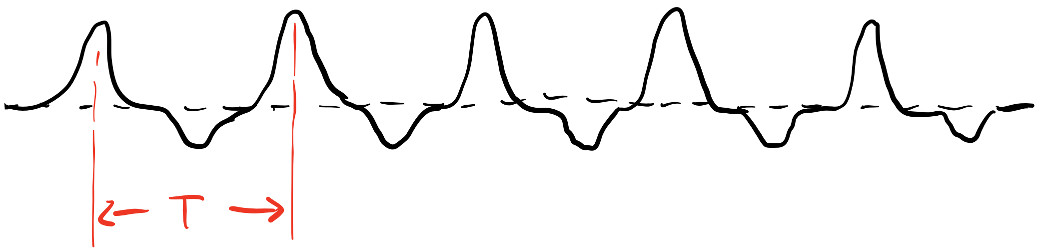
PHYS 232 January 12, 2024

Last Time:



<u>Claim</u>:

Any periodic fcn $f(x)$ can be expressed as an infinite sum of sines & cosines:

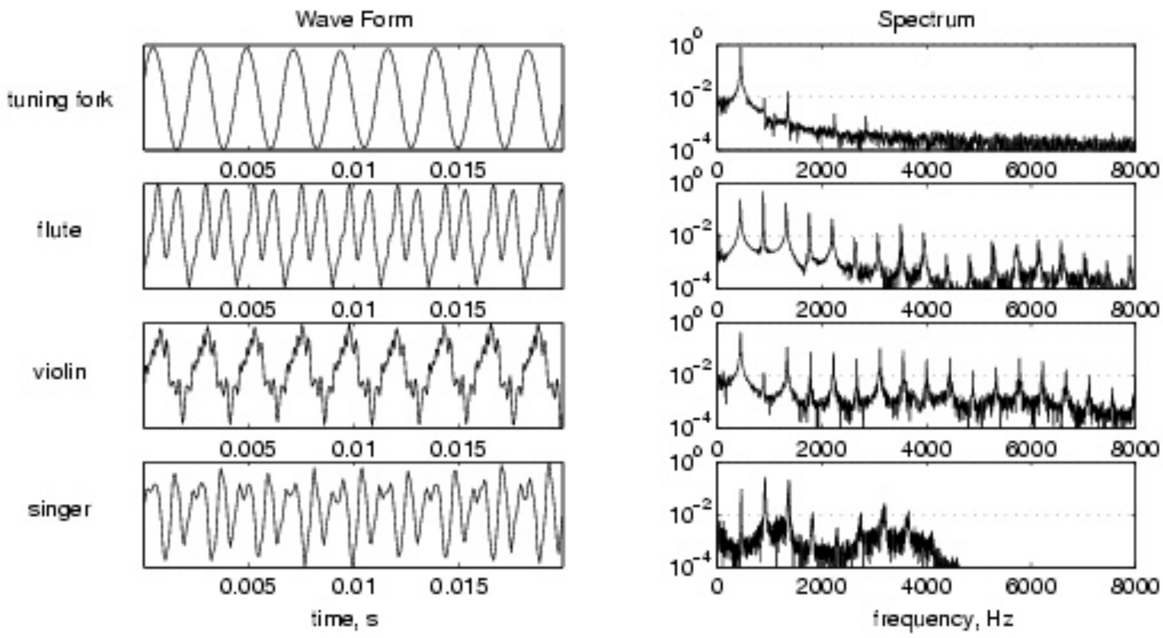
$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

Need to find the a_0, a_n, & b_n coefficients.

Waveforms of various instruments.

Taken from:
https://amath.colorado.edu/pub/matlab/music



waveforms (pressure vs time signals) from various instruments playing the same note.

"frequency spectrum" of the instrument recordings. Shows at which frequencies the air pressure is oscillating and the relative strengths of the various frequency components.

