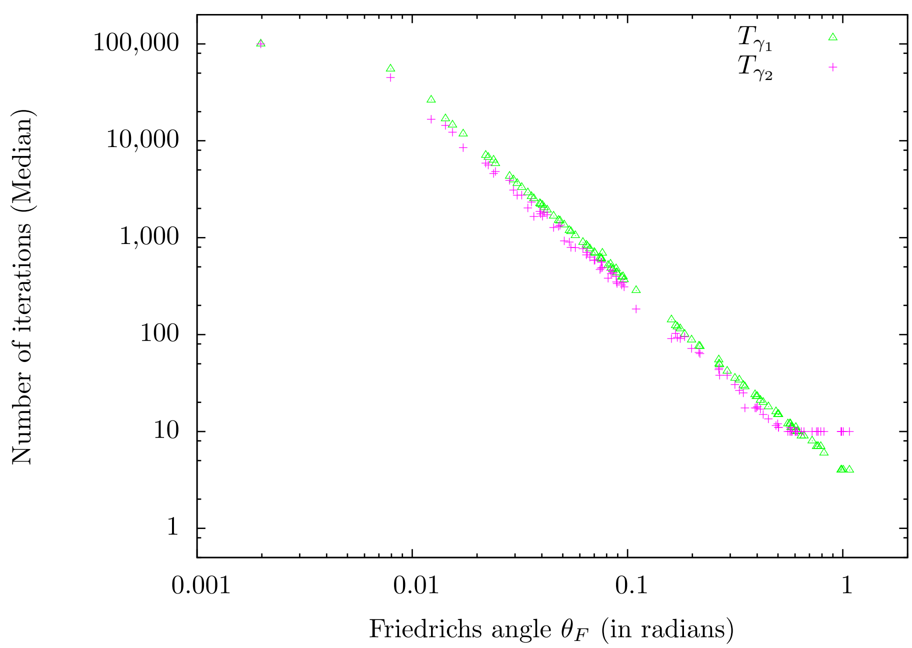


Figure 3: $T_\gamma$ with $\gamma_1 = \frac{1}{1+\sin^2\theta_F}$ ("best"); and $\gamma_2 = 1.5$.

Finally, in Table 1, for each primary category $W_i$, $i = 1, \ldots, 4$ we record the median, the mean, and the standard deviation of the number of iterations required for the algorithms to terminate. The table clearly supports these observations above. In general, the results suggest that all algorithms are more preferable than MAP from the point of view of number of iterations.

| Primary category | | $W_1$ | $W_2$ | $W_3$ | $W_4$ |
|---|---|---|---|---|---|
| Number of instances | | 250 | 250 | 250 | 250 |
| $B_T$ | Median | 1139 | 169 | 13.5 | 5 |
| | Mean | 6002.5 | 206.7 | 22.9 | 5.1 |
| | Std | 19437.1 | 163.8 | 21.6 | 2 |
| $S_{\mu_1}$ | Median | 1404 | 226.5 | 16 | 5 |
| | Mean | 6586.8 | 260.1 | 27.8 | 5.8 |
| | Std | 19396.5 | 195.7 | 26 | 2.1 |
| $S_{\mu_2}$ | Median | 2318.5 | 359.5 | 24.5 | 7 |
| | Mean | 8096.6 | 417.5 | 43.5 | 7.6 |
| | Std | 19657.3 | 326.9 | 42.5 | 3.5 |
| $S_{\mu_3}$ | Median | 1697.5 | 272 | 18.5 | 7 |
| | Mean | 6980.3 | 307.4 | 32 | 6.7 |
| | Std | 19426.9 | 221.5 | 30.3 | 1.5 |
| $T_{\gamma_1}$ | Median | 3636.5 | 611 | 42 | 9 |
| | Mean | 11571 | 684.7 | 64.5 | 8.8 |
| | Std | 21298.1 | 265.9 | 59.3 | 3.3 |
| $T_{\gamma_2}$ | Median | 2704.5 | 481.5 | 32.5 | 10 |
| | Mean | 9788.2 | 528.2 | 48.8 | 10.2 |
| | Std | 20599 | 223.1 | 44.2 | 0.6 |
| MAP | Median | 4058.5 | 722.5 | 49 | 10 |
| | Mean | 12683 | 793.1 | 74 | 10.2 |
| | Std | 22531.1 | 334.7 | 66.4 | 4.1 |
| DR | Median | 1231 | 448.5 | 83.5 | 17.5 |
| | Mean | 1395.2 | 511.3 | 92 | 17.4 |
| | Std | 847.9 | 203.6 | 56.1 | 7.1 |

Table 1: Median, mean, and standard deviation of number of iterations.

