

## Firmly nonexpansive and Kirszbraum-Valentine extensions: a constructive approach via monotone operator theory

Heinz H. Bauschke and Xianfu Wang

*Dedicated to Alex Ioffe and Simeon Reich,  
on the occasion of their 70th and 60th birthdays*

November 28, 2008 (originally submitted on July 8, 2008)

ABSTRACT. Utilizing our recent proximal-average based results on the constructive extension of monotone operators, we provide a novel approach to the celebrated Kirszbraum-Valentine Theorem and to the extension of firmly nonexpansive mappings.

### 1. Introduction

Throughout,  $X$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and associated norm  $\| \cdot \|$ .

DEFINITION 1.1. Let  $S \subseteq X$  and let  $T: S \rightarrow X$ . Then  $T$  is *nonexpansive* if

$$(1) \quad (\forall x \in S)(\forall y \in S) \quad \|Tx - Ty\| \leq \|x - y\|.$$

See, e.g. [14, 15] for further information on nonexpansive mappings. Let us recall the celebrated Kirszbraum-Valentine Theorem (see [16, 28, 29]), which states that every nonexpansive mapping can be extended to the entire space.

FACT 1.2 (Kirszbraum-Valentine). Let  $S \subseteq X$  and let  $T: S \rightarrow X$  be nonexpansive. Then there exists a mapping  $\tilde{T}: X \rightarrow X$  such that  $\tilde{T}$  is nonexpansive and  $\tilde{T}|_S = T$ .

In this note, we present constructive approaches to the Kirszbraum-Valentine extension theorem for nonexpansive mappings. Our main tool is the proximal average, which is used to provide an explicit maximal monotone extension of any

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2000 *Mathematics Subject Classification*. Primary 46C05, 47H07, 47H09; Secondary 90C25.

*Key words and phrases*. Extension of a monotone operator, firmly nonexpansive mapping, Kirszbraum-Valentine extension theorem, nonexpansive mapping, proximal average.

Heinz Bauschke was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program.

Xianfu Wang was partially supported by the Natural Sciences and Engineering Research Council of Canada.

monotone operator. In Section 2, we give maximal monotone extensions with “small domains” under some mild constraint qualifications. Our extension results for firmly nonexpansive and for nonexpansive mappings are presented in Section 3. In some cases, it is possible to provide constructive extensions with an optimally localized range.

## 2. Firmly Nonexpansive Mappings and Monotone Operators

DEFINITION 2.1. Let  $S \subseteq X$  and let  $F: S \rightarrow X$ . Then  $F$  is *firmly nonexpansive* if

$$(2) \quad (\forall x \in S)(\forall y \in S) \quad \|Fx - Fy\|^2 \leq \langle Fx - Fy, x - y \rangle.$$

The Cauchy-Schwarz inequality implies that every firmly nonexpansive operator is nonexpansive. The next result, which is well known (see, e.g., [14, Theorem 12.1] or [15, Proposition 11.2 on page 42]), provides a bijection between nonexpansive and firmly nonexpansive mappings.

FACT 2.2. Let  $S \subseteq X$ , let  $T: S \rightarrow X$ , and let  $F: S \rightarrow X$ . Suppose that  $F = \frac{1}{2}\text{Id} + \frac{1}{2}T$ . Then  $T$  is nonexpansive  $\Leftrightarrow F$  is firmly nonexpansive.

REMARK 2.3. Note (see also [17]) the linear relationship between  $\text{gra}T$  and  $\text{gra}F$ . In fact, the linear operator  $(x, y) \mapsto (x, \frac{1}{2}x + \frac{1}{2}y)$  provides a bijection from  $\text{gra}T$  to  $\text{gra}F$  with inverse  $(x, y) \mapsto (x, 2y - x)$ .

DEFINITION 2.4. Let  $A: X \rightrightarrows X$ , i.e.,  $A$  is a set-valued operator from  $X$  to the power set of  $X$ . Denote the *graph* of  $A$  by  $\text{gra}A := \{(x, x^*) \in X \times X \mid x^* \in Ax\}$ . Then  $A$  is *monotone* if

$$(3) \quad (\forall (x, x^*) \in \text{gra}A)(\forall (y, y^*) \in \text{gra}A) \quad \langle x - y, x^* - y^* \rangle \geq 0.$$

If  $(x, x^*) \in X \times X$  and the operator with graph  $\{(x, x^*)\} \cup \text{gra}A$  is monotone, then  $(x, x^*)$  is *monotonically related* to  $\text{gra}A$ . If  $A$  is monotone and every proper extension of  $A$  fails to be monotone, then  $A$  is *maximal monotone*. The *inverse* operator  $A^{-1}$  is defined via  $\text{gra}A^{-1} := \{(x^*, x) \in X \times X \mid x^* \in Ax\}$ .

