Chapter 2
New Demiclosedness Principles for (firmly) nonexpansive operators

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Dedicated to Jonathan Borwein on the occasion of his 60th Birthday

Summary: The demiclosedness principle is one of the key tools in nonlinear analysis and fixed point theory. In this note, this principle is extended and made more flexible by two mutually orthogonal affine subspaces. Versions for finitely many (firmly) nonexpansive operators are presented. As an application, a simple proof of the weak convergence of the Douglas-Rachford splitting algorithm is provided.

Key words: Demiclosedness principle, Douglas-Rachford algorithm, firmly nonexpansive mapping, maximal monotone operator, nonexpansive mapping, proximal algorithm, resolvent, splitting algorithm.
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2.1 Introduction

Throughout this paper, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle \text{ and induced norm } \|\cdot\|. \tag{2.1}$$

We shall assume basic notation and results from Fixed Point Theory and from Monotone Operator Theory; see, e.g., [2, 4, 8, 15, 16, 20, 21, 22, 24]. The *graph* of a maximally monotone operator $A\colon X \rightrightarrows X$ is denoted by $\operatorname{gra} A$, its *resolvent* $(A+\mathrm{Id})^{-1}$ by J_A, its set of zeros by $\operatorname{zer} A = A^{-1}(0)$, and we set $R_A = 2J_A - \mathrm{Id}$, where Id is the identity operator. Weak convergence is indicated by $\rightharpoonup$.

Let $T\colon X \to X$. Recall that T is *firmly nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2 \le \|x-y\|^2. \tag{2.2}$$

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It is well know that T is firmly nonexpansive if and only if $R = 2T - \mathrm{Id}$ is *nonexpansive*, i.e.,

$$(\forall x \in X)(\forall y \in X) \quad \|Rx - Ry\| \leq \|x - y\|. \tag{2.3}$$

Clearly, every firmly nonexpansive operator is nonexpansive. Building on work by Minty [19], Eckstein and Bertsekas [13] clearly linked firmly nonexpansive mappings to maximally monotone operators—the key result is the following: T is firmly nonexpansive if and only if $T = J_A$ for some maximally monotone operator A (namely, $T^{-1} - \mathrm{Id}$). This implies also a correspondence between maximally monotone operators and nonexpansive mappings (see [14] and [17]). Thus, finding a zero of A is equivalent to finding a fixed point of J_A. Furthermore, the graph of any maximally monotone operator is beautifully described by the associated *Minty parametrization*:

$$\mathrm{gra} A = \big\{(J_A x, x - J_A x) \mid x \in X\big\}. \tag{2.4}$$

The most prominent example of firmly nonexpansive mappings are projectors, i.e., resolvents of normal cone operators associated with nonempty closed convex subsets of X. Despite being (firmly) nonexpansive and hence Lipschitz continuous, even projectors do not interact well with the weak topology as was first observed by Zarantonello [25]:

Example 2.1. Suppose that $X = \ell_2(\mathbb{N})$, set $C = \big\{x \in X \mid \|x\| \leq 1\big\}$, and denote the sequence of standard unit vectors in X by $(e_n)_{n\in\mathbb{N}}$. Set $(\forall n \in \mathbb{N})$ $z_n = e_0 + e_n$. Then

$$z_n \rightharpoonup e_0 \quad \text{yet} \quad P_C z_n \rightharpoonup \tfrac{1}{\sqrt{2}} e_0 \neq e_0 = P_C e_0. \tag{2.5}$$

The following classical demiclosedness principle dates back to the 1960s and work by Browder [6]. It comes somewhat as a surprise in view of the previous example.

Fact 2.2 (Demiclosedness Principle). Let S be a nonempty closed convex subset of X, let $T\colon S \to X$ be nonexpansive, let $(z_n)_{n\in\mathbb{N}}$ be a sequence in S converging weakly to z, and suppose that $z_n - Tz_n \to x$. Then $z - Tz = x$.