### 5.2 Comparison of computation time and memory

In Figures 4, 5, and 6, the horizontal axis represents the Friedrichs angle between two subspaces; and the vertical axis represents the (median) run-time in milliseconds[4], more specifically, the median is computed over 10 instances of one pair of subspaces.

Again, the outcome is similar to the previous section: In Figure 4, we compare $B_T$, the "best" versions $S_{\mu_1}$ and $T_{\gamma_1}$, MAP, and DR. We see that $B_T$ is generally the fastest when $\theta_F > 0.02$. While when $\theta_F \leq 0.02$, DR seems to be the fastest. The results in Figure 5 suggest that the "best" version $S_{\mu_1}$ is somewhat faster than $S_{\mu_2}$ and $S_{\mu_3}$. Finally, it is not clear that the "best" version $T_{\gamma_1}$ is more favorable than $T_{\gamma_2}$ as seen in Figure 6.

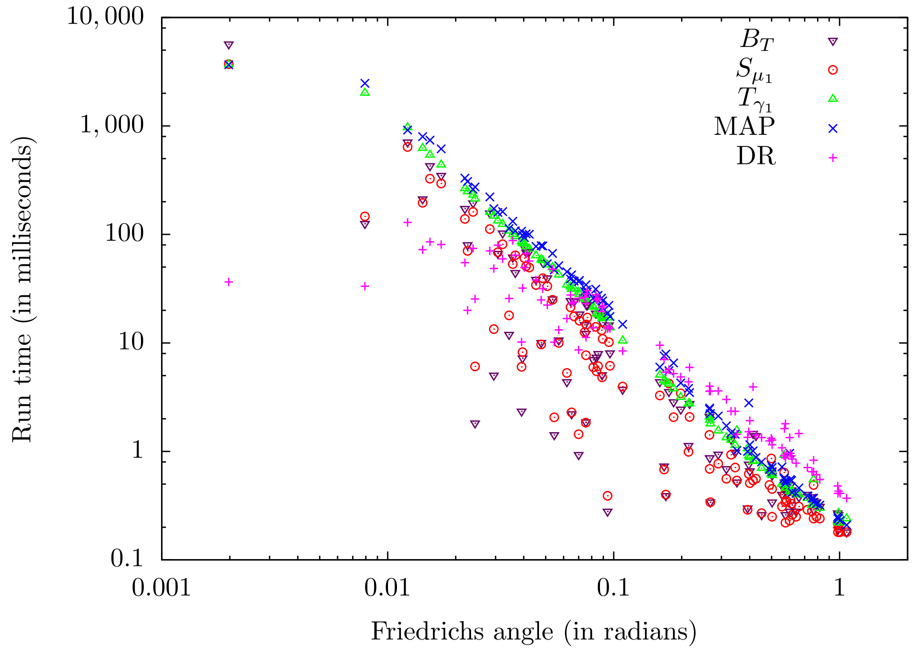


Figure 4: $B_T$ is generally the fastest for large $\theta_F$, while DR is the fastest for small $\theta_F$.

[4]one millisecond is $10^{-3}$ seconds.

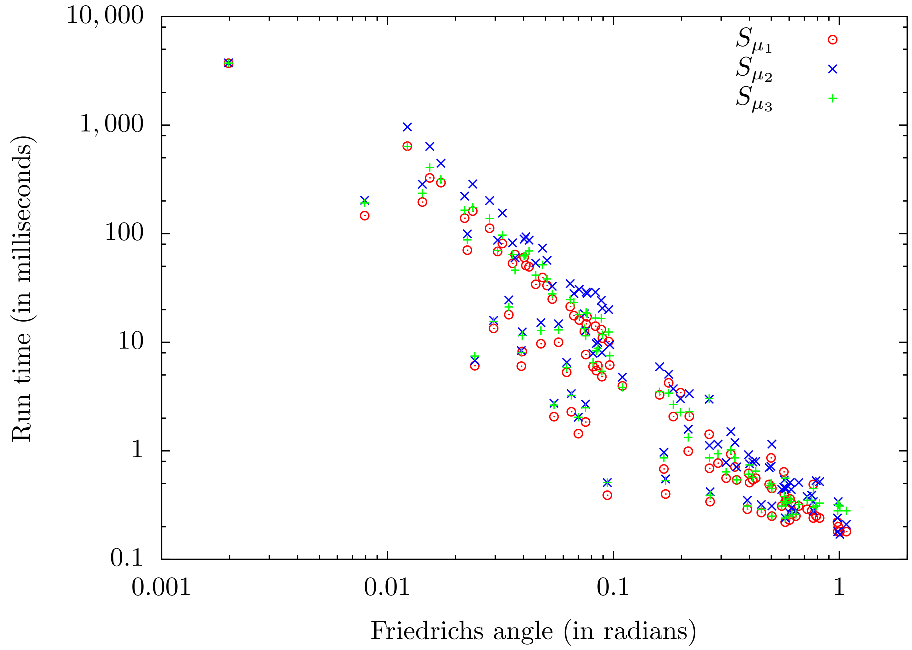


Figure 5: $S_\mu$ with $\mu_1 = \frac{2}{\sin^2\theta_F + \sin^2\theta_p}$ ("best"); $\mu_2 = \frac{1}{\sin^2\theta_p}$; and $\mu_3 = \frac{1}{2} + \frac{1}{\sin^2\theta_p}$.

