General Resolvents for Monotone Operators: Characterization and Extension

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Abstract

Monotone operators, especially in the form of subdifferential operators, are of basic importance in optimization. It is well known since Minty, Rockafellar, and Bertsekas-Eckstein that in Hilbert space, monotone operators can be understood and analyzed from the alternative viewpoint of firmly nonexpansive mappings, which were found to be precisely the resolvents of monotone operators. For example, the proximal mappings in the sense of Moreau are precisely the resolvents of subdifferential operators. More general notions of "resolvent", "proximal mapping" and "firmly nonexpansive" have been studied. One important class, popularized chiefly by Alber, by Kamimura and Takahashi, and by Kohsaka and Takahashi, is based on the normalized duality mapping. Furthermore, Censor and Lent pioneered the use of the gradient of a well behaved convex functions in a Bregman-distance based framework. It is known that resolvents are firmly nonexpansive, but the converse has been an open problem for the latter framework.

In this note, we build on the very recent characterization of maximal monotonicity due to Martínez-Legaz to provide a framework for studying resolvents in which firmly nonexpansive mappings are always resolvents. This framework includes classical resolvents, resolvents based on the normalized duality mapping, resolvents based on Bregman distances, and even resolvents based on (nonsymmetric) rotators. As a by-product of recent work on the proximal average, we obtain a constructive Kirszbraun-Valentine extension result for generalized firmly nonexpansive mappings. Several examples illustrate our results.

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1 Introduction

Throughout this paper, we assume that X is a real reflexive Banach space, with continuous dual space X^*, with pairing $\langle \cdot, \cdot \rangle$, with norm $\| \cdot \|$, and with duality mapping $J = \partial \frac{1}{2} \| \cdot \|^2$, where "$\partial$" stands for the subdifferential operator from Convex Analysis. Notation not explicitly defined here is standard and as in, e.g., [33, 34, 38].

Recall that A is a set-valued operator from X to X^*, written $A\colon X \rightrightarrows X^*$, if A is a mapping from X to the power set of X^*, i.e., $(\forall x \in X)\ Ax \subseteq X^*$. The *graph* of A is $\operatorname{gra} A = \big\{(x, x^*) \in X \times X^* \mid x^* \in Ax\big\}$. Such a mapping is monotone if $(\forall (x, x^*) \in \operatorname{gra} A)\ (\forall (y, y^*) \in \operatorname{gra} A)\ \langle x - y, x^* - y^* \rangle \geq 0$, and maximal monotone if it cannot be properly extended without destroying monotonicity. The *domain* of A is $\operatorname{dom} A = \big\{x \in X \mid Ax \neq \varnothing\big\}$ and the *range* of A is $\operatorname{ran} A = A(X) = \bigcup_{x \in X} Ax$. The *inverse* of A is the operator $A^{-1}\colon X^* \rightrightarrows X$, defined via $\operatorname{gra} A^{-1} = \big\{(x^*, x) \in X^* \times X \mid (x, x^*) \in \operatorname{gra} A\big\}$.

Monotone operators have turned out to be ubiquitous in modern optimization and analysis; see, e.g., [10, 13, 34, 35]. Due to their set-valuedness, there has always been considerable interest to describe and study monotone operators from a more classical point of view. For ease of discussion, let us momentarily assume that X is a Hilbert space. A key tool is the so-called *resolvent* $(A+\mathrm{Id})^{-1}$ associated with a given monotone operator A. This resolvent is not only always single-valued, but also *firmly nonexpansive* (and thus Lipschitz continuous); moreover, the resolvent has full domain X precisely when A is maximal monotone. Resolvents can be used to parametrize the graph of A, and the inverse-resolvent identity provides a useful and elegant expression for the resolvent of A^{-1} in terms of the resolvent for A. More general resolvents have been studied. Alber [1], Kamimura and Takahashi [22] (see also [5, 37]), and Kohsaka and Takahashi [24, 25, 26] initiated the systematic study of resolvents based on the duality mapping J. Building on work by Bregman [9] on generalized distances, Censor and Zenios analyzed proximal mappings [15] (see also [16]). For either generalization, it is known that every resolvent is firmly nonexpansive.

The aim of this note is to present a very general framework for resolvents and firmly nonexpansive mappings in which the two classes coincide. We also study parametrizations of the graph, inverse resolvents, and extensions of firmly nonexpansive mappings. Various examples illustrate our results.

The paper is organized as follows. In Section 2, we review the crucial characterization due to Martínez-Legaz (Fact 2.1) and then fix a monotone operator F upon which the various general notions are based. Section 3 discusses F-firmly nonexpansive mappings, and Section 4 F-resolvents. It is then proved that F-resolvents are F-firmly nonexpansive (Corollary 4.3); the converse implication (Proposition 5.1) is established in Section 5. The parametrization of the graph à la Minty is obtained in Section 6, while the resolvent of the inverse is discussed in Section 7. Section 8 deals with the constructive extension of a given F-firmly nonexpansive mapping. The final Section 9 provides additional examples and a foray into algorithms.

2 Characterizations of maximality

Fact 2.1 (Martínez-Legaz) (See [28, Theorem 8].) *Let $F\colon X \rightrightarrows X^*$ be a maximal monotone operator such that its Fitzpatrick function* [19]

$$X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \sup_{(y,y*)\in \operatorname{gra} F} \big(\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\big) \tag{1}$$

is real-valued, and let $A\colon X \rightrightarrows X^*$ be monotone. Then the following hold.