Remark 2.3. One might inquire whether or not the following even less restrictive demiclosedness principle holds:

$$\left.\begin{array}{r} z_n \rightharpoonup z \\ z_n - Tz_n \rightharpoonup x \end{array}\right\} \overset{?}{\Rightarrow} \; z - Tz = x. \tag{2.6}$$

However, this is generalization is false: indeed, suppose that X, C, and $(z_n)_{n\in\mathbb{N}}$ are as in Example 2.1, and set $T = \mathrm{Id} - P_C$, which is (even firmly) nonexpansive. Then $z_n \rightharpoonup e_0$ and $z_n - Tz_n = P_C z_n \rightharpoonup \frac{1}{\sqrt{2}} e_0$ yet $e_0 - Te_0 = P_C e_0 = e_0 \neq \frac{1}{\sqrt{2}} e_0$.

The aim of this note is to provide new versions of the demiclosedness principle and illustrate their usefulness. The remainder of this paper is organized as follows. Section 2.2 presents new demiclosedness principles for one (firmly) nonexpansive operator. Multi-operator versions are provided in Section 2.3. The weak convergence of the Douglas-Rachford algorithm is rederived with a very transparent proof in Section 2.4.

2.2 Demiclosedness Principles

Fact 2.4 (Brezis). (See [5, Proposition 2.5 on page 27], [23, Lemma 4], or [2, Corollary 20.49].) Let $A\colon X \rightrightarrows X$ be maximally monotone, let $(x,u) \in \operatorname{gra} A$, and let $(x_n,u_n)_{n\in\mathbb{N}}$ be a sequence in $X \times X$ such that $(x_n,u_n) \rightharpoonup (x,u)$ and $\overline{\lim}\langle x_n,u_n\rangle \leq \langle x,u\rangle$. Then $\langle x_n,u_n\rangle \to \langle x,u\rangle$ and $(x,u) \in \operatorname{gra} A$.

Theorem 2.5. (See also [2, Proposition 20.50].) Let $A\colon X \rightrightarrows X$ be maximally monotone, let $(x,u) \in X \times X$, and let C and D be closed affine subspaces of X such that $D-D = (C-C)^{\perp}$. Furthermore, let $(x_n,u_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{gra} A$ such that

$$(x_n,u_n) \rightharpoonup (x,u) \quad \text{and} \quad (x_n,u_n) - P_{C\times D}(x_n,u_n) \to (0,0). \tag{2.7}$$

Then $(x,u) \in (C\times D)\cap \operatorname{gra} A$ and $\langle x_n,u_n\rangle \to \langle x,u\rangle$.

Proof. Set $V = C-C$, which is a closed linear subspace. Since $x_n - P_C x_n \to 0$, we have $P_C x_n \rightharpoonup x$ and thus $x \in C$. Likewise, $u \in D$ and hence

$$C = x+V \quad \text{and} \quad D = u+V^{\perp}. \tag{2.8}$$

It follows that

$$P_C\colon z \mapsto P_V z + P_{V^\perp} x \quad \text{and} \quad P_D\colon z \mapsto P_{V^\perp} z + P_V u. \tag{2.9}$$

Therefore, since P_V and $P_{V^\perp}$ are weakly continuous,

$$\begin{aligned}
\langle x_n,u_n\rangle &= \langle P_V x_n + P_{V^\perp} x_n, P_V u_n + P_{V^\perp} u_n\rangle && \text{(2.10a)}\\
&= \langle P_V x_n, P_V u_n\rangle + \langle P_{V^\perp} x_n, P_{V^\perp} u_n\rangle && \text{(2.10b)}\\
&= \langle P_V x_n, u_n - P_{V^\perp} u_n\rangle + \langle x_n - P_V x_n, P_{V^\perp} u_n\rangle && \text{(2.10c)}\\
&= \langle P_V x_n, u_n - (P_D u_n - P_V u)\rangle && \text{(2.10d)}\\
&\qquad + \langle x_n - (P_C x_n - P_{V^\perp} x), P_{V^\perp} u_n\rangle && \text{(2.10e)}\\
&= \langle P_V x_n, u_n - P_D u_n\rangle + \langle P_V x_n, P_V u\rangle && \text{(2.10f)}\\
&\qquad + \langle x_n - P_C x_n, P_{V^\perp} u_n\rangle + \langle P_{V^\perp} x, P_{V^\perp} u_n\rangle && \text{(2.10g)}\\
&\to \langle P_V x, P_V u\rangle + \langle P_{V^\perp} x, P_{V^\perp} u\rangle && \text{(2.10h)}\\
&= \langle x,u\rangle. && \text{(2.10i)}
\end{aligned}$$