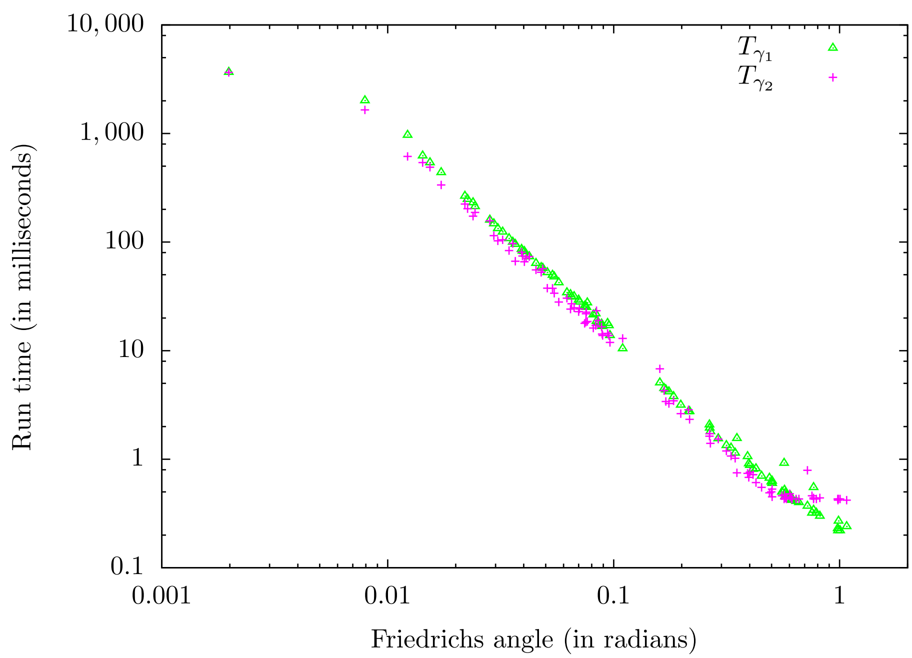


Figure 6: $T_\gamma$ with $\gamma_1 = \frac{1}{1+\sin^2\theta_F}$ ("best"); and $\gamma_2 = 1.5$.

In the next three figures, we records the (median) memory allocated[5] for these algorithms. It appears that the amount of memory allocated is proportional to the run-time.

[5]the amount of memory that an algorithm has accessed during each instance.

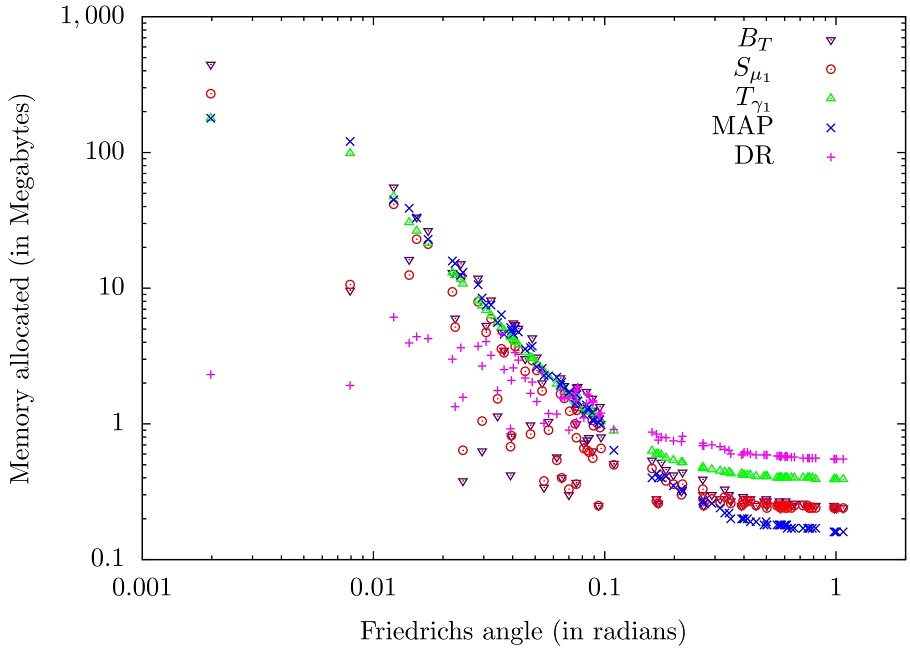


Figure 7: Memory allocated for $B_T$, $S_{\mu_1}$, $T_{\gamma_1}$, MAP, and DR.