Fourier Series: $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots$

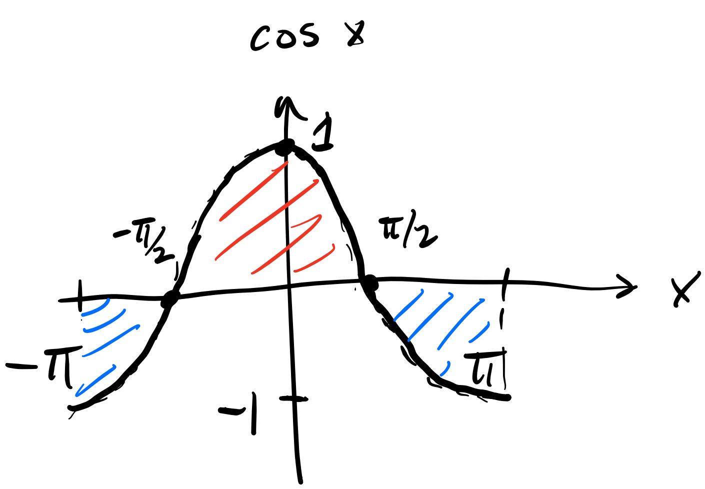
$+ b_1 \sin x + b_2 \sin 2x + \dots$

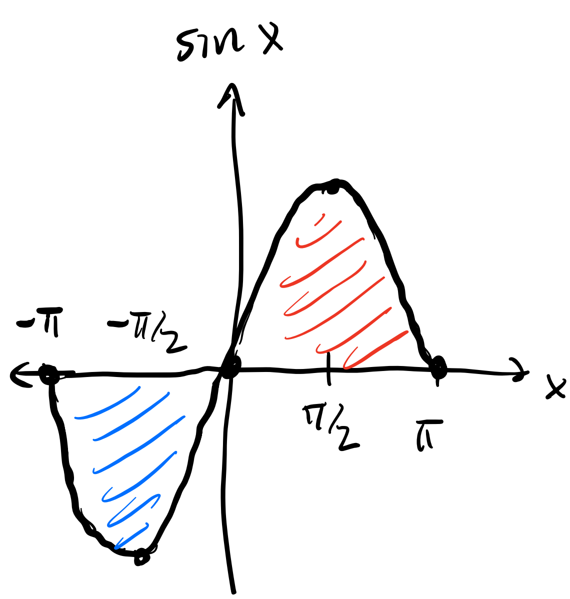
To find a_0, let's try evaluating the following integral:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \cancelto{0}{\int_{-\pi}^{\pi} a_1 \cos x\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} a_2 \cos 2x\,dx} + \dots \right.$$

$$\left. \cancelto{0}{\int_{-\pi}^{\pi} b_1 \sin x\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} b_2 \sin 2x\,dx} + \dots \right]$$





Only the first term in the integral survives:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\,\frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi} dx}_{2\pi} = a_0$$

$\therefore$ The a_0 coefficient is determined from

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Claim #2

$$* \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

when $n = 1, 2, 3, \ldots$

When we sub in $f(x) = \frac{a_0}{2} + \sum \left(a_n \cos nx + b_n \sin nx\right)$

we end up having to evaluate integrals like

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx$$

Consider $\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin mx \cos nx \, dx$

Easiest to evaluate if we express the trig fcns as complex exponentials.

Recall Euler's eq'n $e^{\pm jx} = \cos x \pm j \sin x$

$\therefore e^{jx} + e^{-jx} = 2\cos x$

$\therefore \cos x = \frac{e^{jx} + e^{-jx}}{2}$

Likewise:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{jmx} - e^{-jmx}}{2j}\right)\left(\frac{e^{jnx} + e^{-jnx}}{2}\right) dx$$

If we multiply out all the exponential terms, we get 4 terms of the form

e^{jkx} where k is an integer

$k = m+n, \ m-n, \ n-m, \ -n-m$

$m \neq n$ case.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jkx} dx = \frac{1}{2\pi}\frac{1}{jk} e^{jkx}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\frac{1}{jk}\Big[\underbrace{e^{jk\pi} - e^{-jk\pi}}_{2j \sin k\pi = 0}\Big] = 0$$

$m = n = 0$ case $\quad \sin mx = 0$
$\cos nx = 1$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

$m = n \neq 0$ case.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left(\frac{e^{jnx} - e^{-jnx}}{2j}\right)}_{\sin mx \text{ when } m=n} \underbrace{\left(\frac{e^{jnx} + e^{-jnx}}{2}\right)}_{\cos nx} dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left(\frac{e^{2jnx} \cancel{+1} \cancel{-1} - e^{-2jnx}}{4j}\right)}_{\frac{\sin 2nx}{2}} dx$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \sin 2nx \, dx = 0$$

since $\sin 2nx$ is symmetric about x-axis on $-\pi < x < \pi$.

(1) $$\boxed{\therefore\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \forall\, m, n}$$

Can likewise show that:

(2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \neq 0 \\ \frac{1}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

(3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \neq 0 \\ \frac{1}{2} & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$$

(2) & (3) homework.

Try $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\Big[\cancel{\int_{-\pi}^{\pi} \frac{a_0}{2}\cos nx\,dx} \to 0 \quad + \cancel{\int_{-\pi}^{\pi} a_1 \cos x \cos nx\,dx} \to 0 \text{ by } ③$$

$$+ \cancel{\int_{-\pi}^{\pi} a_2 \cos 2x \cos nx\,dx} \to 0 \text{ by } ③ \quad + \cdots$$

$a_n 2\pi\left(\frac{1}{2}\right)$ by ③

$$+ \boxed{\int_{-\pi}^{\pi} a_n \cos nx \cos nx\,dx} \quad \cancel{+ \cdots} \to 0$$

$$+ \cancel{\int_{-\pi}^{\pi} b_1 \sin x \cos nx\,dx} \to 0 \text{ by } ① \quad + \cancel{\int_{-\pi}^{\pi} b_2 \sin 2x \cos nx\,dx} \to 0 \text{ by } ①$$

$$+ \cdots + \cancel{\int_{-\pi}^{\pi} b_n \sin nx \cos nx\,dx} \to 0 \text{ by } ① \quad \cancel{+ \cdots} \to 0 \Big]$$

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n} = \frac{1}{\cancel{\pi}}\left[\cancel{\pi}\, a_n\right]$$