(i) *If A is maximal monotone, then* $\operatorname{ran}(A + F) = X^*$.

(ii) *If F is single-valued, strictly monotone, and* $\operatorname{ran}(A+F) = X^*$*, then A is maximal monotone.*

Lemma 2.2 *Let $F\colon X \rightrightarrows X^*$ be a maximal monotone operator. Then the Fitzpatrick function of F is real-valued* $\Leftrightarrow$ $(\operatorname{dom} F) \times (\operatorname{ran} F) = X \times X^*$ *and F is 3*-monotone, i.e.,*

$$\big(\forall (x, x^*) \in (\operatorname{dom} F) \times (\operatorname{ran} F)\big) \quad \sup_{(y,y^*)\in \operatorname{gra} F} \langle x - y, y^* - x^*\rangle < +\infty. \tag{2}$$

Proof. "$\Rightarrow$": This follows from [28, Corollary 3]. "$\Leftarrow$": Clear. ■

Theorem 2.3 *Let $F\colon X \to X^*$ be maximal monotone, strictly monotone, 3*-monotone, and surjective, and let $A\colon X \rightrightarrows X^*$ be monotone. Then*

$$A \text{ is maximal monotone} \quad \Leftrightarrow \quad \operatorname{ran}(A + F) = X^*. \tag{3}$$

Proof. Since $\operatorname{dom} F = X$, $\operatorname{ran} F = X^*$, and F is 3*-monotone, Lemma 2.2 implies that the Fitzpatrick function of F is real-valued. The characterization now follows from Fact 2.1. ■

Corollary 2.4 *Let $f\colon X \to \mathbb{R}$ be Gâteaux differentiable everywhere, strictly convex, and cofinite, and let $A\colon X \rightrightarrows X^*$ be monotone. Then A is maximal monotone* $\Leftrightarrow$ $\operatorname{ran}(A + \nabla f) = X^*$.

Proof. Indeed, $\operatorname{dom} \nabla f = X$ (by assumption), ∇f is maximal monotone (as a subdifferential), strictly monotone (as f is strictly convex), 3*-monotone (as a subdifferential), and $\operatorname{ran} \nabla f = \operatorname{dom} f^* = X^*$ (by assumption). The result thus follows from Theorem 2.3. ■

Remark 2.5 Some comments on Corollary 2.4 are in order.

(i) If f is not cofinite, then the implication "$\Rightarrow$" fails: indeed, suppose that $X = \mathbb{R}$, let $f = \exp$, and set $A \equiv 0$. Then A is maximal monotone, yet $\operatorname{ran}(A + \nabla f) = \operatorname{ran} \nabla f = \left]0, +\infty\right[\neq \mathbb{R}$.

(ii) If f does not have full domain then the implication "$\Leftarrow$" fails: this time, suppose that $X = \mathbb{R}$, let f be the negative entropy function, and set $A = \operatorname{Id}|_{[0,+\infty[}$. Then $A + \nabla f = \operatorname{Id} + \nabla f$ is surjective (which is seen either directly or from Corollary 2.7), but A is not maximal monotone.

Corollary 2.6 (Rockafellar) (See [31, Corollary on page 78], and also [36] for another proof.) *Suppose that X is strictly convex and smooth, and let $A\colon X \rightrightarrows X^*$ be monotone. Then A is maximal monotone* $\Leftrightarrow \operatorname{ran}(A+J) = X^*$.

Proof. To say that the Banach space X is strictly convex and smooth means precisely that $\frac{1}{2}\|\cdot\|^2$ is strictly convex and Gâteaux differentiable. Since $\frac{1}{2}\|\cdot\|^2$ is cofinite (the conjugate being the corresponding halved energy for the dual norm), the result is clear from Corollary 2.4. ■

Specializing Corollary 2.6 further gives another classical case.

Corollary 2.7 (Minty) (See [29].) *Suppose that X is a Hilbert space, and let $A\colon X \rightrightarrows X$ be monotone. Then A is maximal monotone* $\Leftrightarrow \operatorname{ran}(A+\operatorname{Id}) = X$.

3 F-firmly nonexpansive operators

From now on, we assume that

$$F\colon X \to X^* \text{ is maximal monotone, strictly monotone, } 3^*\text{-monotone, and surjective.} \tag{4}$$

There are many examples of operators satisfying our standing assumptions (4) on F.

Example 3.1 Each of the following describes a situation where (4) holds.

(i) $F = \operatorname{Id}$, when X is a Hilbert space.

(ii) $F = J$, when X is strictly convex and smooth.

(iii) $F = \nabla\big(\frac{1}{p}\|\cdot\|^p\big)$, when X is strictly convex and smooth, and $p \in \,]1,+\infty[$,

(iv) $F = \nabla f$, when $f\colon X \to \mathbb{R}$ is differentiable everywhere, strictly convex, and cofinite.

(v) F is the counter-clockwise rotator by an angle in $[0, \pi/2[$, when $X = \mathbb{R}^2$.