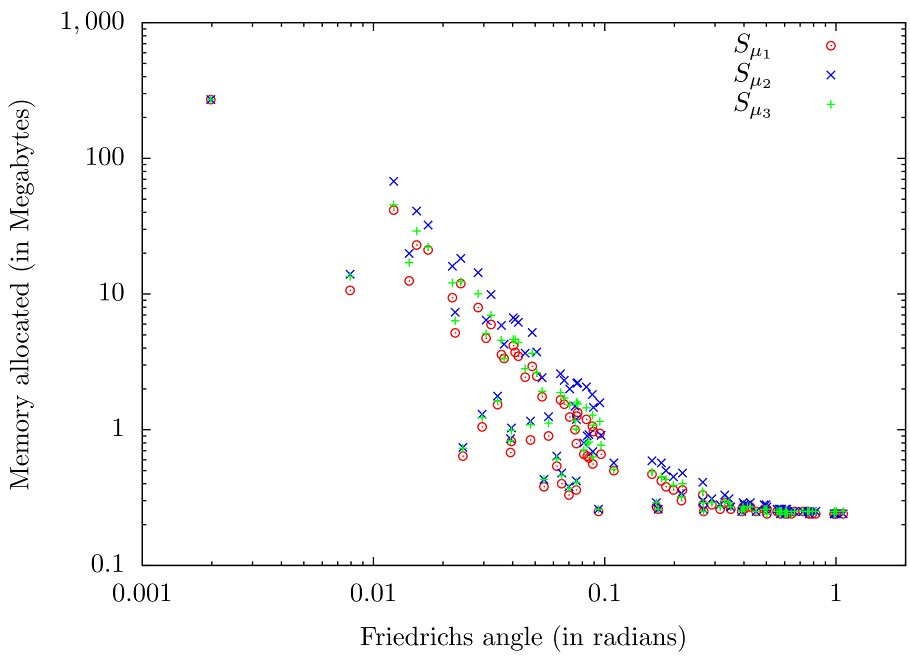


Figure 8: Memory allocated for $S_{\mu_i}$, $i = 1, 2, 3$.

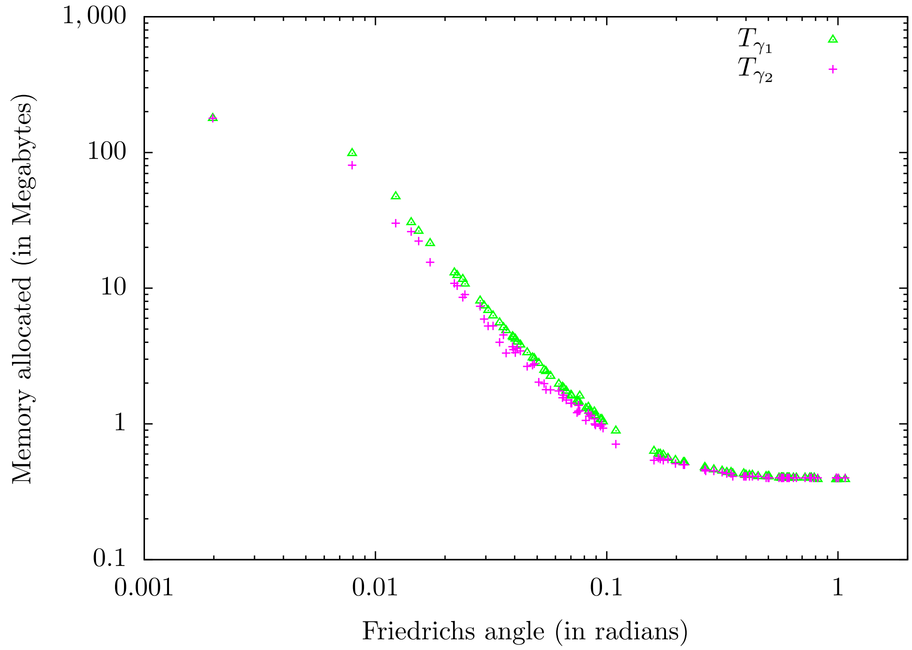


Figure 9: Memory allocated for $T_{\gamma_i}$, $i = 1, 2$.

