Chapter 1
Self-dual Smooth Approximations of Convex Functions via the Proximal Average

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Abstract The proximal average of two convex functions has proven to be a useful tool in convex analysis. In this note, we express the Goebel self-dual smoothing operator in terms of the proximal average, which allows us to give a different proof of self duality. We also provide a novel self-dual smoothing operator. Both operators are illustrated by smoothing the norm.

Key words: approximation, convex function, Fenchel conjugate, Goebel smoothing operator, Moreau envelope, proximal average.
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1.1 Introduction

Let X be the standard Euclidean space $\mathbb{R}^n$, with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. It will be convenient to set

$$\mathsf{q} = \tfrac{1}{2}\|\cdot\|^2. \tag{1.1}$$

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Now let $f\colon X \to \,]-\infty,+\infty]$ be convex, lower semicontinuous, and proper. Since many convex functions are nonsmooth, it is natural to ask: How can one approximate f with a smooth function?

The most famous and very useful answer to this question is provided by the *Moreau envelope* [15, 17], which, for $\lambda > 0$, is defined by [1]

$$e_\lambda f = f \,\square\, \lambda^{-1}\mathfrak{q}. \tag{1.2}$$

It is well known that $e_\lambda f$ is smooth (i.e., continuously differentiable) and that $\lim_{\lambda\to 0^+} e_\lambda f = f$ point-wise; see, e.g., [17, Theorem 1.25 and Theorem 2.26]. Parenthetically, other approaches to smoothing are Ghomi's integral convolution method [9], Seeger's ball rolling technique [18], and Teboulle's entropic proximal mappings [19].

Let us now consider the norm, which is nonsmooth at the origin.

Example 1.1.1 (Moreau envelope of the norm) Let $\lambda \in \,]0,1[$, set $f = \|\cdot\|$, and denote the closed unit ball by C. Then, for x and x^* in X, we have [2]

$$e_\lambda f(x) = \begin{cases} \dfrac{\|x\|^2}{2\lambda}, & \text{if } \|x\| \leq \lambda; \\[2ex] \|x\| - \dfrac{\lambda}{2}, & \text{if } \|x\| > \lambda, \end{cases} \tag{1.3}$$

$\big(e_\lambda f\big)^* = \iota_C + \lambda\mathfrak{q}$, and $e_\lambda\,(f^*)(x^*) = (2\lambda)^{-1}\cdot\big(\max\{0, \|x^*\|-1\}\big)^2$. Consequently, $\big(e_\lambda f\big)^* \neq e_\lambda\,(f^*)$.

Proof. Either a straight-forward computation or [17, Example 11.26(a)] yields

$$f^* = \iota_C. \tag{1.4}$$

Next, if $y \in X$, then

$$e_{1/\lambda}\,\iota_C(y) = \inf_{c\in C} \lambda\mathfrak{q}(y-c) \tag{1.5}$$

$$= \frac{\lambda}{2} d_C^2(y) \tag{1.6}$$

$$= \frac{\lambda}{2}\cdot \begin{cases} \big(\|y\|-1\big)^2, & \text{if } \|y\| > 1; \\[2ex] 0, & \text{if } \|y\| \leq 1, \end{cases} \tag{1.7}$$

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[1] The symbol "$\square$" denotes *infimal convolution*: $(f_1 \,\square\, f_2)(x) = \inf_y \big(f_1(y) + f_2(x-y)\big)$.

[2] Here, ι_C is the *indicator function* defined by $\iota_C(x) = 0$, if $x \in C$; $\iota_C(x) = +\infty$, if $x \notin C$, $f^*(x^*) = \sup_{x\in X}\big(\langle x, x^*\rangle - f(x)\big)$ is the *Fenchel conjugate* of f, and $d_C(x) = \inf_{c\in C}\|x-c\| = \big(\|\cdot\| \,\square\, \iota_C\big)(x)$ is the *distance function*.

and thus

$$e_{1/\lambda}\,\iota_C\big(x/\lambda\big) = \frac{\lambda}{2}\cdot\begin{cases}\big(\|x/\lambda\|-1\big)^2, & \text{if } \|x\|>\lambda;\\[1ex] 0, & \text{if } \|x\|\le\lambda.\end{cases} \tag{1.8}$$