Maximal monotone operators play a critical role in modern Analysis and Optimization; see, e.g., [10, 25, 26, 27, 30].

The following result, brought out fully by Eckstein and Bertsekas [12], has its roots in the seminal works by Minty [19] and by Rockafellar [24].

FACT 2.5. Let  $A: X \rightrightarrows X$ , let  $S \subseteq X$ , and let  $F: S \rightarrow X$ . Suppose that  $F = (A + \text{Id})^{-1}$ ; equivalently, that  $A = F^{-1} - \text{Id}$ . Then the following hold.

- (i)  $A$  is monotone  $\Leftrightarrow F$  is firmly nonexpansive.
- (ii)  $A$  is maximal monotone  $\Leftrightarrow F$  is firmly nonexpansive and  $S = X$ .

REMARK 2.6. Using the notation of Fact 2.5, we recall the linear relationship (see [17]) between the graphs of  $F$  and  $A$ . Indeed, the linear operator  $(x, y) \mapsto (y, x - y)$  provides a bijection from  $\text{gra}F$  to  $\text{gra}A$  with inverse  $(x, y) \mapsto (x + y, x)$ . Recalling Remark 2.3, we observe the linear relationship from  $\text{gra}T$  to  $\text{gra}A$  via  $(x, y) \mapsto \frac{1}{2}(x + y, x - y)$  with inverse  $(x, y) \mapsto (x + y, x - y)$ .

COROLLARY 2.7. Let  $F: X \rightarrow X$  be firmly nonexpansive. Then  $\overline{\text{ran}}F$  is convex.

PROOF. Set  $A := F^{-1} - \text{Id}$  and observe that  $\text{dom}A = \text{ran}F$ . The conclusion now follows from Fact 2.5.(ii) and [26, Theorem 18.6].  $\square$

REMARK 2.8. Suppose that  $X = \mathbb{R}^2$ , let  $P_A$  be the projector onto the line  $\mathbb{R} \times \{1\}$  and let  $P_B$  be the projector onto the closed unit ball. Then  $P_A$  and  $P_B$  are both (firmly) nonexpansive (see, e.g., [14, Theorem 12.2] or [15, page 17]). Set  $T := P_B P_A$ . Then  $T$  is a nonexpansive mapping defined on the entire Euclidean plane; however,  $\overline{\text{ran}} T$  is equal to the the closed upper half circle, which is not convex.

The following result, originally obtained with the help of the *proximal average*, the *Fitzpatrick function*, and other tools from *Convex Analysis*, is of key importance. We refer the reader to [2, 3, 4, 5, 7] for further information on the proximal average, to [11, 13, 18, 20, 27] and the references therein for results on the Fitzpatrick function, and to [24, 25, 30] for the basic theory of Convex Analysis. The notation  $\tilde{A}$  for the maximal monotone extension of a monotone operator  $A: X \rightrightarrows X$  will be used from now on.

FACT 2.9. (See [7, Fact 5.6 and Theorem 5.7].) Let  $A: X \rightrightarrows X$  be monotone. Recall that the *Fitzpatrick function* of  $A$  is the function on  $X \times X$  defined by

$$(4) \quad \Phi_A: (x, x^*) \mapsto \sup_{(a, a^*) \in \text{gra } A} (\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle),$$

with *Fenchel conjugate*

$$(5) \quad \Phi_A^*: (y^*, y) \mapsto \sup_{(x, x^*) \in X \times X} (\langle x, y^* \rangle + \langle y, x^* \rangle - \Phi_A(x, x^*)).$$

Set

$$(6) \quad \Psi_A: (x, x^*) \mapsto \min_{(x, x^*) = \frac{1}{2}(y+z, y^*+z^*)} \left( \frac{1}{2} \Phi_A(y, y^*) + \frac{1}{2} \Phi_A^*(z^*, z) + \frac{1}{8} (\|y - z\|^2 + \|y^* - z^*\|^2) \right),$$

which is the *proximal average* between  $\Phi_A$  and the (transpose of the)  $\Phi_A^*$ , and define  $\tilde{A}: X \rightrightarrows X$  via

$$(7) \quad \text{gra } \tilde{A} = \{(x, x^*) \in X \times X \mid (x^*, x) \in \partial \Psi_A(x, x^*)\},$$

where “ $\partial$ ” denotes the *subdifferential operator* from Convex Analysis. Then the following hold.