The result now follows from Fact 2.4. ■

Remark 2.6. Theorem 2.5 generalizes [1, Theorem 2], which corresponds to the case when C is a closed linear subspace and $D = C^{\perp}$. A referee pointed out that

Theorem 2.5 may be obtained from [1, Theorem 2] by a translation argument. However, the above proof of Theorem 2.5 is different and *much simpler* than the proof of [1, Theorem 2].

Corollary 2.7 (firm nonexpansiveness principle). Let $F\colon X \to X$ be firmly nonexpansive, let $(z_n)_{n\in\mathbb{N}}$ be a sequence in X such that $(z_n)_{n\in\mathbb{N}}$ converges weakly to $z \in X$, suppose that $Fz_n \rightharpoonup x \in X$, and that C and D are closed affine subspaces of X such that $D-D = (C-C)^{\perp}$, $Fz_n - P_C Fz_n \to 0$, $(z_n - Fz_n) - P_D(z_n - Fz_n) \to 0$. Then $x \in C$, $z \in x+D$, and $x = Fz$.

Proof. Set $A = F^{-1} - \mathrm{Id}$ so that $J_A = F$. By (2.4), A is maximally monotone and

$$(x_n, u_n)_{n\in\mathbb{N}} := (Fz_n, z_n - Fz_n)_{n\in\mathbb{N}} \tag{2.11}$$

is a sequence in gra$A$ that converges weakly to $(x, z-x)$. Thus, by Theorem 2.5, $x \in C$, $z - x \in D$, and $z - x \in Ax$. Therefore, $z \in x + Ax$, i.e., $x = J_A z = Fz$. ■

Corollary 2.8 (nonexpansiveness principle). Let $T\colon X \to X$ be nonexpansive, let $(z_n)_{n\in\mathbb{N}}$ be a sequence in X such that $z_n \rightharpoonup z$, suppose that $Tz_n \rightharpoonup y$, and that C and D are closed affine subspaces of X such that $D - D = (C-C)^{\perp}$, $z_n + Tz_n - P_C z_n - P_C T z_n \to 0$, and $z_n - Tz_n - P_D z_n - P_D(-Tz_n) \to 0$. Then $\frac{1}{2}z + \frac{1}{2}y \in C$, $\frac{1}{2}z - \frac{1}{2}y \in D$, and $y = Tz$.

Proof. Set $F = \frac{1}{2}\,\mathrm{Id} + \frac{1}{2}T$, which is firmly nonexpansive. Then $Fz_n \rightharpoonup \frac{1}{2}z + \frac{1}{2}y =: x$. Since P_C is affine, we get

$$z_n + Tz_n - P_C z_n - P_C T z_n \to 0 \tag{2.12a}$$
$$\Leftrightarrow z_n + Tz_n - 2\big(\tfrac{1}{2}P_C z_n + \tfrac{1}{2}P_C T z_n\big) \to 0 \tag{2.12b}$$
$$\Leftrightarrow z_n + Tz_n - 2P_C\big(\tfrac{1}{2}z_n + \tfrac{1}{2}Tz_n\big) \to 0 \tag{2.12c}$$
$$\Leftrightarrow 2Fz_n - 2P_C F z_n \to 0 \tag{2.12d}$$
$$\Leftrightarrow Fz_n - P_C F z_n \to 0. \tag{2.12e}$$