Finally, in the following tables, for each primary category $W_i$, $i = 1, \ldots, 4$ we record the usual statistics information of the run time and the memory allocated required for the algorithms to terminate. The table clearly supports these observations above. In general, the results suggest that all algorithms are more preferable than MAP.

| Algorithms | Min | 1st quartile | Median | 3rd quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.22 | 29.52 | 69.93 | 160.32 | 5741.93 | 344.97 | 1105.47 |
| $S_{\mu_1}$ | 3.93 | 30.39 | 62.7 | 145.04 | 3801.8 | 253.46 | 722.83 |
| $S_{\mu_2}$ | 0.2 | 40.38 | 91.13 | 215.78 | 3780.42 | 312.98 | 738.89 |
| $S_{\mu_3}$ | 0.18 | 27.84 | 69.76 | 171.87 | 3751.75 | 264.13 | 723.73 |
| $T_{\gamma_1}$ | 54.41 | 84.32 | 133.8 | 272.9 | 3699.6 | 428.89 | 780.63 |
| $T_{\gamma_2}$ | 15.14 | 69.66 | 103.07 | 239.27 | 3689.59 | 367.76 | 752.67 |
| MAP | 22.02 | 97.44 | 148.68 | 358.67 | 3774.56 | 477.22 | 828.51 |
| DR | 3.8 | 31.57 | 52.07 | 75.36 | 201.65 | 57.12 | 34.43 |

Table 2a: Run-time (in milliseconds) for 250 instances in category $W_1$

| Algorithms | Min | $1^{st}$ quartile | Median | $3^{rd}$ quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.24 | 2.37 | 5.3 | 12.28 | 445.78 | 26.97 | 86.6 |
| $S_{\mu_1}$ | 0.49 | 2.14 | 4.02 | 10.5 | 271.07 | 18.06 | 52.53 |
| $S_{\mu_2}$ | 0.24 | 2.78 | 6.5 | 15.18 | 271.07 | 22.15 | 53.24 |
| $S_{\mu_3}$ | 0.24 | 2.17 | 4.82 | 12.25 | 271.07 | 19.13 | 52.62 |
| $T_{\gamma_1}$ | 2.97 | 4.33 | 6.87 | 13.11 | 178.9 | 21.03 | 38.02 |
| $T_{\gamma_2}$ | 1.12 | 3.35 | 5.2 | 11.41 | 178.9 | 17.85 | 36.77 |
| MAP | 1.26 | 4.61 | 7.39 | 16.7 | 178.67 | 22.79 | 40.22 |
| DR | 0.72 | 1.99 | 2.73 | 3.79 | 9.74 | 3.02 | 1.51 |

Table 2b: Memory allocated (in Mb) for 250 instances in category $W_1$

| Algorithms | Min | $1^{st}$ quartile | Median | $3^{rd}$ quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.17 | 4.78 | 11.8 | 19.35 | 50.46 | 13.43 | 10.68 |
| $S_{\mu_1}$ | 0.35 | 4.73 | 9.47 | 15.91 | 39.66 | 11.21 | 8.55 |
| $S_{\mu_2}$ | 0.4 | 6.64 | 14.42 | 27.19 | 74.7 | 18.18 | 14.69 |
| $S_{\mu_3}$ | 0.37 | 5.37 | 11.23 | 17.96 | 59.53 | 12.98 | 10.01 |
| $T_{\gamma_1}$ | 12.88 | 19.11 | 25.66 | 33.49 | 70.71 | 28.28 | 11.54 |
| $T_{\gamma_2}$ | 3.34 | 15.21 | 20.11 | 27.92 | 52.39 | 22.26 | 9.36 |
| MAP | 5.51 | 22.85 | 29.99 | 40.14 | 98.17 | 33.33 | 15.3 |
| DR | 3.34 | 14.12 | 21.55 | 27.55 | 57.06 | 21.78 | 9.42 |

Table 3a: Run-time (in milliseconds) for 250 instances in category $W_2$

| Algorithms | Min | $1^{st}$ quartile | Median | $3^{rd}$ quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.24 | 0.59 | 0.98 | 1.61 | 4.03 | 1.15 | 0.73 |
| $S_{\mu_1}$ | 0.25 | 0.55 | 0.84 | 1.21 | 2.8 | 0.93 | 0.53 |
| $S_{\mu_2}$ | 0.25 | 0.65 | 1.2 | 1.93 | 4.84 | 1.36 | 0.88 |
| $S_{\mu_3}$ | 0.25 | 0.6 | 0.96 | 1.43 | 3.35 | 1.06 | 0.6 |
| $T_{\gamma_1}$ | 1.03 | 1.22 | 1.47 | 1.83 | 2.89 | 1.6 | 0.47 |
| $T_{\gamma_2}$ | 0.55 | 1.03 | 1.23 | 1.53 | 2.55 | 1.32 | 0.4 |
| MAP | 0.4 | 1.13 | 1.44 | 1.88 | 3.42 | 1.56 | 0.6 |
| DR | 0.68 | 1.17 | 1.4 | 1.68 | 2.66 | 1.44 | 0.36 |