- (i)  $(\forall (x, x^*) \in X \times X) \Psi_A(x, x^*) \geq \langle x, x^* \rangle$ .
- (ii)  $(\forall (x, x^*) \in X \times X) x^* \in \tilde{A}x \Leftrightarrow \Psi_A(x, x^*) = \langle x, x^* \rangle$ .
- (iii)  $\tilde{A}$  is a maximal monotone extension of  $A$ .

It is convenient (see, e.g., Example 3.3 below) to be able to use the following alternative description of  $\tilde{A}$ .

THEOREM 2.10. *Let  $A: X \rightrightarrows X^*$  be monotone. Define  $B: X \rightrightarrows X^*$  via*

$$(8) \quad x^* \in Bx \Leftrightarrow (x, x^*) = \frac{1}{2}(x_1 + x_2, x_1^* + x_2^*),$$

where

$$(9) \quad (x_2^*, x_2) \in \partial \Phi_A(x_1, x_1^*) \quad \text{and} \quad x_1 - x_2 = x_1^* - x_2^*.$$

Then  $B = \tilde{A}$ .

PROOF. Take  $(x, x^*) \in X \times X$ .

“ $\text{gra } \tilde{A} \subseteq \text{gra } B$ ”: Suppose that  $(x, x^*) \in \text{gra } \tilde{A}$ . Then  $\Psi_A(x, x^*) = \langle x^*, x \rangle$  by Fact 2.9.(ii) and, by (7), there exist  $(x_1, x_1^*)$  and  $(x_2, x_2^*)$  in  $X \times X$  such that

$$(10) \quad (x, x^*) = \frac{1}{2}(x_1 + x_2, x_1^* + x_2^*)$$

and

$$(11) \quad \frac{1}{4}\langle x_1 + x_2, x_1^* + x_2^* \rangle = \frac{1}{2}\Phi_A(x_1, x_1^*) + \frac{1}{2}\Phi_A^*(x_2^*, x_2) + \frac{1}{8}(\|x_1 - x_2\|^2 + \|x_1^* - x_2^*\|^2).$$

The Fenchel-Young inequality yields  $\Phi_A(x_1, x_1^*) + \Phi_A^*(x_2^*, x_2) \geq \langle (x_1, x_1^*), (x_2^*, x_2) \rangle = \langle x_1, x_2^* \rangle + \langle x_1^*, x_2 \rangle$ . Hence, (11) results in

$$(12) \quad \frac{1}{4}\langle x_1 + x_2, x_1^* + x_2^* \rangle \geq \frac{1}{2}\langle x_1, x_2^* \rangle + \frac{1}{2}\langle x_1^*, x_2 \rangle + \frac{1}{8}(\|x_1 - x_2\|^2 + \|x_1^* - x_2^*\|^2);$$

equivalently,

$$(13) \quad \frac{1}{2}\|x_1 - x_2\|^2 + \frac{1}{2}\|x_1^* - x_2^*\|^2 \leq \langle x_1 - x_2, x_1^* - x_2^* \rangle.$$

Thus,  $x_1^* - x_2^* \in \partial(\frac{1}{2}\|\cdot\|^2)(x_1 - x_2)$  and hence

$$(14) \quad x_1^* - x_2^* = x_1 - x_2.$$

In view of (11) and (14), it follows that

$$(15) \quad \frac{1}{2}\langle x_1 + x_2, x_1^* + x_2^* \rangle = \Phi_A(x_1, x_1^*) + \Phi_A^*(x_2^*, x_2) + \frac{1}{2}\langle x_1 - x_2, x_1^* - x_2^* \rangle,$$

i.e.,  $\Phi_A(x_1, x_1^*) + \Phi_A^*(x_2^*, x_2) = \langle x_1, x_2^* \rangle + \langle x_2, x_1^* \rangle = \langle (x_1, x_1^*), (x_2^*, x_2) \rangle$  and hence

$$(16) \quad (x_2^*, x_2) \in \partial\Phi_A(x_1, x_1^*).$$

Combining (10), (16), and (14), we see that  $(x, x^*) \in \text{gra } B$ .