Likewise, since $z_n - Fz_n = z_n - \frac{1}{2}z_n - \frac{1}{2}Tz_n = \frac{1}{2}z_n - \frac{1}{2}Tz_n$, we have

$$z_n - Tz_n - P_D z_n - P_D(-Tz_n) \to 0 \tag{2.13a}$$
$$\Leftrightarrow z_n - Tz_n - 2\big(\tfrac{1}{2}P_D z_n + \tfrac{1}{2}P_D(-Tz_n)\big) \to 0 \tag{2.13b}$$
$$\Leftrightarrow 2(z_n - Fz_n) - 2P_D\big(\tfrac{1}{2}z_n + \tfrac{1}{2}(-Tz_n)\big) \to 0 \tag{2.13c}$$
$$\Leftrightarrow z_n - Fz_n - P_D(z_n - Fz_n) \to 0. \tag{2.13d}$$

Thus, by Corollary 2.7, $x \in C$, $z \in x + D$, and $x = Fz$, i.e., $\frac{1}{2}z + \frac{1}{2}y \in C$, $z \in \frac{1}{2}z + \frac{1}{2}y + D$, and $\frac{1}{2}z + \frac{1}{2}y = Fz = \frac{1}{2}z + \frac{1}{2}Tz$, i.e., $\frac{1}{2}z + \frac{1}{2}y \in C$, $\frac{1}{2}z - \frac{1}{2}y \in D$, and $y = Tz$. ■

Corollary 2.9 (classical demiclosedness principle). Let S be a nonempty closed convex subset of X, let $T\colon S \to X$ be nonexpansive, let $(z_n)_{n\in\mathbb{N}}$ be a sequence in S converging weakly to z, and suppose that $z_n - Tz_n \to x$. Then $z - Tz = x$.

Proof. We may and do assume that $S = X$ (otherwise, consider $T \circ P_S$ instead of T). Set $y = z - x$ and note that $Tz_n \rightharpoonup y$. Now set $C = X$ and $D = \{x/2\}$. Then $D - D = \{0\} = X^\perp = (X - X)^\perp = (D - D)^\perp$, $z_n + Tz_n - P_C z_n - P_C T z_n \equiv 0$ and $z_n - Tz_n - P_D z_n - P_D(-Tz_n) = z_n - Tz_n - x/2 - x/2 \to 0$. Corollary 2.8 implies $y = Tz$, i.e., $z - x = Tz$. ■

2.3 Multi-Operator Demiclosedness Principles

Set

$$I = \{1, 2, \ldots, m\}, \quad \text{where } m \text{ is an integer greater than or equal to 2.} \tag{2.14}$$

We shall work in the product Hilbert space

$$\mathbf{X} = X^I \tag{2.15}$$

with induced inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$ and $\|\mathbf{x}\| = \sqrt{\sum_{i \in I} \|x_i\|^2}$, where $\mathbf{x} = (x_i)_{i \in I}$ and $\mathbf{y} = (y_i)_{i \in I}$ denote generic elements in $\mathbf{X}$.

We start with a multi-operator demiclosedness principle for firmly nonexpansive mappings, which we derive from the corresponding two-operator version (Corollary 2.7). A referee pointed out that Theorem 2.10 is also equivalent to [1, Corollary 3] (see also [23, Lemma 5] for a Banach space extension of [1, Corollary 3]).

Theorem 2.10 (Multi-Operator Demiclosedness Principle for Firmly Nonexpansive Operators). Let $(F_i)_{i \in I}$ be a family of firmly nonexpansive operators on X, and let, for each $i \in I$, $(z_{i,n})_{n \in \mathbb{N}}$ be a sequence in X such that for all i and j in I,

$$z_{i,n} \rightharpoonup z_i \ \text{ and } \ F_i z_{i,n} \rightharpoonup x \tag{2.16a}$$

$$\sum_{i \in I} (z_{i,n} - F_i z_{i,n}) \to -mx + \sum_{i \in I} z_i \tag{2.16b}$$

$$F_i z_{i,n} - F_j z_{j,n} \to 0. \tag{2.16c}$$

Then $F_i z_i = x$, for every $i \in I$.