By [17, Example 11.26(b) on page 495], we obtain

$$\begin{aligned} e_\lambda f(x) &= \frac{1}{\lambda}\mathfrak{q}(x) - e_{1/\lambda} f^*\big(x/\lambda\big) && (1.9)\\ &= \frac{1}{2\lambda}\|x\|^2 - \frac{\lambda}{2}\cdot\begin{cases}\dfrac{\|x\|^2}{\lambda^2}-\dfrac{2\|x\|}{\lambda}+1, & \text{if } \|x\|>\lambda;\\[1ex] 0, & \text{if } \|x\|\le\lambda\end{cases} && (1.10)\\ &= \begin{cases}\|x\|-\dfrac{\lambda}{2}, & \text{if } \|x\|>\lambda;\\[1ex] \dfrac{\|x\|^2}{2\lambda}, & \text{if } \|x\|\le\lambda\end{cases} && (1.11)\end{aligned}$$

and $\big(e_\lambda f\big)^* = f^* + \lambda\mathfrak{q} = \iota_C + \lambda\mathfrak{q}$. Alternatively, one may use [7, Example 2.16], which provides the proximal mapping of f, and then use the proximal mapping calculus to obtain these results. Finally, a referee pointed out that (11) can also be derived by reducing the computation of the Moreau envelope to

$$\begin{aligned} e_\lambda f(x) &= (f\,\square\,\lambda^{-1}\mathfrak{q})(x) && (1.12)\\ &= \inf_y \big(f(y)+\lambda^{-1}\mathfrak{q}(x-y)\big) && (1.13)\\ &= \inf_y \Big(\|y\| + \frac{1}{2\lambda}\big(\|x\|^2+\|y\|^2-2\langle x,y\rangle\big)\Big) && (1.14)\\ &= \inf_{\eta\ge 0}\inf_{\|y\|=\eta} \Big(\|y\| + \frac{1}{2\lambda}\big(\|x\|^2+\|y\|^2-2\langle x,y\rangle\big)\Big) && (1.15)\\ &= \inf_{\eta\ge 0}\inf_{\|y\|=\eta} \Big(\eta + \frac{1}{2\lambda}\big(\|x\|^2+\eta^2-2\eta\|x\|\big)\Big) && (1.16)\\ &= \frac{\|x\|^2}{2\lambda} + \frac{1}{2\lambda}\inf_{\eta\ge 0}\Big(\eta^2+2\big(\lambda-\|x\|\big)\eta\Big), && (1.17)\end{aligned}$$

which can now be treated by one-dimensional calculus. ■

While the Moreau envelope has many desirable properties, we see from Example 1.1 that the smooth approximation $e_\lambda f$ is not *self-dual* in the sense that

$$\big(e_\lambda f\big)^* \neq e_\lambda\,(f^*). \tag{1.18}$$

It is perhaps surprising that self-dual smoothing operators even exist. The first example appears in [11]. Specifically, Goebel defined

$$G_\lambda f = (1-\lambda^2)e_\lambda f + \lambda \mathsf{q} \tag{1.19}$$

and proved that

$$\big(G_\lambda f\big)^* = G_\lambda (f^*), \tag{1.20}$$

i.e., *Fenchel conjugation and Goebel smoothing commute!* For applications of the Goebel smoothing operator, see [11].

The purpose of this note is two-fold. First, we present a different representation of the Goebel smoothing operator which allows us to prove self-duality using the Fenchel conjugation formula for the proximal average. Second, the proximal average is also utilized to obtain a novel smoothing operator. Both smoothing operators are computed explicitly for the norm. The formulas derived show that the new smoothing operator is distinct from the one provided by Goebel.

For f_1 and f_2, two functions from X to $\left]-\infty,+\infty\right]$ that are convex, lower semicontinuous and proper, and for two strictly positive convex coefficients ($\lambda_1+\lambda_2=1$), the *proximal average* is defined by

$$\operatorname{pav}(f_1,f_2;\lambda_1,\lambda_2) = \big(\lambda_1(f_1+\mathsf{q})^* + \lambda_2(f_2+\mathsf{q})^*\big)^* - \mathsf{q}. \tag{1.21}$$

The proximal average, which is actually a convex function, has been a useful tool for constructing primal-dual symmetric antiderivatives [5] and for extending monotone operators [2]; see also [3, 4, 6, 11, 12] for further information and applications. One of the key properties is the *Fenchel conjugation formula*

$$\operatorname{pav}(f_1,f_2;\lambda_1,\lambda_2)^* = \operatorname{pav}(f_1^*,f_2^*;\lambda_1,\lambda_2); \tag{1.22}$$

see [6, Theorem 6.1], [4, Theorem 4.3], or [3, Theorem 5.1].

