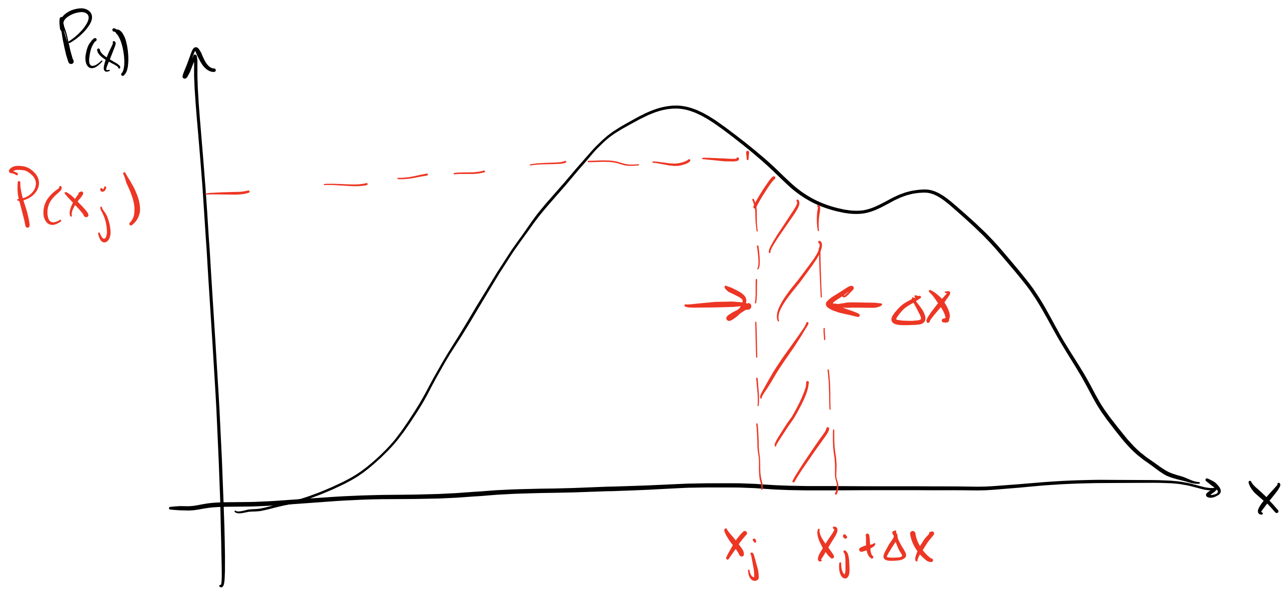
PHYS 232 Jan. 24, 2024

Last Time:

discrete distributions	continuous distributions
no. of possible outcomes $\rightarrow n$: $\sum_{j=1}^{n} P(x_j) = 1$	$\int_{\text{all } x} P(x)\,dx = 1$ $[P(x)] = \frac{1}{[x]}$
$\mu = \sum_{j=1}^{n} x_j P(x_j)$	$\mu = \int_{\text{all } x} x\, P(x)\,dx$
$\sigma^2 = \sum_{j=1}^{n} (x_j - \mu)^2 P(x_j)$	$\sigma^2 = \int_{\text{all } x} (x-\mu)^2 P(x)\,dx$
$= \sum_{j=1}^{n} x_j^2 P(x_j) - \mu^2$	$= \int_{\text{all } x} x^2 P(x)\,dx - \mu^2$

Recall that if a quantity x follows a prob. dist'n $P(x)$, then prob. of meas x between x_j & $x_j + \Delta x$ is:

$$P(x_j)\Delta x = \text{shaded area}$$



Try an example w/ discrete dist'n.

Role a pair of dice

$D_1 = 1, 2, 3, 4, 5, 6$

$D_2 = 1, 2, 3, 4, 5, 6$

outcome ① If $|D_1 - D_2| \geq 3$

then $M = \$1$.

outcome ② If $1 \leq |D_1 - D_2| \leq 2$

then $M = \$2$

outcome ③ If $D_1 = D_2$

then $M = \$3$

Have to pay \$1.75 for each play.
Is it worth it to play?

If $\langle M \rangle > \$1.75$ should play.

$$\langle M \rangle = \sum_{i=1}^{n=3} M_i P(M_i)$$

Need to determine $P(M_i)$.

$D_1 \backslash D_2$	1	2	3	4	5	6
6	1	1	1	2	2	3
5	1	1	2	2	3	2
4	1	2	2	3	2	2
3	2	2	3	2	2	1
2	2	3	2	2	1	1
1	3	2	2	1	1	1

36 entries

6 ways to get $M_3 = \$3$ $\quad P(M_3) = \frac{6}{36}$

$= \frac{1}{6}$

18 ways to get $M_2 = \$2$ $\quad P(M_2) = \frac{18}{36}$

$P(M_2) = \frac{1}{2}$

12 ways to get $M_i = \$1$ $P(M_1) = \frac{12}{36}$

$= \frac{1}{3}$

$$\langle M \rangle = \sum_{i=1}^{n=3} M_i P(M_i)$$

$$= \$1 \frac{1}{3} + \$2 \frac{1}{2} + \$3 \frac{1}{6}$$

$$= \frac{2+6+3}{6} = \frac{11}{6}$$

$$= \$1 \frac{5}{6}$$

$$= \$1.8333\ldots$$

more than $1.75 ...
so profitable.