Table 3b: Memory allocated (in Mb) for 250 instances in category $W_2$

| Algorithms | Min | 1st quartile | Median | 3rd quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.11 | 0.6 | 0.93 | 1.9 | 23.87 | 1.58 | 2.25 |
| $S_{\mu_1}$ | 0.24 | 0.52 | 0.7 | 1.53 | 27.9 | 1.54 | 2.89 |
| $S_{\mu_2}$ | 0.13 | 0.7 | 1.03 | 2.47 | 20.77 | 1.89 | 2.23 |
| $S_{\mu_3}$ | 0.2 | 0.53 | 0.84 | 2.09 | 22.32 | 1.52 | 2.16 |
| $T_{\gamma_1}$ | 0.6 | 0.93 | 1.78 | 3.23 | 26.91 | 2.81 | 3.22 |
| $T_{\gamma_2}$ | 0.42 | 0.72 | 1.34 | 2.84 | 32.04 | 2.47 | 3.62 |
| MAP | 0.37 | 1.11 | 2.06 | 4.1 | 28.45 | 3.44 | 3.82 |
| DR | 0.64 | 1.8 | 3.47 | 5.56 | 26.93 | 4.52 | 4.0 |

Table 4a: Run-time (in milliseconds) for 250 instances in category $W_3$

| Algorithms | Min | 1st quartile | Median | 3rd quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.23 | 0.27 | 0.29 | 0.35 | 0.66 | 0.33 | 0.1 |
| $S_{\mu_1}$ | 0.24 | 0.26 | 0.27 | 0.33 | 0.55 | 0.3 | 0.07 |
| $S_{\mu_2}$ | 0.24 | 0.28 | 0.3 | 0.39 | 0.74 | 0.35 | 0.11 |
| $S_{\mu_3}$ | 0.24 | 0.26 | 0.28 | 0.35 | 0.64 | 0.32 | 0.08 |
| $T_{\gamma_1}$ | 0.41 | 0.42 | 0.46 | 0.54 | 0.89 | 0.5 | 0.1 |
| $T_{\gamma_2}$ | 0.4 | 0.41 | 0.44 | 0.5 | 0.87 | 0.47 | 0.08 |
| MAP | 0.17 | 0.2 | 0.24 | 0.34 | 0.89 | 0.28 | 0.12 |
| DR | 0.54 | 0.61 | 0.68 | 0.76 | 1.0 | 0.7 | 0.1 |

Table 4b: Memory allocated (in Mb) for 250 instances in category $W_3$

| Algorithms | Min | 1st quartile | Median | 3rd quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.16 | 0.28 | 0.35 | 0.44 | 9.29 | 0.4 | 0.58 |
| $S_{\mu_1}$ | 0.13 | 0.23 | 0.28 | 0.36 | 18.02 | 0.48 | 1.64 |
| $S_{\mu_2}$ | 0.12 | 0.28 | 0.38 | 0.49 | 17.66 | 0.49 | 1.11 |
| $S_{\mu_3}$ | 0.2 | 0.28 | 0.32 | 0.37 | 0.78 | 0.34 | 0.11 |
| $T_{\gamma_1}$ | 0.2 | 0.32 | 0.43 | 0.52 | 19.49 | 0.84 | 2.57 |
| $T_{\gamma_2}$ | 0.37 | 0.42 | 0.44 | 0.49 | 18.49 | 0.7 | 1.93 |
| MAP | 0.16 | 0.32 | 0.46 | 0.57 | 18.36 | 0.64 | 1.63 |
| DR | 0.29 | 0.6 | 0.84 | 1.08 | 19.22 | 1.07 | 1.72 |

Table 5a: Run-time (in milliseconds) for 250 instances in category $W_4$

| Algorithms | Min | $1^{st}$ quartile | Median | $3^{rd}$ quartile | Max | Mean | Std |
|---|---|---|---|---|---|---|---|
| $B_T$ | 0.24 | 0.25 | 0.25 | 0.26 | 0.51 | 0.25 | 0.02 |
| $S_{\mu_1}$ | 0.24 | 0.24 | 0.24 | 0.25 | 0.44 | 0.25 | 0.01 |
| $S_{\mu_2}$ | 0.24 | 0.24 | 0.25 | 0.26 | 0.29 | 0.25 | 0.01 |
| $S_{\mu_3}$ | 0.24 | 0.25 | 0.25 | 0.25 | 0.26 | 0.25 | 0.01 |
| $T_{\gamma_1}$ | 0.39 | 0.4 | 0.4 | 0.4 | 0.6 | 0.4 | 0.01 |
| $T_{\gamma_2}$ | 0.4 | 0.4 | 0.4 | 0.4 | 0.41 | 0.4 | 0.01 |
| MAP | 0.16 | 0.17 | 0.17 | 0.18 | 0.31 | 0.17 | 0.01 |
| DR | 0.54 | 0.56 | 0.57 | 0.58 | 0.79 | 0.57 | 0.02 |