We use standard convex analysis calculus and notation as, e.g., in [16, 17, 21]. In Section 2, we consider the Goebel smoothing operator from the proximal-average view point. The new smoothing operator is presented in Section 3.

1.2 The Goebel smoothing operator

Definition 1.2.1 (Goebel smoothing operator) *Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex, lower semicontinuous and proper, and let $\lambda \in \left]0,1\right[$. Then the* Goebel smoothing operator [11] *is defined by*

$$G_\lambda f = (1-\lambda^2)e_\lambda f + \lambda \mathsf{q}. \tag{1.23}$$

Note that (23) and standard properties of the Moreau envelope imply that point-wise

$$\lim_{\lambda\to 0^+} G_\lambda f = f \tag{1.24}$$

and that each $G_\lambda f$ is smooth.

Our first main result provides two alternative descriptions of the Goebel smoothing operator. The first description, item 1 in Theorem 2.2, shows a pleasing reformulation in terms of the proximal average. The second description, item 2 in Theorem 2.2, is less appealing but has the advantage of providing a different proof of the *self-duality*, item 3, observed by Goebel.

Theorem 1.2.2 *Let* $f\colon X \to \,]-\infty,+\infty]$ *be convex, lower semicontinuous and proper, and let* $\lambda \in \,]0,1[$*. Then the following hold*[3].

1. $G_\lambda f = (1+\lambda)\,\mathrm{pav}\,(f,0;1-\lambda,\lambda) + \lambda\mathfrak{q}$.
2. $G_\lambda f = (1+\lambda)^2\,\mathrm{pav}\left(f,\mathfrak{q};\frac{1-\lambda}{1+\lambda},\frac{2\lambda}{1+\lambda}\right)\circ(1+\lambda)^{-1}\,\mathrm{Id}$.
3. **(Goebel)** $\big(G_\lambda f\big)^* = G_\lambda\,(f^*)$.

Proof. Let $x \in X$. Then, using (21) and standard convex calculus, we obtain

$$\left((1+\lambda)^2\,\mathrm{pav}\left(f,\mathfrak{q};\frac{1-\lambda}{1+\lambda},\frac{2\lambda}{1+\lambda}\right)\circ(1+\lambda)^{-1}\,\mathrm{Id}\right)(x) \tag{1.25}$$

$$= (1+\lambda)^2\left(\left(\frac{1-\lambda}{1+\lambda}(f+\mathfrak{q})^* + \frac{2\lambda}{1+\lambda}(\mathfrak{q}+\mathfrak{q})^*\right)^* - \mathfrak{q}\right)\left(\frac{x}{1+\lambda}\right) \tag{1.26}$$

$$= (1+\lambda)^2\left(\frac{1-\lambda}{1+\lambda}(f+\mathfrak{q})^* + \frac{\lambda}{1+\lambda}\mathfrak{q}\right)^*\left(\frac{x}{1+\lambda}\right) - \mathfrak{q}(x) \tag{1.27}$$

$$= (1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\mathfrak{q}\Big)^*(x) - \mathfrak{q}(x) \tag{1.28}$$

$$= (1+\lambda)\left(\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\big(0+\mathfrak{q}\big)^*\Big)^* - \mathfrak{q}\right)(x) + \lambda\mathfrak{q}(x) \tag{1.29}$$

$$= \Big((1+\lambda)\,\mathrm{pav}\,(f,0;1-\lambda,\lambda) + \lambda\mathfrak{q}\Big)(x). \tag{1.30}$$