“ $\text{gra } B \subseteq \text{gra } \tilde{A}$ ”: Suppose that  $(x, x^*) \in \text{gra } B$ . Then there exist  $(x_1, x_1^*)$  and  $(x_2, x_2^*)$  in  $X \times X$  such that (10) and (9) hold. Note that (9) yields

$$(17) \quad \Phi_A(x_1, x_1^*) + \Phi_A^*(x_2^*, x_2) = \langle x_1, x_2^* \rangle + \langle x_1^*, x_2 \rangle$$

and

$$(18) \quad \frac{1}{2}\|x_1 - x_2\|^2 + \frac{1}{2}\|x_1^* - x_2^*\|^2 = \langle x_1 - x_2, x_1^* - x_2^* \rangle.$$

Using (10), (17), and (18), we obtain

$$(19) \quad \langle x, x^* \rangle = \frac{1}{4}\langle x_1 + x_2, x_1^* + x_2^* \rangle$$

$$(20) \quad = \frac{1}{2}\Phi_A(x_1, x_1^*) + \frac{1}{2}\Phi_A^*(x_2^*, x_2) + \frac{1}{4}\langle x_1^* - x_2^*, x_1 - x_2 \rangle$$

$$(21) \quad = \frac{1}{2}\Phi_A(x_1, x_1^*) + \frac{1}{2}\Phi_A^*(x_2^*, x_2) + \frac{1}{8}(\|x_1 - x_2\|^2 + \|x_1^* - x_2^*\|^2).$$

Using this, (6), and Fact 2.9.(i), we estimate

$$(22) \quad \begin{aligned} \langle x, x^* \rangle &= \frac{1}{2}\Phi_A(x_1, x_1^*) + \frac{1}{2}\Phi_A^*(x_2^*, x_2) + \frac{1}{8}(\|x_1 - x_2\|^2 + \|x_1^* - x_2^*\|^2) \\ &\geq \Psi_A(x, x^*) \\ &\geq \langle x, x^* \rangle, \end{aligned}$$

and we see that  $\Psi_A(x, x^*) = \langle x, x^* \rangle$ . By Fact 2.9(ii),  $(x, x^*) \in \text{gra } \tilde{A}$ .  $\square$

EXAMPLE 2.11. Let  $(a, a^*) \in X \times X$  and let  $A: X \rightrightarrows X$  be given by  $\text{gra } A = \{(a, a^*)\}$ . Then  $\text{gra } \tilde{A} = \{(x, x + a^* - a)\}_{x \in X}$ .

PROOF. In view of (4), we see that

$$(23) \quad \Phi_A: X \rightarrow \mathbb{R}: (x, x^*) \mapsto \langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle = \langle (x, x^*), (a^*, a) \rangle - \langle a, a^* \rangle.$$

Since  $\partial\Phi_A \equiv (a^*, a)$ , Theorem 2.10 implies

$$(24) \quad \text{gra } \widetilde{A} = \left\{ \frac{1}{2}(x_1 + a, x_1 - a + 2a^*) \mid x_1 \in X \right\}.$$

The change of variable  $x = \frac{1}{2}(x_1 + a)$  now yields the result.  $\square$

Note that in Example 2.11 the domain of the extension  $\widetilde{A}$  is the entire space  $X$  while the domain of the given operator is a singleton. This raises the question on finding extensions with the smallest possible domain. Let  $A: X \rightrightarrows X$  be monotone and set  $D := \overline{\text{conv}} \text{ dom } A$ . Denote the *normal cone* operator to  $D$  by  $N_D$ ; i.e.,  $N_D = \partial\iota_D$  so that  $\text{dom } N_D = D$  and  $(\forall x \in D) N_D(x) = \{x^* \in X \mid \sup\langle D - x, x^* \rangle \leq 0\}$ . Since  $(\forall x \in D) 0 \in N_D(x)$ , the operator

$$(25) \quad A + N_D$$

is a monotone extension of  $A$ . However,  $A + N_D$  may fail to be maximal monotone: consider, e.g., the case when  $A$  is the zero operator restricted to the *open* unit ball. The following result will aid us in our quest to provide a sufficient condition for (25) to be maximal monotone.