Proof. It is clear that (i)–(iv) become increasingly less restrictive; for (iv), the 3^* monotonicity follows from [11] (see also [39, Section 32.21]). Finally, see [4] for (v). ■

Definition 3.2 *Let $C \subseteq X$, and let $T\colon C \to X$. Then T is F-*firmly nonexpansive *if*

$$(\forall x \in C)(\forall y \in C) \quad \langle Tx - Ty, FTx - FTy\rangle \le \langle Tx - Ty, Fx - Fy\rangle. \tag{5}$$

Remark 3.3 While it is tempting to ponder set-valued extension of F-firm nonexpansiveness, it turns out that this leads one back to the single-valued case: let $T\colon X \rightrightarrows X$ satisfy

$$\big(\forall (x,u) \in \operatorname{gra} T\big)\big(\forall (y,v) \in \operatorname{gra} T\big) \quad \langle u - v, Fu - Fv\rangle \le \langle u - v, Fx - Fy\rangle, \tag{6}$$

and suppose that $\{(x,u_1),(x,u_2)\} \subseteq \operatorname{gra} T$. The monotonicity of F and (6) yield

$$0 \leq \langle u_1 - u_2, Fu_1 - Fu_2 \rangle \leq \langle u_1 - u_2, Fx - Fx \rangle = 0. \tag{7}$$

Hence $\langle u_1 - u_2, Fu_2 - Fu_2 \rangle = 0$ and thus $u_1 = u_2$ by strict monotonicity of F.

Example 3.4 (classical firm nonexpansiveness) Suppose that X is a Hilbert space and that $F = \operatorname{Id}$. Let $C \subseteq X$ and let $T\colon C \to X$. Then T is Id-firmly nonexpansive $\Leftrightarrow$

$$(\forall x \in C)(\forall y \in C) \quad \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \tag{8}$$

i.e., T is *firmly nonexpansive* in the classical Hilbert space sense (see, e.g., [20, 21]).

Example 3.5 ("firmly nonexpansive type") Suppose that X is strictly convex and smooth. Let $C \subseteq X$ and let $T\colon C \to X$. Following Kohsaka and Takahashi [25], we say that the operator T is of *firmly nonexpansive type* if

$$(\forall x \in C)(\forall y \in C) \quad \langle Tx - Ty, JTx - JTy \rangle \leq \langle Tx - Ty, Jx - Jy \rangle. \tag{9}$$

Example 3.6 ("D-firm") Let $f\colon X \to \mathbb{R}$ be differentiable everywhere, strictly convex, and cofinite, let $C \subseteq X$, and let $T\colon C \to X$. Following [3], we say that the operator T is *D-firm* if

$$(\forall x \in C)(\forall y \in C) \quad \langle Tx - Ty, \nabla f(Tx) - \nabla f(Ty) \rangle \leq \langle Tx - Ty, \nabla f(x) - \nabla f(y) \rangle. \tag{10}$$

The "D" in D-firm stems from the fact that if we let

$$D\colon X \times X \to \mathbb{R}\colon (x,y) \mapsto f(x) - f(y) - \langle x - y, \nabla f(y) \rangle \tag{11}$$

be the *Bregman distance* (see [9, 14, 16] for further information) associated with f, then (10) is equivalent to

$$(\forall x \in C)(\forall y \in C) \quad D(Tx,Ty) + D(Ty,Tx) \leq D(Tx,y) + D(Ty,x) - D(Tx,x) - D(Ty,y); \tag{12}$$

see also [3, Proposition 3.5(iv)]. Note that if X is strictly convex and smooth, and $f = \frac{1}{2}\|\cdot\|^2$, then T is D-firm $\Leftrightarrow$ T is of firmly nonexpansive type. In this sense, the notion of D-firmness is significantly more general than that of firmly nonexpansive type.

In the next section, we turn to the construction of examples of F-firmly nonexpansive operators.

4 F-resolvents are F-firmly nonexpansive ...

In the setting of Hilbert space, as in Example 3.4, it is well known that resolvents of monotone operators are firmly nonexpansive. More generally, operators that are of firmly nonexpansive type or even D-firm may be obtained similarly. Most generally, we will show in this section that F-resolvents give similarly rise to F-firmly nonexpansive operators.

Definition 4.1 *Let $A\colon X \rightrightarrows X^*$. Then the composition*

$$(A+F)^{-1}F \tag{13}$$

is the F-resolvent *of A.*

Proposition 4.2 *Let $A\colon X \rightrightarrows X^*$, let $T_A = (A+F)^{-1}F$ be its associated F-resolvent, and let $x \in X$. Then the following hold.*

(i) $\operatorname{dom} T_A = F^{-1}(\operatorname{ran}(A+F))$ *and* $\operatorname{ran} T_A = \operatorname{dom} A$.

(ii) $x \in T_A x \Leftrightarrow 0 \in Ax$.

(iii) *If A is monotone, then T_A is at most single-valued and F-firmly nonexpansive.*

(iv) *If A is monotone, then: A is maximal monotone $\Leftrightarrow \operatorname{dom} T_A = X$.*

Proof. (i): $x \in \operatorname{dom} T_A \Leftrightarrow Fx \in \operatorname{dom}(A+F)^{-1} \Leftrightarrow Fx \in \operatorname{ran}(A+F) \Leftrightarrow x \in F^{-1}(\operatorname{ran}(A+F))$. Furthermore, $\operatorname{ran} T_A = \operatorname{dom} F^{-1}(A+F) = \operatorname{dom} A$.

(ii): $x \in T_A x \Leftrightarrow Fx \in (A+F)x = Ax + Fx \Leftrightarrow 0 \in Ax$.