We have verified that (28) as well as the right sides of 1 and 2 coincide. Starting from (28) and again applying standard convex caluclus, we see that

$$(1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)^* + \lambda\mathfrak{q}\Big)^*(x) - \mathfrak{q}(x) \tag{1.31}$$

$$= (1+\lambda)\Big(\Big((1-\lambda)\big(f+\mathfrak{q}\big)^*\Big)^*\,\Box\,\big(\lambda\mathfrak{q}\big)^*\Big)(x) - \mathfrak{q}(x) \tag{1.32}$$

$$= (1+\lambda)\left((1-\lambda)\big(f+\mathfrak{q}\big)\left(\frac{\cdot}{1-\lambda}\right)\Box\,\frac{1}{\lambda}\mathfrak{q}\right)(x) - \mathfrak{q}(x) \tag{1.33}$$

[3] Here $\mathrm{Id}\colon X \to X : x \mapsto x$ is the *identity operator*.

$$= (1+\lambda)\inf_y\left((1-\lambda)\big(f+\mathfrak{q}\big)\left(\frac{y}{1-\lambda}\right)+\frac{1}{\lambda}\mathfrak{q}(x-y)\right)-\mathfrak{q}(x) \tag{1.34}$$

$$= (1+\lambda)\inf_y\left((1-\lambda)f\left(\frac{y}{1-\lambda}\right)+(1-\lambda)\mathfrak{q}\left(\frac{y}{1-\lambda}\right)+\frac{1}{\lambda}\mathfrak{q}(x-y)-\frac{1}{1+\lambda}\mathfrak{q}(x)\right) \tag{1.35}$$

$$= (1-\lambda^2)\inf_y\left(f\left(\frac{y}{1-\lambda}\right)+\mathfrak{q}\left(\frac{y}{1-\lambda}\right)+\frac{1}{\lambda(1-\lambda)}\mathfrak{q}(x-y)-\frac{1}{1-\lambda^2}\mathfrak{q}(x)\right). \tag{1.36}$$

Simple algebra shows that for every $y \in X$,

$$\mathfrak{q}\left(\frac{y}{1-\lambda}\right)+\frac{1}{\lambda(1-\lambda)}\mathfrak{q}(x-y)-\frac{1}{1-\lambda^2}\mathfrak{q}(x)=\frac{1}{\lambda}\mathfrak{q}\left(x-\frac{y}{1-\lambda}\right)+\frac{\lambda}{1-\lambda^2}\mathfrak{q}(x). \tag{1.37}$$

Therefore,

$$(1+\lambda)\Big((1-\lambda)\big(f+\mathfrak{q}\big)^*+\lambda\mathfrak{q}\Big)^*(x)-\mathfrak{q}(x) \tag{1.38}$$

$$= (1-\lambda^2)\inf_y\left(f\left(\frac{y}{1-\lambda}\right)+\mathfrak{q}\left(\frac{y}{1-\lambda}\right)+\frac{1}{\lambda(1-\lambda)}\mathfrak{q}(x-y)-\frac{1}{1-\lambda^2}\mathfrak{q}(x)\right) \tag{1.39}$$

$$= (1-\lambda^2)\inf_y\left(f\left(\frac{y}{1-\lambda}\right)+\frac{1}{\lambda}\mathfrak{q}\left(x-\frac{y}{1-\lambda}\right)+\frac{\lambda}{1-\lambda^2}\mathfrak{q}(x)\right) \tag{1.40}$$

$$= (1-\lambda^2)\inf_z\left(f(z)+\frac{1}{\lambda}\mathfrak{q}(x-z)+\frac{\lambda}{1-\lambda^2}\mathfrak{q}(x)\right) \tag{1.41}$$

$$= \big((1-\lambda^2)e_\lambda f+\lambda\mathfrak{q}\big)(x) \tag{1.42}$$

$$= G_\lambda f(x), \tag{1.43}$$

which completes the proof of 1 and 2.

3: In view of the conjugate formula $(\beta^2 h\circ(\beta^{-1}\,\mathrm{Id}))^* = \beta^2 h^*\circ(\beta^{-1}\,\mathrm{Id})$, 2, and (22), we obtain

$$\big(G_\lambda f\big)^* = \left((1+\lambda)^2\operatorname{pav}\left(f,\mathfrak{q};\frac{1-\lambda}{1+\lambda},\frac{2\lambda}{1+\lambda}\right)\circ(1+\lambda)^{-1}\,\mathrm{Id}\right)^* \tag{1.44}$$