FACT 2.12. (See [7, Theorem 2.14 and Theorem 2.15].) Let  $A: X \rightrightarrows X$  be monotone, and set  $D := \overline{\text{conv}} \text{ dom } A$ . Then the following hold.

$$(26) \quad \begin{aligned} & \text{(i) } (x, x^*) \in X \times X \text{ is monotonically related to } \text{gra}(A + N_D) \Leftrightarrow \\ & (x, x^*) \text{ is monotonically related to } \text{gra } A \quad \text{and} \quad x \in \bigcap_{a \in \text{dom } A} (a + T_D(a)), \end{aligned}$$

where  $T_D(a) = N_D^\ominus(a)$  is the *polar* (negative dual) cone of  $N_D(a)$ .

$$(ii) \quad \text{If } \text{conv } \text{dom } A \text{ is closed, then } \bigcap_{a \in \text{dom } A} (a + T_D(a)) = D.$$

THEOREM 2.13. Let  $A: X \rightrightarrows X$  be monotone, set  $D := \overline{\text{conv}} \text{ dom } A$  and  $A_D := A + N_D$ . Then the following hold.

$$(27) \quad \begin{aligned} & \text{(i) } \widetilde{A}_D \text{ is a maximal monotone extension of } A, \text{ and} \\ & \overline{\text{dom}} \widetilde{A}_D = \overline{\text{conv}} \text{ dom } \widetilde{A}_D \subseteq \bigcap_{a \in \text{dom } A} (a + T_D(a)). \end{aligned}$$

$$(ii) \quad \text{If } \text{conv } \text{dom } A \text{ is closed, then } \overline{\text{dom}} \widetilde{A}_D = D.$$

PROOF. Note that  $\text{dom } A_D = \text{dom } A \cap \text{dom } N_D = \text{dom } A \cap D = \text{dom } A$ .

(i): By Fact 2.9.(iii),  $\widetilde{A}_D$  is a maximal monotone extension of  $A_D$ . On the other hand,  $A_D$  is a monotone extension of  $A$ . Altogether,  $\widetilde{A}_D$  is a maximal monotone extension of  $A$ . Fact 2.12.(i) yields  $\text{dom } \widetilde{A}_D \subseteq \bigcap_{a \in \text{dom } A} (a + T_D(a))$ . Since the last intersection is closed and convex, we obtain the inclusion in (27). Finally, the maximal monotonicity of  $\widetilde{A}_D$  and [26, Theorem 18.6] imply that  $\overline{\text{conv}} \text{ dom } \widetilde{A}_D = \overline{\text{dom}} \widetilde{A}_D$ .

(ii): Combining (i) and Fact 2.12.(ii), we deduce that

$$(28) \quad D = \overline{\text{conv}} \text{ dom } A \subseteq \overline{\text{conv}} \text{ dom } \widetilde{A}_D = \overline{\text{dom}} \widetilde{A}_D \subseteq \bigcap_{a \in \text{dom } A} (a + T_D(a)) = D.$$

The proof is complete.  $\square$

REMARK 2.14. In Theorem 2.13.(ii), the set  $\text{conv dom } A$  is closed whenever one of the following holds.

- (i)  $\text{dom } A$  is a finite subset of  $X$ .
- (ii)  $X$  is finite-dimensional, and  $\text{dom } A$  is closed and bounded.

We conclude this section with a second approach that is tailored to finite-dimensional spaces.

THEOREM 2.15. *Suppose that  $X$  is finite-dimensional, and let  $A: X \rightrightarrows X$  be monotone. Suppose that (after translation if necessary)  $0 \in \text{ri } \overline{\text{conv}} \text{ dom } A = \text{ri conv dom } A$ , i.e.,  $0 \in \text{int}_Y \overline{\text{conv}} \text{ dom } A$ , where  $Y := \text{aff dom } A = \text{span dom } A$  is a closed subspace of  $X$ . Let  $P: X \rightarrow X$  be the linear orthogonal projector onto  $Y$ , and let  $Q: X \rightarrow Y: x \mapsto Px$ . Then  $P^* = P$  and  $Q^*: Y \rightarrow X: y \mapsto y$ . Finally, set  $D := \overline{\text{conv}} \text{ dom } A$ . Then the following hold.*