(iii): Suppose that A is monotone. Since F is strictly monotone, it follows that $A+F$ is strictly monotone, which in turn implies that $(A+F)^{-1}$ is at most single-valued. Since F is single-valued, we deduce that the composition $(A+F)^{-1}F$ is at most single-valued. Using (i), we set $C = \operatorname{dom} T_A = F^{-1}(\operatorname{ran}(A+F))$. Let $y \in C$, i.e., $Fy \in \operatorname{ran}(A+F)$. Then there exists $v \in X$ such that $Fy \in (A+F)v$. Hence $Fy - Fv \in Av$ and $v \in (A+F)^{-1}Fy = T_A y$, i.e., $v = T_A y$ and so

$$(T_A y, Fy - FT_A y) \in \operatorname{gra} A. \tag{14}$$

Let $z \in C$. A similar argugment shows that there exists $w = T_A z \in X$ such that $Fz - Fw \in Az$ and $w = T_A z$. Since A is monotone, $0 \le \langle v - w, (Fy - Fv) - (Fz - Fw)\rangle = \langle T_A y - T_A z, (Fy - Fz) - (FT_A y - FT_A z)\rangle$, i.e.,

$$\langle T_A y - T_A z, FT_A y - FT_A z\rangle \le \langle T_A y - T_A z, Fy - Fz\rangle. \tag{15}$$

This verifies that T_A is F-firmly nonexpansive.

(iv): Suppose that A is monotone. Using Theorem 2.3, the bijectivity of F, and (i), we obtain the equivalences: A is maximal monotone $\Leftrightarrow \operatorname{ran}(A+F) = X^* \Leftrightarrow F^{-1}(\operatorname{ran}(A+F)) = X \Leftrightarrow \operatorname{dom} T_A = X$. ■

Corollary 4.3 *Let $A\colon X \rightrightarrows X^*$ be maximal monotone, and let $T_A = (A+F)^{-1}F$ be its associated F-resolvent. Then $T_A\colon X \to X$ is F-firmly nonexpansive. If X is finite-dimensional, then T_A is continuous.*

Proof. In view of Proposition 4.2, we only have to establish the continuity of T_A in the finite-dimensional case. Since F and $(A+F)^{-1}$ are single-valued maximal monotone operators with full domain, it follows from [34, Theorem 12.63(c)] that they are continuous, and so is their composition $(A+F)^{-1}F = T_A$. ■

Example 4.4 Let $F = \nabla f$ be as in Example 3.1(iv). Then the F-resolvent of a maximal monotone operator A becomes the "D-resolvent" considered in [17, 3], and the counterpart of Proposition 4.2 is [3, Proposition 3.8]. If A is a subdifferential operator, then one obtains "D-prox operators"; see, e.g., [15, 3]. Finally, if $A = N_C$, where C is a nonempty closed convex subset of X, then we obtain Bregman projections; see, e.g., [2].

Example 4.5 Suppose that X is strictly convex and smooth, and let $F = J$ be as in Example 3.1(ii). We then recover the resolvent $(A+J)^{-1}J$ (see, e.g., [23, 25]), and the counterpart of Proposition 4.2 is [25, Lemma 2.3]. If A is specialized to the normal cone operator N_C, where C is a nonempty closed convex subset of X, then the resolvent becomes the generalized projection operators studied, e.g., in [1, 25].

Example 4.6 (Minty-Rockafellar) Suppose X is a Hilbert space and A is maximal monotone. Then the standard resolvent $(A+\mathrm{Id})^{-1}$ is firmly nonexpansive and it has full domain. This is classical and goes back to Minty [29] and to Rockafellar [32].

5 . . . and vice versa

Eckstein and Bertsekas [18] observed that the converse of Example 4.6 holds, i.e., that every firmly nonexpansive operator (with full domain) must be the resolvent of the corresponding (maximal) monotone operator. As we now show, this is also the case for F-firmly nonexpansive operators.

Proposition 5.1 *Let $C \subseteq X$, let $T\colon C \to X$, and set $A_T = FT^{-1} - F$. Then the following hold.*

(i) *The F-resolvent of A_T is T.*

(ii) *If T is F-firmly nonexpansive, then A_T is monotone.*

(iii) *If T is F-firmly nonexpansive, then: $C = X \Leftrightarrow A_T$ is maximal monotone.*

Proof. (i): $A_T = FT^{-1} - F \Rightarrow A_T + F = FT^{-1} \Rightarrow (A_T+F)^{-1} = (FT^{-1})^{-1} = TF^{-1} \Rightarrow (A_T+F)^{-1}F = TF^{-1}F = T$.

(ii): Suppose that T is F-firmly nonexpansive. Take $(u,u^*), (v,v^*)$ in $\operatorname{gra} A_T$. Then $u^* \in A_T u = FT^{-1}u - Fu \Leftrightarrow u^* + Fu \in FT^{-1}u \Leftrightarrow u \in (FT^{-1})^{-1}(u^* + Fu) \Leftrightarrow u = TF^{-1}(u^* + Fu)$, and analogously $v = TF^{-1}(v^* + Fv)$. Since T is F-firmly nonexpansive, we estimate

$$\begin{aligned}\langle u - v, Fu - Fv\rangle &= \langle TF^{-1}(u^* + Fu) - TF^{-1}(v^* + Fv), FTF^{-1}(u^* + Fu) - FTF^{-1}(v^* + Fv)\rangle \\ &\le \langle TF^{-1}(u^* + Fu) - TF^{-1}(v^* + Fv), FF^{-1}(u^* + Fu) - FF^{-1}(v^* + Fv)\rangle \\ &= \langle u - v, (u^* + Fu) - (v^* + Fv)\rangle. \end{aligned} \tag{16}$$

Hence, $0 \le \langle u - v, u^* - v^*\rangle$, as required.