Exercise for student, show that

$$\sigma^2 = \sum_{i=1}^{n} (M_i - \mu)^2 P(M_i)$$

$$\sigma^2 = \frac{17}{36} \quad \text{or} \quad \sigma = \$0.687$$

Continuous dist'n example.

Prob. that an electron is a dist. r from the nucleus of a hydrogen atom is:

$$P(r)\,dr = C r^2 e^{-r/R}\,dr$$

where C & R are constants.

(a) Use the requirement $\int P(r)\,dr = 1$ to find the value of C.

$$\int P(r)\,dr = \int_0^\infty C r^2 e^{-r/R}\,dr = 1$$

$$\therefore \frac{1}{C} = \int_{r=0}^\infty r^2 e^{-r/R}\,dr$$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

Integrate by parts.

$$u = r^2 \qquad du = 2r\,dr$$

$$v = -Re^{-r/R} \qquad dv = e^{-r/R}\,dr$$

$$\frac{1}{C} = \underbrace{r^2(-R)e^{-r/R}\Big|_0^\infty}_{0} - \int_0^\infty (-R)2re^{-r/R}\,dr$$

$$\frac{1}{C} = +2R\int_0^\infty re^{-r/R}\,dr$$

IBP again $\quad u = r \quad du = dr$

$v = -Re^{-r/R} \quad dv = e^{-r/R}\,dr$

$$\frac{1}{C} = 2R\left[\underbrace{-rRe^{-r/R}\Big|_0^\infty}_{0} + R\int_0^\infty e^{-r/R}\,dr\right]$$

$$= 2R^2\left(-Re^{-r/R}\right)\Big|_0^\infty$$

$$= 2R^2\left(0 - (-R)e^0\right)$$

$$= 2R^3$$

$$\boxed{\therefore C = \frac{1}{2R^3}}$$

normalization constant.

(b) find avg. dist. $\langle r \rangle$ that electron is from nucleus.

$$\langle r \rangle = \int_0^\infty r\, P(r)\, dr$$

$$= \int_0^\infty r \left(C r^2 e^{-r/R} \right) dr$$

$$= C \int_0^\infty r^3 e^{-r/R}\, dr$$

IBP $\quad u = r^3 \qquad du = 3r^2\, dr$

$\qquad v = -Re^{-r/R} \qquad dv = e^{-r/R}\, dr$

$$\langle r \rangle = C \left[\underbrace{-r^3 R e^{-r/R} \Big|_0^\infty}_{0} + 3R \underbrace{\int_0^\infty r^2 e^{-r/R}\, dr}_{\frac{1}{C} \text{ from (a)}} \right]$$

$$\therefore \underbrace{\boxed{\langle r \rangle = 3R}}_{\mu}$$ average dist. of e^- from nucleus in H.

(c) Find σ for electron in H atom.

$$\sigma^2 = \int_0^\infty (r-\mu)^2 P(r)\, dr$$

$$= \int_0^\infty (r^2 - 2\mu r + \mu^2) P(r)\, dr$$

$$= \int_0^\infty r^2 P(r)\, dr - 2\mu \underbrace{\int_0^\infty r P(r)\, dr}_{\mu = \langle r \rangle}$$

$$+ \mu^2 \underbrace{\int_0^\infty P(r)\, dr}_{1}$$

$$\sigma^2 = \underbrace{\int_0^\infty r^2 P(r)\,dr}_{\langle r^2\rangle} - \underbrace{\mu^2}_{\langle r\rangle}$$

$$\boxed{\therefore \sigma^2 = \langle r^2\rangle - \langle r\rangle^2}$$

Need to find $\langle r^2\rangle$

(know $\langle r\rangle = 3R$)

$$\langle r^2\rangle = \int_0^\infty r^2 P(r)\,dr$$

$$= c\int_0^\infty r^4 e^{-r/R}\,dr$$

IBP :(

Know from (b) that

$$\int_0^\infty r^3 e^{-r/R}\, dr = \frac{3R}{c}$$

$$= \frac{3R}{1/2R^3} = 6R^4$$

Show that $\langle r^2 \rangle = 12R^2$

then $\sigma^2 = \langle r^2 \rangle - \langle r \rangle^2$

$$= 12R^2 - (3R)^2$$

$$= 3R^2$$

$$\boxed{\therefore \sigma = \sqrt{3}R}$$

Know that a meas. has a 68% prob. of falling within on standard deviation of mean. → For Gaussian dist'n.