$$= (1+\lambda)^2\left(\operatorname{pav}\left(f,\mathfrak{q};\frac{1-\lambda}{1+\lambda},\frac{2\lambda}{1+\lambda}\right)\right)^*\circ(1+\lambda)^{-1}\,\mathrm{Id} \tag{1.45}$$

$$= (1+\lambda)^2 \operatorname{pav}\left(f^*, \mathsf{q}^*; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right) \circ (1+\lambda)^{-1} \operatorname{Id} \tag{1.46}$$

$$= (1+\lambda)^2 \operatorname{pav}\left(f^*, \mathsf{q}; \frac{1-\lambda}{1+\lambda}, \frac{2\lambda}{1+\lambda}\right) \circ (1+\lambda)^{-1} \operatorname{Id} \tag{1.47}$$

$$= G_\lambda (f^*). \tag{1.48}$$

The proof is complete. ■

Remark 1.2.3 Theorem 2.2(1)&(2) gives two representations of the Goebel smoothing operator in terms of the proximal average. Goebel [10] discovered a converse formula, which we state next without proof:

$$\operatorname{pav}(f, \mathsf{q}; \lambda, 1-\lambda) = \frac{(2-\lambda)^2}{4} G_{\lambda/(2-\lambda)} f \circ \left(\frac{2}{2-\lambda} \operatorname{Id}\right). \tag{1.49}$$

Example 1.2.4 Let $\lambda \in \,]0,1[$ and set $f = \|\cdot\|$. Then, for every $x \in X$,

$$G_\lambda f(x) = \begin{cases} \dfrac{\|x\|^2}{2\lambda}, & \text{if } \|x\| \le \lambda; \\[2ex] \dfrac{\lambda\|x\|^2}{2} + (1-\lambda^2)\|x\| - \dfrac{\lambda(1-\lambda^2)}{2}, & \text{if } \|x\| > \lambda. \end{cases} \tag{1.50}$$

Proof. Combine (23) and (3). ■

1.3 A new smoothing operator

We now provide a novel smoothing operator that has a very simple expression in terms of the proximal average.

Definition 1.3.1 (new smoothing operator) *Let $f\colon X \to \,]-\infty, +\infty]$ be convex, lower semicontinuous and proper, and let $\lambda \in \,]0,1[$. Then the $S_\lambda f$ is defined by*

$$S_\lambda f = \operatorname{pav}(f, \mathsf{q}; 1-\lambda, \lambda). \tag{1.51}$$

Theorem 1.3.2 *Let $f\colon X \to \,]-\infty, +\infty]$ be convex, lower semicontinuous and proper, and let $\lambda \in \,]0,1[$. Set $\mu = \lambda/(2-\lambda)$. Then the following hold.*

1. $S_\lambda f = (1-\lambda) e_\mu f \circ \left(\frac{2}{2-\lambda} \operatorname{Id}\right) + \mu \mathsf{q}$.
2. $\left(S_\lambda f\right)^* = S_\lambda (f^*)$.

Proof. 1: Let $x \in X$. Then, using (51), (21) and standard convex calculus, we obtain

$$
\begin{aligned}
&\big(S_\lambda f\big)(x)\\
&= \big((1-\lambda)(f+\mathfrak{q})^* + \lambda(\mathfrak{q}+\mathfrak{q})^*\big)^*(x) - \mathfrak{q}(x) &(1.52)\\
&= \big((1-\lambda)(f+\mathfrak{q})^* + \tfrac{\lambda}{2}\mathfrak{q}\big)^*(x) - \mathfrak{q}(x) &(1.53)\\
&= \left((1-\lambda)(f+\mathfrak{q})\Big(\frac{\cdot}{1-\lambda}\Big) \,\Box\, \frac{2}{\lambda}\mathfrak{q}\right)(x) - \mathfrak{q}(x) &(1.54)\\
&= \inf_y \left((1-\lambda)f\Big(\frac{y}{1-\lambda}\Big) + (1-\lambda)\mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{2}{\lambda}\mathfrak{q}(x-y) - \mathfrak{q}(x)\right) &(1.55)\\
&= (1-\lambda)\inf_y \left(f\Big(\frac{y}{1-\lambda}\Big) + \mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{2}{\lambda(1-\lambda)}\mathfrak{q}(x-y) - \frac{1}{1-\lambda}\mathfrak{q}(x)\right). &(1.56)
\end{aligned}
$$