(iii): Suppose that T is F-firmly nonexpansive. By (ii), A_T is monotone. Using (i) and Proposition 4.2(iv), we obtain: A_T is maximal monotone $\Leftrightarrow$ $\operatorname{dom} T = C = X$. ■

Corollary 5.2 *Let $A\colon X \rightrightarrows X^*$ with associated F-resolvent $T_A = (A + F)^{-1}F$, let $C \subseteq X$, let $T\colon C \to X$, and set $A_T = FT^{-1} - F$. Assume that $T_A = T$; equivalently, that $A_T = A$. Then A is (maximal) monotone $\Leftrightarrow$ T is F-firmly nonexpansive (and $C = X$).*

Proof. Combine Proposition 4.2 and Proposition 5.1. ■

Specializing to $F = J$, where X is strictly convex and smooth, one obtains the following result related to [26, Proposition 3.1].

Corollary 5.3 (Kohsaka-Takahashi) *Suppose that X is strictly convex and smooth and that $F = J$. Let $C \subseteq X$, let $T\colon C \to X$, and set $A_T = JT^{-1} - J$. Then T is J-firmly nonexpansive $\Leftrightarrow$ A_T is monotone.*

In the setting of Hilbert space, Corollary 5.2 recovers the following result, which appeared first in [18, Theorem 2].

Corollary 5.4 (Eckstein-Bertsekas) *Suppose that X is a Hilbert space and that $F = \mathrm{Id}$, let $A\colon X \rightrightarrows X$, and denote the* Id*-resolvent of A by T_A. Then A is (maximal) monotone $\Leftrightarrow$ T_A is firmly nonexpansive (with full domain).*

6 Minty parametrization

Theorem 6.1 (F-Minty parametrization) *Let $A\colon X \rightrightarrows X^*$ be monotone, let $T_A = (A+F)^{-1}F$ be its associated F-resolvent, and set $C = \operatorname{dom} T_A$. Then*

$$\Psi\colon C \to \operatorname{gra} A\colon x \mapsto \big(T_A x, Fx - FT_A x\big) \tag{17}$$

is a bijection with

$$\Psi^{-1}\colon \operatorname{gra} A \to C\colon (u, u^*) \mapsto F^{-1}(u^* + Fu). \tag{18}$$

Moreover, the following hold.

(i) *If F, F^{-1}, T_A are continuous, then so are Ψ and Ψ^{-1}.*

(ii) *If X is finite-dimensional and A is maximal monotone, then $F, F^{-1}, T_A, \Psi, \Psi^{-1}$ are continuous.*

(iii) *If X is finite-dimensional and F is linear, then $F, F^{-1}, T_A, \Psi, \Psi^{-1}$ are Lipschitz continuous.*

Proof. It follows from (14) that $(\forall y \in C)\ (T_A y, Fy - FT_A y) \in \operatorname{gra} A$. Hence $\operatorname{ran}\Psi \subseteq \operatorname{gra} A$. Now take $(u, u^*) \in \operatorname{gra} A$ and set $x = F^{-1}(u^* + Fu)$. Then $u^* \in Au \Rightarrow u^* + Fu \in (A+F)u \Rightarrow x = F^{-1}(u^* + Fu) \in F^{-1}(\operatorname{ran}(A+F)) = \operatorname{dom} T_A = C$ by Proposition 4.2(i). Furthermore, $T_A x = (A+F)^{-1} F F^{-1}(u^* + Fu) = (A+F)^{-1}(u^* + Fu) = u$ and thus $Fx - FT_A x = FF^{-1}(u^* + Fu) - Fu = u^*$. Therefore, $(u, u^*) = \Psi(x)$ and hence $\operatorname{ran}\Psi = \operatorname{gra} A$. On the other hand, let y and z be in C such that $\Psi(y) = \Psi(z)$. Then $T_A x = T_A y$ and $Fx - FT_A x = Fy - FT_A y$, hence that $Fx = Fy$ and thus $x = y$. It follows that Ψ is injective. Altogether, Ψ is a bijection between C and $\operatorname{gra} A$. The beginning of this proof implies the formula for Ψ^{-1}. We now turn to the continuity assertions.

(i): This statement is clear from the formulae (17) and (18).

(ii): Suppose that X is finite-dimensional. In Corollary 4.3, we observed that T_A is continuous; by using once again [34, Theorem 12.63(c)], we obtain continuity of F and F^{-1}. Now apply (i).

(iii): It is clear that F and F^{-1} are Lipschitz continuous. Denote the smallest eigenvalue of the symmetric part of F by λ. Then $\lambda > 0$. Since T_A is F-firmly nonexpansive by Proposition 4.2(iii), we estimate

$$\begin{aligned}(\forall x \in C)(\forall y \in C) \quad \lambda\|T_A x - T_A y\|^2 &\leq \langle T_A x - T_A y, F(T_A x - T_A y)\rangle \\ &= \langle T_A x - T_A y, FT_A x - FT_A y\rangle \\ &\leq \langle T_A x - T_A y, Fx - Fy\rangle \\ &\leq \|T_A x - T_A y\|\,\|F\|\,\|x - y\|;\end{aligned}$$

consequently, $\|T_A x - T_A y\| \leq (\|F\|/\lambda)\|x - y\|$. The formulae (17) and (18) show that Ψ and Ψ^{-1} are Lipschitz continuous as well. ■

Remark 6.2 When $F = J$, the inclusion $\operatorname{ran}\Psi \subseteq \operatorname{gra} A$ in Theorem 6.1 was already noted by Kohsaka and Takahashi (see [24, page 242] and also [22, page 942]).