- (i) *The composition  $PA: Y \rightrightarrows Y$  is monotone.*
- (ii) *If  $B: Y \rightrightarrows Y$  is a maximal monotone extension of  $PA$  (e.g.,  $B = \widetilde{PA}$ ), then*

$$(29) \quad \widehat{B} := Q^*BQ + N_D: X \rightrightarrows X$$

*is a maximal monotone extension of  $A$  and  $\overline{\text{dom}} \widehat{B} = D$ .*

PROOF. (i): Take  $(y_1, y_1^*)$  and  $(y_2, y_2^*)$  in  $\text{gra } A$ . Since  $P^* = P$ ,  $\text{dom } A \subseteq Y$ , and  $A$  is monotone, it follows that  $\langle y_1 - y_2, Py_1^* - Py_2^* \rangle = \langle Py_1 - Py_2, y_1^* - y_2^* \rangle = \langle y_1 - y_2, y_1^* - y_2^* \rangle \geq 0$ .

(ii): Let  $B: Y \rightrightarrows Y$  be a maximal monotone extension of  $PA$ . Using [26, Theorem 18.8] and since  $\text{span dom } B = \text{span dom } A = Y$ , we see that

$$(30) \quad \text{int}_Y \text{dom } B = \text{int}_Y \text{conv dom } B = \text{int}_Y \overline{\text{conv}} \text{ dom } B$$

and that  $\text{int}_Y \text{conv dom } A = \text{int}_Y \overline{\text{conv}} \text{ dom } A$ . Thus

$$(31) \quad \text{int}_Y \text{dom } B = \text{int}_Y \overline{\text{conv}} \text{ dom } B \supseteq \text{int}_Y \overline{\text{conv}} \text{ dom } A = \text{int}_Y \text{conv dom } A \ni 0,$$

and hence  $0 \in \text{int}_Y \text{dom } B = \text{ri dom } B$ . Since  $0 \in Y \cap \text{ri dom } B = \text{ran } Q \cap \text{ri dom } B$ , [25, Theorem 12.43] implies that  $Q^*BQ: X \rightrightarrows X$  is maximal monotone. Because  $\text{dom } Q^*BQ = Q^{-1} \text{dom } B$ ,  $Q0 = 0 \in \text{int}_Y \text{dom } B$ , and  $Q: X \rightarrow Y$  is continuous, we see that  $0 \in \text{int dom } Q^*BQ$ . On the other hand,  $0 \in \text{ri } D = \text{ri dom } N_D$ . Altogether,

$$(32) \quad 0 \in \text{int dom } Q^*BQ \cap \text{ri dom } N_D.$$

Thus, by [25, Corollary 12.44],

$$(33) \quad \widehat{B} = Q^*BQ + N_D \quad \text{is maximal monotone.}$$

We shall now show that  $\widehat{B}$  is an extension of  $A$ . To this end, take  $(a, a^*) \in \text{gra } A$ . Since  $a \in \text{dom } A \subseteq Y$ , we have  $Qa = a$ . Recalling that  $B$  extends  $PA$ , we deduce that  $Pa^* \in PAa \subseteq Ba = BQa = Q^*BQa$  and so

$$(34) \quad Pa^* \in Q^*BQa.$$

On the other hand,  $a \in \text{dom } A \subseteq D \subseteq Y$ , which implies  $Y^\perp \subseteq N_D(a)$  and further

$$(35) \quad a^* - Pa^* \in N_D(a).$$

Adding (34) and (35) yields

$$(36) \quad a^* \in (Q^*BQ + N_D)(a) = \widehat{B}a.$$

Hence  $(a, a^*) \in \text{gra } \widehat{B}$  and we conclude that

$$(37) \quad \widehat{B} \text{ is a maximal monotone extension of } A.$$

By [26, Theorem 18.6],  $\overline{\text{conv}} \text{ dom } \widehat{B} = \overline{\text{dom}} \widehat{B} \subseteq \overline{\text{dom}} N_D = D$ . Furthermore, (37) implies that  $\text{dom } A \subseteq \text{dom } \widehat{B}$  and thus  $D = \overline{\text{conv}} \text{ dom } A \subseteq \overline{\text{conv}} \text{ dom } \widehat{B}$ . Therefore,  $D = \overline{\text{dom}} \widehat{B}$ .  $\square$

Let us now illustrate and compare Theorem 2.13 and Theorem 2.15.