Simple algebra shows that for every $y \in X$,

$$
\mathfrak{q}\Big(\frac{y}{1-\lambda}\Big) + \frac{2}{\lambda(1-\lambda)}\mathfrak{q}(x-y) - \frac{1}{1-\lambda}\mathfrak{q}(x) = \frac{2-\lambda}{\lambda}\mathfrak{q}\Big(\frac{2x}{2-\lambda} - \frac{y}{1-\lambda}\Big) + \frac{\lambda}{(1-\lambda)(2-\lambda)}\mathfrak{q}(x). \quad (1.57)
$$

Therefore,

$$
\begin{aligned}
&\big(S_\lambda f\big)(x) &(1.58)\\
&= (1-\lambda)\inf_y \left(f\Big(\frac{y}{1-\lambda}\Big) + \frac{2-\lambda}{\lambda}\mathfrak{q}\Big(\frac{2x}{2-\lambda} - \frac{y}{1-\lambda}\Big) + \frac{\lambda}{(1-\lambda)(2-\lambda)}\mathfrak{q}(x)\right) &(1.59)\\
&= (1-\lambda)\inf_z \left(f(z) + \frac{2-\lambda}{\lambda}\mathfrak{q}\Big(\frac{2x}{2-\lambda} - z\Big)\right) + \frac{\lambda}{2-\lambda}\mathfrak{q}(x) &(1.60)\\
&= (1-\lambda)\Big(f \,\Box\, \frac{1}{\mu}\mathfrak{q}\Big)\Big(\frac{2x}{2-\lambda}\Big) + \mu\mathfrak{q}(x), &(1.61)
\end{aligned}
$$

as claimed.

2: Using (51) and (22), we get

$$\begin{aligned}\big(S_\lambda f\big)^* &= \big(\operatorname{pav}(f, \mathsf{q}\,; 1-\lambda, \lambda)\big)^* && (1.62)\\ &= \operatorname{pav}(f^*, \mathsf{q}^*\,; 1-\lambda, \lambda) && (1.63)\\ &= \operatorname{pav}(f^*, \mathsf{q}\,; 1-\lambda, \lambda) && (1.64)\\ &= S_\lambda\,(f^*). && (1.65)\end{aligned}$$

The proof is complete. ■

Note that Theorem 3.2(1) and standard properties of the Moreau envelope imply that point-wise

$$\lim_{\lambda \to 0^+} S_\lambda f = f \tag{1.66}$$

and that each $S_\lambda f$ is smooth.

Example 1.3.3 Let $\lambda \in\,]0,1[$ and set $f = \|\cdot\|$. Then, for every $x \in X$,

$$S_\lambda f(x) = \begin{cases} \dfrac{(2-\lambda)\|x\|^2}{2\lambda}, & \text{if } \|x\| \le \dfrac{\lambda}{2};\\[2ex] \dfrac{\lambda\|x\|^2}{2(2-\lambda)} + \dfrac{2(1-\lambda)}{2-\lambda}\|x\| - \dfrac{\lambda(1-\lambda)}{2(2-\lambda)}, & \text{if } \|x\| > \dfrac{\lambda}{2}. \end{cases} \tag{1.67}$$

Proof. Combine (3) and Theorem 3.2(1). ■

Remark 1.3.4 Let $f = \|\cdot\|$. The explicit formulas provided in Example 2.4 and Example 3.3 imply that $G_\alpha f \neq S_\beta f$, for *all* α and β in $]0,1[$. Thus, the smoothing operator defined by (51) is indeed new and different from the Goebel smoothing operator.

Remark 1.3.5 It would be desirable to obtain further explicit formulas beyond the example of the norm. Given a more complicated function f, the explicit computation of the smoothing operators $G_\lambda f$ and $S_\lambda f$ may not be easy. However, computational convex analysis provides tools [8, 13, 14] to compute the Moreau envelope numerically which — due to the Moreau envelope formulations (23) and Theorem 3.2(1) — make it possible to compute the smoothing operators $G_\lambda f$ and $S_\lambda f$ numerically. It would also be interesting to extend the present results to infinite-dimensional settings. Promising starting points for this endeavour are [1, 21]. Finally, self-dual regularizations of maximal monotone operators are studied in [20].

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