7 Resolvent of the inverse

In this section, we discuss the possibility of computing the F^{-1}-resolvent of A^{-1} in terms of the F-resolvent of A.

Theorem 7.1 (inverse-resolvent fixed point equation) *Let $A\colon X \rightrightarrows X^*$ be monotone, let $T_A = (A+F)^{-1}F$ be the its associated F-resolvent, let $T_{A^{-1}} = (A^{-1} + F^{-1})^{-1}F^{-1}$ be the F^{-1}-*

resolvent of A^{-1}*, let* $x^* \in \operatorname{dom} T_{A^{-1}} = F(\operatorname{ran}(A^{-1} + F^{-1})$*, and let* $y^* \in X^*$*. Then*

$$y^* = T_{A^{-1}}x^* \quad \Leftrightarrow \quad y^* = F\Big(F^{-1}x^* - T_A\Big(F^{-1}\big(y^* + F(F^{-1}x^* - F^{-1}y^*)\big)\Big)\Big). \tag{19}$$

Proof. The identity for $\operatorname{dom} T_{A^{-1}}$ follows from Proposition 4.2(i). For convenience, set $x = F^{-1}x^*$ and $y = F^{-1}y^*$. We then have the equivalences

$$\begin{aligned} y^* = T_{A^{-1}}x^* &\Leftrightarrow y^* = (F^{-1} + A^{-1})^{-1}F^{-1}x^* \\ &\Leftrightarrow F^{-1}x^* \in (F^{-1} + A^{-1})y^* \\ &\Leftrightarrow x - y \in A^{-1}y^* \\ &\Leftrightarrow y^* \in A(x - y) \\ &\Leftrightarrow y^* + F(x - y) \in (A + F)(x - y) \\ &\Leftrightarrow x - y \in (A + F)^{-1}FF^{-1}\big(y^* + F(x - y)\big) \\ &\Leftrightarrow x - y = T_AF^{-1}\big(y^* + F(x - y)\big) \\ &\Leftrightarrow y = x - T_AF^{-1}\big(y^* + F(x - y)\big) \\ &\Leftrightarrow y^* = F\Big(x - T_AF^{-1}\big(y^* + F(x - y)\big)\Big), \end{aligned} \tag{20}$$

and this last identity is in turn equivalent to the right side of (19). ■

Corollary 7.2 *Suppose that* F *is linear, let* $A\colon X \rightrightarrows X^*$ *be monotone, let* $T_A = (A + F)^{-1}F$ *be its associated* F*-resolvent, and let* $T_{A^{-1}} = (A^{-1} + F^{-1})^{-1}F^{-1}$ *be the* F^{-1}*-resolvent of* A^{-1}*. Then*

$$T_{A^{-1}} = \operatorname{Id} - FT_AF^{-1}. \tag{21}$$

In the classical Hilbert space setting of Example 4.6, one recovers the following well known result [34, Lemma 12.14].

Corollary 7.3 (inverse-resolvent identity) *Suppose that* X *is a Hilbert space and that* $F = \operatorname{Id}$*. Let* $A\colon X \rightrightarrows X^*$ *be maximal monotone. Then* $T_{A^{-1}} = \operatorname{Id} - T_A$*, i.e.,*

$$(A^{-1} + \operatorname{Id})^{-1} = \operatorname{Id} - (A + \operatorname{Id})^{-1}. \tag{22}$$

8 Constructive extension

We now describe how F-firmly nonexpansive operators can be extended to the whole space. This technique was recently utilized in [8] in the setting of Hilbert spaces.

Theorem 8.1 *Let* $C \subseteq X$*, and let* $T\colon C \to X$ *be* F*-firmly nonexpansive. Proceed as follows.*

① *Set* $A = FT^{-1} - F$.

② *Denote the Fitzpatrick function of* A *(see* (1)*) by* Φ.

③ *Compute*

$$\Psi\colon (x,x^*) \mapsto \min_{(y+z,y^*+z^*)=2(x,x^*)} \big(\tfrac{1}{2}\Phi(y,y^*) + \tfrac{1}{2}\Phi^*(z^*,z) + \tfrac{1}{8}\big(\|y-z\|^2 + \|y^*-z^*\|^2\big)\big), \tag{23}$$

which is the proximal average [6] *between* Φ *and* Φ^* *(with the variables transposed).*

④ *Define* $\widetilde{A}\colon X \rightrightarrows X^*$ *via*

$$\operatorname{gra}\widetilde{A} = \big\{(x,x^*) \in X \times X^* \mid \Psi(x,x^*) = \langle x, x^*\rangle\big\}. \tag{24}$$

⑤ *Set* $\widetilde{T} = (\widetilde{A} + F)^{-1}F$.

Then $\widetilde{T}\colon X \to X$ *is* F*-firmly nonexpansive and it extends* T *to the entirety of* X.

Proof. By Proposition 5.1, A is monotone. Hence, using [7, Fact 5.6 and Theorem 5.7], we see that $\widetilde{A}$ is a maximal monotone extension of A. Theorem 6.1 and Proposition 4.2 now show that $\widetilde{T}$ is an F-firmly nonexpansive extension of T to the entire space X. ■

Remark 8.2 Let us comment on Theorem 8.1 further when X is a real Hilbert space.

(i) In this case, Theorem 8.1 becomes [8, Theorem 3.1].

(ii) As explained in [8, Theorem 3.6], one may use Theorem 8.1 to obtain a *constructive* Kirszbraun-Valentine extension of a given nonexpansive operator.