EXAMPLE 2.16. Suppose that  $X = \mathbb{R}$ , let  $A = \text{Id}|_{]-1,1[}$  be the identity operator restricted to the open interval  $]-1,1[$ , and set  $D = \overline{\text{conv}} \text{ dom } A = [-1,1]$ . Since  $\text{dom } A$  is open, we have  $A_D := A + N_D = A$ . Hence

$$(38) \quad \widetilde{A}_D = \widetilde{A} = \text{Id}$$

by [7, Example 5.10]. Let  $B$  be an arbitrary maximal monotone extension of  $A$ . Since  $\text{gra } B$  is closed, it follows that  $B$  extends  $\text{Id}|_D$ . Hence  $B + N_D$  extends  $\text{Id} + N_D$ , but the latter is already maximal monotone. Thus, the operator  $\widehat{B}$  provided by Theorem 2.15.(ii) is

$$(39) \quad \widehat{B} = \text{Id} + N_D.$$

Therefore, Theorem 2.13 and Theorem 2.15 may in general produce different maximal monotone extensions.

### 3. Main Results

We now turn to the extension of (firmly) nonexpansive operators. For recent results in this direction, all based on Zorn's Lemma, see [23], [1], and the references therein.

THEOREM 3.1 (constructive extension). *Let  $S \subseteq X$  and let  $T: S \rightarrow X$  be nonexpansive. Proceed as follows.*

- Step 1. Set  $F := \frac{1}{2} \text{Id} + \frac{1}{2} T$ .
- Step 2. Set  $A := F^{-1} - \text{Id}$ .
- Step 3. Compute  $\widetilde{A}$  as in Fact 2.9.
- Step 4. Set  $\widetilde{F} := (\text{Id} + \widetilde{A})^{-1}$ .
- Step 5. Set  $\widetilde{T} := 2\widetilde{F} - \text{Id}$ .

Then  $\widetilde{T}: X \rightarrow X$  is a nonexpansive extension of  $T$ .

PROOF. Since  $T$  is nonexpansive,  $F$  of Step 1 is firmly nonexpansive (Fact 2.2). By Fact 2.5.(i),  $A$  is monotone. Now Fact 2.9 implies that  $\widetilde{A}$  is a maximal monotone extension of  $A$ . Hence, by Fact 2.5,  $\widetilde{F}: X \rightarrow X$  is firmly nonexpansive. Finally, Fact 2.2 yields that  $\widetilde{T}: X \rightarrow X$  is a nonexpansive extension of  $T$ .  $\square$

EXAMPLE 3.2. Let  $S \subseteq X$  and set  $T = 0|_S$ . Then Theorem 3.1 returns  $\widetilde{T} = 0$ .

PROOF. We use the notation of Theorem 3.1. Then  $F = \frac{1}{2} \text{Id}|_S$  and  $A = \text{Id}|_S$ . By [7, Example 5.10],  $\widetilde{A} = \text{Id}$ . Therefore,  $\widetilde{F} = \frac{1}{2} \text{Id}$  and  $\widetilde{T} = 0$ .  $\square$

EXAMPLE 3.3. Let  $d$  and  $d^*$  be in  $X$ , and set  $S = \{d\}$ . Consider the nonexpansive operator  $T: S \rightarrow X$  given by  $Td = d^*$ . Then Theorem 3.1 returns  $\widetilde{T} \equiv d^*$ .

PROOF. Adopting the notation of Theorem 3.1, we see that  $\text{gra } \widetilde{F} = \{(d, e)\}$ , where  $e = \frac{1}{2}(d + d^*)$ . Hence  $\text{gra } A = \{(e, d - e)\}$  and Example 2.11 yields  $A: x \mapsto x + (d - e) - e = x - d^*$ . It follows that  $\widetilde{F}: x^* \mapsto \frac{1}{2}(x^* + d^*)$ . Therefore,  $\widetilde{T} \equiv d^*$ .  $\square$

We now translate the results on maximal monotone extensions of the previous section to the setting of (firmly) nonexpansive mappings. The next two results are *constructive* counterparts to [1, Corollary 5] provided some assumption on the range is satisfied.