Table 5b: Memory allocated (in Mb) for 250 instances in category $W_4$

# 6 Conclusion

We have presented quantifications on the optimal linear convergence rates of relaxed alternative projection methods and generalized Douglas-Rachford splitting methods for two subspaces in a finite dimensional space. For this purpose, it is necessary to study the optimal linear convergence rate of a matrix. We derive a complete characterization when the matrix has the optimal convergence rate in term of semi-simpleness of all the subdominant eigenvalues and the unit eigenvalue. Combining with the principal angles between two subspaces, we provide an analysis of optimal linear convergence rates for relaxed alternating projection, partial relaxed alternating projection and generalized Douglas-Rachford methods for two subspaces. The optimal linear convergence rate is explicitly given in terms of the relaxation parameter and principal angles between two subspaces. It turns out that the partial relaxed alternating projection method and its nonlinear version could obtain the smallest convergence rate among these ones, which are demonstrated by numerical performances. Our results not only recover but also significantly extend currently known results in the literature. In future research one may similarly investigate Jacobi, Gauss-Seidel, and especially successive over-relaxation methods. Understanding further the partial relaxed alternating projection method for two sets is also an intriguing project.

# Acknowledgments

The authors would like to thank two referees for useful comments. HHB was partially supported by a Discovery Grant and an Accelerator Supplement of the Natural Sciences and Engineering Research Council of Canada (NSERC) and by the Canada Research Chair Program. JYBC was partially supported by CNPq grants 303492/2013-9, 474160/2013-0 and 202677/2013-3 and by projects CAPES-MES-CUBA 226/2012 and UNIVERSAL FAPEG/CNPq. JYBC is grateful for the warm hospitality of the Irving K. Barber School of Arts and Sciences at the University of British Columbia for his visit 08/2013–01/2015. TTAN was partially supported by a postdoctoral fellowship of the Pacific Institute for the Mathematical Sciences and by NSERC grants of HHB and XW. HMP was partially supported by NSERC grants of HHB and XW, and by a University of Massachusetts at Lowell internal grant. XW was partially supported by a Discovery Grant of NSERC.

## References


[1] F.J. Aragón Artacho, J.M. Borwein and M.K. Tam, Recent results on Douglas-Rachford methods for combinatorial optimization problems, *J. Optim. Theory Appl.* 163, pp. 1–30, 2014.

[2] H.H. Bauschke and J.M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.* 38, pp. 367–426, 1996.

[3] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2011.

[4] H.H. Bauschke, P.L. Combettes, and S.G. Kruk, Extrapolation algorithm for affine-convex feasibility problems, *Numer. Algorithms* 4, pp. 239–274, 2006.

[5] H.H. Bauschke, J.Y. Bello Cruz, T.T.A. Nghia, H.M. Phan, and X. Wang, The rate of linear convergence of the Douglas-Rachford algorithm for subspaces is the cosine of the Friedrichs angle, *J. Approx. Theory* 185, pp. 63–79, 2014.

[6] H.H. Bauschke, F. Deutsch, H. Hundal, and S.-H. Park, Accelerating the convergence of the method of alternating projections, *Trans. Amer. Math. Soc.* 355, pp. 3433–3461, 2003.

[7] H.H. Bauschke and S.G. Kruk, Reflection-projection method for convex feasibility problems with an obtuse cone, *J. Optim. Theory Appl.* 120, pp. 503–531, 2004.

[8] H.H. Bauschke, M.N. Dao, D. Noll, and H.M. Phan, Proximal point algorithm, Douglas-Rachford algorithm and alternating projections: a case study, *Journal of Convex Analysis*, to appear.

[9] A. Björck and G.H. Golub, Numerical methods for computing angles between linear subspaces, *Math. Comp.* 27, pp. 579–594, 1973.

[10] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, 1994.

[11] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Springer, 2012.

[12] A. Cegielski and A. Suchocka, Relaxed alternating projection methods, *SIAM J. Optim.* 19, pp. 1093–1106, 2008.