9 Examples

We begin with the F-resolvent of the identity, where F is a counter-clockwise rotator in the Euclidean plane.

Example 9.1 Suppose that $X = \mathbb{R}^2$, let $\theta \in \big[0, \frac{\pi}{2}\big[$, and set

$$F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{25}$$

Then

$$(\operatorname{Id} + F)^{-1}F = \frac{1}{2}\begin{pmatrix} 1 & -\frac{\sin\theta}{1+\cos\theta} \\ \frac{\sin\theta}{1+\cos\theta} & 1 \end{pmatrix}. \tag{26}$$

The most important example of a standard resolvent is the projector onto a nonempty closed convex set C, which arises as the resolvent of the normal cone operator $N_C = \partial\iota_C$. As it turns out, a generalized projector is obtained in the general F-resolvent setting.

Theorem 9.2 (F-projector) *Let $C \subseteq X$ be nonempty, closed, and convex, denote the F-resolvent of N_C by P_C, and assume that $y \in \operatorname{int} C$. Then $\operatorname{ran} P_C = \operatorname{Fix} P_C = C$, $P_C^2 = P_C$, and $P_C^{-1}y = \{y\}$.*

Proof. Note that $\operatorname{ran} P_C = \operatorname{dom} N_C = C$ by Proposition 4.2(i); furthermore, $\operatorname{Fix} P_C = N_C^{-1}0 = C$ by Proposition 4.2(ii). Finally, since $y \in \operatorname{int} C$, $N_C y = \{0\}$ and therefore $P_C^{-1}y = \big((N_C+F)^{-1}F\big)^{-1}y = F^{-1}(N_C+F)y = F^{-1}(0+Fy) = y$. ■

For the purpose of illustration, let us now compute some generalized projectors when F is the rotator from Example 9.1. The following result is clear from Theorem 9.2.

Example 9.3 Suppose that $X = \mathbb{R}^2$, let θ and F be as in Example 9.1, let $C \subseteq \mathbb{R}^2$, and denote the F-resolvent of N_C by P_C.

(i) If $C = \{0\}$, then $P_C = 0$.

(ii) If $C = \mathbb{R}^2$, then $P_C = \operatorname{Id}$.

Example 9.4 Suppose that $X = \mathbb{R}^2$, let θ and F be as in Example 9.1, set $C = \mathbb{R} \times \{0\}$, and denote the F-resolvent of N_C by P_C. Then

$$P_C = \begin{pmatrix} 1 & -\tan\theta \\ 0 & 0 \end{pmatrix}. \tag{27}$$

Proof. Let $x = (x_1, x_2) \in \mathbb{R}^2$, and set $y = P_C x$. Then $y \in C$ and $Fx \in N_C y + Fy = (\{0\} \times \mathbb{R}) + Fy$. Thus $F(x-y) \in \{0\} \times \mathbb{R}$. Write $y = (y_1, 0)$. We then have $(x_1 - y_1)\cos\theta - x_2 \sin\theta = 0$. Hence $y_1 = x_1 - x_2 \tan\theta$, and (27) holds. ■

Example 9.5 Suppose that $X = \mathbb{R}^2$ with the Euclidean norm, let θ and F be as in Example 9.1, let $C = \big\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\big\}$ be the closed unit ball, denote the F-resolvent of N_C by P_C, and set $\alpha = \sqrt{\|x\|^2 - \sin^2\theta} - \cos\theta$. Then

$$P_C \colon \mathbb{R}^2 \to C \colon x \mapsto \begin{cases} x, & \text{if } x \in C; \\ \frac{1}{\|x\|^2}(\operatorname{Id} + \alpha F)x, & \text{if } x \notin C. \end{cases} \tag{28}$$

Moreover,

$$\big(\forall z \in \mathbb{R}^2\big) \quad P_C^{-1}z = \begin{cases} z, & \text{if } \|z\| < 1; \\ z + [0, +\infty[\cdot F^* z, & \text{if } \|z\| = 1; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{29}$$

Proof. Let $x \in \mathbb{R}^2$. We consider two cases.
Case 1: $\|x\| \leq 1$. Then $x \in C$ and so $P_C x = x$ by Theorem 9.2.
Case 2: $\|x\| > 1$. Set $y = P_C x = (N_C + F)^{-1} F x$. Assume that $\|y\| < 1$. Then $Fx = N_C y + Fy = 0 + Fy = Fy$. Hence $x = y$, which is absurd. Thus

$$\|y\| = 1. \tag{30}$$

and therefore

$$N_C(y) = [0, +\infty[\, \cdot y. \tag{31}$$

It follows that there exists $\alpha \in [0, +\infty[$ such that $Fx = \alpha y + Fy = (\alpha \operatorname{Id} + F)y$. Since $x \neq y$, we see that $\alpha > 0$. Moreover, by the orthogonality of F, it follows that

$$(\alpha \operatorname{Id} + F)^{-1} F = \tfrac{1}{\alpha^2 + 2\alpha \cos\theta + 1} (\alpha F + \operatorname{Id}) \tag{32}$$

and that

$$F^*(\alpha \operatorname{Id} + F^*)^{-1} (\alpha \operatorname{Id} + F)^{-1} F = \tfrac{1}{\alpha^2 + 2\alpha \cos\theta + 1} \operatorname{Id}. \tag{33}$$