Proof. Set $\mathbf{x} = (x)_{i \in I}$, $\mathbf{z} = (z_i)_{i \in I}$, $(\mathbf{z}_n) = (z_{i,n})_{n \in \mathbb{N}}$, and $\mathbf{C} = \big\{(y)_{i \in I} \mid y \in X\big\}$. Then $\mathbf{z}_n \rightharpoonup \mathbf{z}$ and $\mathbf{C}$ is a closed subspace of $\mathbf{X}$ with $\mathbf{C}^\perp = \big\{(y_i)_{i \in I} \mid \sum_{i \in I} y_i = 0\big\}$. Furthermore, we set $\mathbf{D} = \mathbf{z} - \mathbf{x} + \mathbf{C}^\perp$ so that $(\mathbf{C} - \mathbf{C})^\perp = \mathbf{C}^\perp = \mathbf{D} - \mathbf{D}$, and also $\mathbf{F} \colon (y_i)_{i \in I} \mapsto (F y_i)_{i \in I}$. Then $\mathbf{F}$ is firmly nonexpansive on $\mathbf{X}$, and $\mathbf{F}\mathbf{z}_n \rightharpoonup \mathbf{x}$. Now (2.16c) implies

$$(\forall i \in I) \quad F_i z_{i,n} - \frac{1}{m} \sum_{j \in I} F_j z_{j,n} \to 0, \tag{2.17}$$

which—when viewed in $\mathbf{X}$—means that $\mathbf{F}\mathbf{z}_n - P_{\mathbf{C}} \mathbf{F}\mathbf{z}_n \to 0$. Similarly, using (2.16b),

$$\mathbf{z}_n - \mathbf{F}\mathbf{z}_n - P_{\mathbf{D}}(\mathbf{z}_n - \mathbf{F}\mathbf{z}_n) = \mathbf{z}_n - \mathbf{F}\mathbf{z}_n - P_{\mathbf{z} - \mathbf{x} + \mathbf{C}^\perp}(\mathbf{z}_n - \mathbf{F}\mathbf{z}_n) \tag{2.18a}$$

$$
\begin{aligned}
&= \mathbf{z}_n - \mathbf{F}\mathbf{z}_n - \big(\mathbf{z} - \mathbf{x} + P_{\mathbf{C}^\perp}\big(\mathbf{z}_n - \mathbf{F}\mathbf{z}_n - (\mathbf{z} - \mathbf{x})\big)\big) && (2.18b)\\
&= \big(\operatorname{Id} - P_{\mathbf{C}^\perp}\big)(\mathbf{z}_n - \mathbf{F}\mathbf{z}_n) - \big(\operatorname{Id} - P_{\mathbf{C}^\perp}\big)(\mathbf{z} - \mathbf{x}) && (2.18c)\\
&= P_{\mathbf{C}}(\mathbf{z}_n - \mathbf{F}\mathbf{z}_n) - P_{\mathbf{C}}(\mathbf{z} - \mathbf{x}) && (2.18d)\\
&= \Big(\tfrac{1}{m}\sum_{i\in I}\big(z_{i,n} - F_i z_{i,n} - z_i + x\big)\Big)_{j\in I} && (2.18e)\\
&\to 0. && (2.18f)
\end{aligned}
$$

Therefore, by Corollary 2.7, $\mathbf{x} = \mathbf{F}\mathbf{z}$. ■

Theorem 2.11 (Multi-Operator Demiclosedness Principle for Nonexpansive Operators). Let $(T_i)_{i\in I}$ be a family of nonexpansive operators on X, and let, for each $i \in I$, $(x_{i,n})_{n\in\mathbb{N}}$ be a sequence in X such that for all i and j in I,

$$
\begin{aligned}
& z_{i,n} \rightharpoonup z_i \ \text{ and } \ T_i z_{i,n} \rightharpoonup y_i && (2.19a)\\
& \sum_{i\in I}\big(z_{i,n} - T_i z_{i,n}\big) \to \sum_{i\in I}\big(z_i - y_i\big) && (2.19b)\\
& z_{i,n} - z_{j,n} + T_i z_{i,n} - T_j z_{j,n} \to 0. && (2.19c)
\end{aligned}
$$

Then $T_i z_i = y_i$, for each $i \in I$.