[13] Y. Censor, T. Elfving, G.T. Herman and T. Nikazad, On diagonally relaxed orthogonal projection methods, *SIAM J. Sci. Comput.* 30, pp. 473–504, 2007/08.

[14] P.L. Combettes and J.C. Pesquet, Proximal splitting methods in signal processing, *Fixed-point Algorithms for Inverse Problems in Science and Engineering*, 185–212, Springer Optim. Appl., 49, Springer, New York, 2011.

[15] P.L. Combettes, Solving monotone inclusions via compositions of nonexpansive averaged operators, *Optimization* 53, pp. 475–504, 2004.

[16] L. Demanet and X. Zhang, Eventual linear convergence of the Douglas-Rachford iteration for basis pursuit, *Math. Comput.* 85, pp. 209–238, 2016.

[17] F. Deutsch, The angle between subspaces of a Hilbert space, in *Approximation Theory, Wavelets and Applications*, S.P. Singh (editor), Kluwer, pp. 107–130, 1995.

[18] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer, 2001.

[19] J. Douglas and H.H. Rachford, On the numerical solution of heat conduction problems in two and three space variables, *Trans. Amer. Math. Soc.* 82, pp. 421–439, 1956.

[20] J. Eckstein and D.P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Program.* 55, pp. 293–318, 1992.

[21] T. Elfving, T. Nikazad and P.C. Hansen, Semi-convergence and relaxation parameters for a class of SIRT algorithms, *Electron. Trans. Numer. Anal.* 37, pp. 321–336, 2010.

[22] R. Escalante and M. Raydan, *Alternating Projection Methods*, Fundamentals of Algorithms 8, SIAM, Philadelphia, PA, 2011.

[23] W.B. Gearhart and M. Koshy, Acceleration schemes for the method of alternating projections, *J. Comp. Appl. Math.* 26, pp. 235–249, 1989.

[24] GNU Plot, `http://sourceforge.net/projects/gnuplot`.

[25] L.G. Gubin, B.T. Polyak, and E.V. Raik, The method of projections for finding the common point of convex sets, *USSR Comp. Math. Math. Phys.* 7, pp. 1–24, 1967.

[26] K. Hensel, Über Potenzreihen von Matrizen, *J. Reine Angew. Math.* 155, pp. 107–110, 1926.

[27] S. Kayalar and H. Weinert, Error bounds for the method of alternating projections, *Math. Control Signals Systems* 1, pp. 43–59, 1996.

[28] S. Kirkland, A cycle-based bound for subdominant eigenvalues of stochastic matrices, *Linear Multilinear Algebra* 57, pp. 247–266, 2009.

[29] R. Hesse, D.R. Luke, and P. Neumann, Alternating projections and Douglas-Rachford for sparse affine feasibility, *IEEE Trans. Signal Process.* 62, pp. 4868–4881, 2014.

[30] R. Hesse and D.R. Luke, Nonconvex notions of regularity and convergence of fundamental algorithms for feasibility problems, *SIAM J. Optim.* 23, pp. 2397–2419, 2013.

[31] The Julia language, `http://julialang.org/`.

[32] P.-L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.* 16, pp. 964–979, 1979.

[33] D.R. Luke, Finding best approximation pairs relative to a convex and prox-regular set in a Hilbert space, *SIAM J. Optim.* 19, pp. 714–739, 2008.

[34] I. Marek, On square roots of M-operators, *Linear Algebra Appl.* 223/224, pp. 501–520, 1995.

[35] I. Marek and D.B. Szyld, Comparison theorems for the convergence factor of iterative methods for singular matrices, *Linear Algebra Appl.* 316, pp. 67–87, 2000.

[36] I. Marek and D.B. Szyld, Comparison of convergence of general stationary iterative methods for singular matrices, *SIAM J. Matrix Anal. Appl.* 24, pp. 68–77, 2002.

[37] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.

[38] C.D. Meyer and R.J. Plemmons, Convergent powers of a matrix with applications to iterative methods for singular linear systems, *SIAM J. Numer. Anal.* 14, pp. 699–705, 1977.

[39] J. Miao and A. Ben-Israel, On principal angles between subspaces in $\mathbb{R}^n$, *Linear Algebra Appl.* 171, pp. 81–98, 1992.

[40] S. Nelson and M. Neumann, Generalizations of the projection method with applications to SOR theory for Hermitian positive semidefinite linear systems, *Numer. Math.* 51, pp. 123–141, 1987.

[41] R. Oldenburger, Infinite powers of matrices and characteristic roots, *Duke Math. J.* 6, pp. 357–361, 1940.

[42] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia, 2003.

[43] G.W. Stewart, *Matrix Algorithms*, Vol. II: Eigensystems, SIAM, Philadelphia, 2001.