Since $1 = \|y\| = \|(\alpha \operatorname{Id} + F)^{-1} F x\|$, we thus have $\alpha^2 + 2\alpha \cos\theta + 1 = \|x\|^2$ and hence $\alpha = -\cos\theta + \sqrt{\cos^2\theta + \|x\|^2 - 1} = -\cos\theta + \sqrt{\|x\|^2 - \sin^2\theta}$. Consequently,

$$y = (\alpha \operatorname{Id} + F)^{-1} F x = \tfrac{1}{\|x\|^2} (\alpha F + \operatorname{Id}) x, \tag{34}$$

which yields (28). Now let $z \in \mathbb{R}^2$. In view of Theorem 9.2, it suffices to consider the case when $z \in \operatorname{bdry} C$, i.e., $\|z\| = 1$ and thus $N_C z = [0, +\infty[\, \cdot z$. Then $P_C^{-1} z = F^{-1}(N_C + F) z = z + F^* [0, +\infty[\, \cdot z = z + [0, +\infty[\, \cdot F^* z$. ■

Example 9.6 Suppose that X is a Hilbert space and that $F = \nabla \frac{1}{p} \| \cdot \|^p$, where $p \in]1, +\infty[$. Let $x \in X$ and set

$$k_p(x) = \begin{cases} 0, & \text{if } x = 0; \\ \text{the unique solution of } k^{p-1} + k/\|x\|^{p-2} = 1 \text{ in }]0,1[, & \text{if } x \neq 0. \end{cases} \tag{35}$$

Let $T_p = (\operatorname{Id} + F)^{-1} F$ be the F-resolvent of Id. Then $T_p(x) = k_p(x) x$. Moreover,

$$\lim_{p \to 1^+} T_p(x) = 0 \quad \text{and} \quad \lim_{p \to +\infty} T_p(x) = \begin{cases} 0, & \text{if } \|x\| < 1; \\ x, & \text{if } \|x\| \geq 1. \end{cases} \tag{36}$$

Proof. The statements are clear if $x = 0$, so we assume that $x \neq 0$. Set $y = T_p(x)$. Then $y \neq 0$, $Fx = \|x\|^{p-2} x$ and $Fy = \|y\|^{p-2} y$. Furthermore, $Fx \in (\operatorname{Id} + F) y = y + Fy \Leftrightarrow \|x\|^{p-2} x = (1 + \|y\|^{p-2}) y$, which implies that $y = kx$, where $k \in]0, +\infty[$ satisfies $\|x\|^{p-2} = k + k^{p-1} \|x\|^{p-2} \Leftrightarrow k^{p-1} + k/\|x\|^{p-2} = 1$. The remaining statements follow using Calculus. ■

Remark 9.7 Consider Example 9.6 when X is finite-dimensional. By Corollary 4.3, T_p is continuous; however, the limiting (in the pointwise sense) operator $\lim_{p \to +\infty} T_p$ is not continuous.

We now turn to an algorithmic result on iterating F-resolvents.

Theorem 9.8 *Suppose that X is a Hilbert space and that F is linear. Let $T = (\mathrm{Id} + F)^{-1}F$ be the F-resolvent of* Id*, let $x_0 \in X$, and set $(\forall n \in \mathbb{N})\ x_{n+1} = Tx_n$. Then $\|T\| < 1$ and hence $x_n \to 0$.*

Proof. Set $\alpha = 1/\|F\|$. Then $(\forall x \in X)\ \|Fx\| \leq \|x\|/\alpha$; equivalently,

$$(\forall y \in X) \quad \|F^{-1}y\| \geq \alpha\|y\|. \tag{37}$$

Observe that

$$T = (\mathrm{Id} + F)^{-1}F = (\mathrm{Id} + F^{-1})^{-1}, \tag{38}$$

let $x \in X$, and set $y = Tx$. Then $x = T^{-1}y = (\mathrm{Id} + F^{-1})y$; thus, by monotonicity of F^{-1} and (37), we obtain

$$\|x\|^2 = \|y + F^{-1}y\|^2 = \|y\|^2 + \|F^{-1}y\|^2 + 2\langle y, F^{-1}y\rangle \geq \|y\|^2 + \|F^{-1}y\|^2 \geq (1 + \alpha^2)\|y\|^2. \tag{39}$$

Hence $\|y\|^2 = \|Tx\|^2 \leq \|x\|^2/(1 + \alpha^2)$ and so

$$\|T\| \leq 1/\sqrt{1 + \alpha^2} < 1. \tag{40}$$

By the Banach Contraction Mapping Principle, we see that $(x_n)_{n\in\mathbb{N}} = (T^n x_0)_{n\in\mathbb{N}}$ converges in norm to 0, which is the unique fixed point of T. Alternatively, observe that $T = (\mathrm{Id} + F^{-1})^{-1}$ and apply [12, Theorem 2.4.(e)]. ■

Remark 9.9 Let us conclude by interpreting Theorem 9.8 and outlining possible future research directions. Resolvent iterations are important for finding zeros of subdifferential operators — that is, minimizers — or more generally for finding zeros of maximal monotone operators. When $F = \mathrm{Id}$, this brings us to the classical setting of the proximal point algorithm [27, 32]; when $F = J$, where X is uniformly convex and uniformly smooth, see [25] and references therein, and when $F = \nabla f$, this goes back to [15]. It would be very interesting to build a general convergence theory for iterating F-resolvents. The difficulty lies in the absence of a potential function like the Bregman distance (11). However, Theorem 9.8 shows that it may be possible to create a theory in the present general framework, since this result shows that resolvent iterations do converge to the unique zero of the maximal monotone operator Id. This promises to be an exciting topic for further research.

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