Proof. Set $(\forall i \in I)$ $F_i = \frac{1}{2}\operatorname{Id} + \frac{1}{2}T_i$. Then F_i is firmly nonexpansive and $F_i z_{i,n} \rightharpoonup \frac{1}{2}z_i + \frac{1}{2}y_i$, for every $i \in I$. By (2.19c), $0 \leftarrow 2F_i z_{i,n} - 2F_j z_{j,n} = (z_{i,n} + T_i z_{i,n}) - (z_{j,n} + T_j z_{j,n}) \rightharpoonup (z_i + y_i) - (z_j + y_j)$, for all i and j in I. It follows that $x = \frac{1}{2}z_i + \frac{1}{2}y_i$ is *independent* of $i \in I$. Furthermore,

$$
\begin{aligned}
\sum_{i\in I}\big(z_{i,n} - F_i z_{i,n}\big) &= \sum_{i\in I}\tfrac{1}{2}\big(z_{i,n} - T_i z_{i,n}\big) && (2.20a)\\
&\to \sum_{i\in I}\tfrac{1}{2}\big(z_i - y_i\big) && (2.20b)\\
&= \sum_{i\in I}\big(\tfrac{1}{2}z_i - \big(x - \tfrac{1}{2}z_i\big)\big) && (2.20c)\\
&= -mx + \sum_{i\in I} z_i. && (2.20d)
\end{aligned}
$$

Therefore, the conclusion follows from Theorem 2.10. ■

2.4 Application to Douglas-Rachford splitting

In this section, we assume that A and B are maximally monotone operators on X such that

$$\operatorname{zer}(A+B) = (A+B)^{-1}(0) \neq \varnothing. \tag{2.21}$$

We set

$$T = \frac{1}{2}\,\mathrm{Id} + \frac{1}{2}R_B R_A = J_B(2J_A - \mathrm{Id}) + (\mathrm{Id} - J_A), \tag{2.22}$$

which is the Douglas-Rachford splitting operator and where $R_A = 2J_A - \mathrm{Id}$ and $R_B = 2J_B - \mathrm{Id}$ are the "reflected resolvents" already considered in Section 2.1. (The term "reflected resolvent" is motivated by the fact that when J_A is a projection operator, then R_A is the corresponding reflection.) See [2], [10] and [11] for further information on this algorithm; and also [3] for some results for operators that are not maximally monotone. One has (see [10, Lemma 2.6(iii)] or [2, Proposition 25.1(ii)])

$$J_A\big(\mathrm{Fix}\,T\big) = \mathrm{zer}(A+B). \tag{2.23}$$

Now let $z_0 \in X$ and define the sequence $(z_n)_{n\in\mathbb{N}}$ by

$$(\forall n \in \mathbb{N}) \quad z_{n+1} = Tz_n. \tag{2.24}$$

This sequence is very useful in determining a zero of $A+B$ as the next result illustrates.

Fact 2.12 (Lions-Mercier). [18] The sequence $(z_n)_{n\in\mathbb{N}}$ converges weakly to some point $z \in X$ such that $z \in \mathrm{Fix}\,T$ and $J_A z \in \mathrm{zer}(A+B)$. Moreover, the sequence $(J_A z_n)_{n\in\mathbb{N}}$ is bounded, and every weak cluster point of this sequence belongs to $\mathrm{zer}(A+B)$.

Since J_A is in general *not* sequentially weakly continuous (see Example 2.1), it is not obvious whether or not $J_A z_n \rightharpoonup J_A z$. However, recently Svaiter provided a relatively complicated proof that in fact weak convergence does hold. As an application, we rederive the most fundamental instance of his result with a considerably simpler and more conceptual proof.

Fact 2.13 (Svaiter). [23] The sequence $(J_A z_n)_{n\in\mathbb{N}}$ converges weakly to $J_A z$.