THEOREM 3.4. *Let  $S \subseteq X$ , and let  $F: S \rightarrow X$  be firmly nonexpansive. Set  $D := \overline{\text{conv}} \text{ran } F$ ,  $A := F^{-1} - \text{Id}$ , and  $A_D := A + N_D$ . Compute  $\widetilde{A}_D$ , and set  $\widetilde{F} := (\text{Id} + \widetilde{A}_D)^{-1}$ . Then the following hold.*

- (i)  $\widetilde{F}: X \rightarrow X$  is a firmly nonexpansive extension of  $F$  such that  $\overline{\text{ran}} \widetilde{F} = \overline{\text{conv}} \text{ran } \widetilde{F} \subseteq \bigcap_{r \in \text{ran } F} (r + T_D(r))$ .
- (ii) If  $\text{conv } \text{ran } F$  is closed, then  $\overline{\text{ran}} \widetilde{F} = D$ .

PROOF. Since  $\text{ran } F = \text{dom } A$ , the result is a direct consequence of Theorem 2.13.  $\square$

THEOREM 3.5. *Suppose that  $X$  is finite-dimensional, let  $S \subseteq X$ , and let  $F: S \rightarrow X$  be firmly nonexpansive. Set  $D := \overline{\text{conv}} \text{ran } F$  and  $Y := \text{span } D$ , and assume that  $0 \in \text{ri } D$ . Denote the orthogonal projector from  $X$  onto  $Y$  by  $P$ . Let  $B: Y \rightrightarrows Y$  be an arbitrary maximal monotone extension of  $P(F^{-1} - \text{Id}): Y \rightrightarrows Y$ , set  $\widehat{B} := BP + N_D: X \rightrightarrows X$ , and  $\widehat{F} := (\text{Id} + \widehat{B})^{-1}$ . Then  $\widehat{F}$  is a firmly nonexpansive extension of  $F$  and  $\overline{\text{ran}} \widehat{F} = D$ .*

PROOF. This is a direct consequence of Theorem 2.15.  $\square$

We conclude this paper with a constructive version of the original Kirszbraun-Valentine result.

THEOREM 3.6 (constructive Kirszbraun-Valentine extension). *Let  $S \subseteq X$  and let  $T: S \rightarrow X$  be nonexpansive. Compute  $\widetilde{T}$  as in Theorem 3.1. Set  $D := \overline{\text{conv}} \text{ran } T$ , and denote the orthogonal projector onto  $D$  by  $P$ . Then  $P\widetilde{T}: X \rightarrow X$  is a nonexpansive extension of  $T$  such that  $\overline{\text{ran}}(P\widetilde{T}) \subseteq D$ .*

PROOF. From Theorem 3.1, we know that  $\widetilde{T}: X \rightarrow X$  is a nonexpansive extension of  $T$ . Since  $P$  is nonexpansive, the result follows.  $\square$

REMARK 3.7. We point out that if the given (firmly) nonexpansive mapping is described by its graph, then Remark 2.3 and Remark 2.6 may be used to go back and forth between the (firmly) nonexpansive mapping and its full-domain extension, and the corresponding monotone operators, respectively. Firmly nonexpansive mappings and their extensions in Banach spaces are studied in [9] and [21]; see also the recent manuscript [8] and the references therein. The domains of accretive and monotone operators in Banach spaces are studied in [22] (cf. Corollary 2.7).

REMARK 3.8. Let  $S \subseteq X$  and let  $F: S \rightarrow X$  be firmly nonexpansive. Then Corollary 2.7 and [1, Corollary 5] — which was proved *nonconstructively* — guarantees the existence of a firmly nonexpansive extension  $G: X \rightarrow X$  of  $F$  such that  $\overline{\text{ran}} G = \overline{\text{conv}} \text{ran } F$ . We do not know whether it is possible to use Theorem 3.6 — or any other constructive result — to obtain such a mapping; see also [1, Remark 8].



### Acknowledgment

The authors thank an anonymous referee for his insightful comments.

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MATHEMATICS, IRVING K. BARBER SCHOOL, UNIVERSITY OF BRITISH COLUMBIA OKANAGAN,  
KELOWNA, BRITISH COLUMBIA V1V 1V7, CANADA.

*E-mail address:* `heinz.bauschke@ubc.ca`

MATHEMATICS, IRVING K. BARBER SCHOOL, UNIVERSITY OF BRITISH COLUMBIA OKANAGAN,  
KELOWNA, BRITISH COLUMBIA V1V 1V7, CANADA.

*E-mail address:* `shawn.wang@ubc.ca`