Proof. By Fact 2.12,

$$z_n \rightharpoonup z \in \mathrm{Fix}\,T. \tag{2.25}$$

Since J_A is (firmly) nonexpansive and $(z_n)_{n\in\mathbb{N}}$ is bounded, the sequence $(J_A z_n)_{n\in\mathbb{N}}$ is bounded as well. Let x be an arbitrary weak cluster point of $(J_A z_n)_{n\in\mathbb{N}}$, say

$$J_A z_{k_n} \rightharpoonup x \in \mathrm{zer}(A+B) \tag{2.26}$$

by Fact 2.12. Set $(\forall n \in \mathbb{N})\ y_n = R_A z_n$. Then

$$y_{k_n} \rightharpoonup y = 2x - z \in X. \tag{2.27}$$

Since the operator T is firmly nonexpansive and $\mathrm{Fix}\,T \neq \varnothing$, it follows from [7] that $z_n - Tz_n \to 0$ (i.e., T is "asymptotically regular"); thus,

$$J_A z_n - J_B y_n = z_n - Tz_n \to 0 \tag{2.28}$$

and hence

$$J_B y_{k_n} \rightharpoonup x. \tag{2.29}$$

Next,

$$\begin{aligned} 0 &\leftarrow J_A z_{k_n} - J_B y_{k_n} && (2.30a)\\ &= z_{k_n} - J_A z_{k_n} + R_A z_{k_n} - J_B y_{k_n} && (2.30b)\\ &= z_{k_n} - J_A z_{k_n} + y_{k_n} - J_B y_{k_n} && (2.30c)\\ &\rightharpoonup z + y - 2x. && (2.30d) \end{aligned}$$

To summarize,

$$\begin{aligned} (z_{k_n}, y_{k_n}) \rightharpoonup (z,y) \quad \text{and} \quad (J_A z_{k_n}, J_B y_{k_n}) &\rightharpoonup (x,x), && (2.31a)\\ (z_{k_n} - J_A z_{k_n}) + (y_{k_n} - J_B y_{k_n}) &\to -2x + z + y = 0, && (2.31b)\\ J_A z_{k_n} - J_B y_{k_n} &\to 0. && (2.31c) \end{aligned}$$

By Theorem 2.10, $J_A z = J_B y = x$. Hence $J_A z_{k_n} \rightharpoonup J_A z$. Since x was an arbitrary weak cluster point of the bounded sequence $(J_A z_n)_{n\in\mathbb{N}}$, we conclude that $J_A z_n \rightharpoonup J_A z$. ■

Motivated by a referee's comment, let us turn towards inexact iterations of T. The following result underlines the usefulness of the multi-operator demiclosedness principle.

Theorem 2.14. Suppose that $(z_n)_{n\in\mathbb{N}}$ is a sequence in X such that $z_n - Tz_n \to 0$ and $z_n \rightharpoonup z$, where $z \in \operatorname{Fix} T$. Then $J_A z_n \rightharpoonup J_A z$.

Proof. Argue exactly as in the proof of Fact 2.13. ■

We now present a prototypical result on inexact iterations; see [9], [10], [11], [13], and [23] for many more results in this direction as well as [2] and also [12].

Corollary 2.15. Suppose that $(z_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ are sequences in X such that

$$\sum_{n\in\mathbb{N}} \|e_n\| < +\infty \qquad \text{and} \qquad (\forall n \in \mathbb{N}) \quad z_{n+1} = e_n + Tz_n. \tag{2.32}$$

Then there exists $z \in \operatorname{Fix} T$ such that $z_n \rightharpoonup z$ and $J_A z_n \rightharpoonup J_A z$.

Proof. Combettes' [9, Proposition 4.2(ii)] yields $z_n - Tz_n \to 0$ while the existence of $z \in \operatorname{Fix} T$ such that $z_n \rightharpoonup z$ is guaranteed by his [9, Theorem 5.2(i)]. Now apply Theorem 2.14. ■

Unfortunately, the author is unaware of any existing actual numerical implementation guaranteeing summable errors; however, these theoretical results certainly increase confidence in the numerical stability of the Douglas-Rachford algorithm.

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