PHYS 232 Feb. 14, 2024

Last Time: "Derivation" of Gaussian distribution.

Start w/ binomial dist'n / random walk & find prob. of taking m more steps right than left when $p = \frac{1}{2}$. $\left(\begin{array}{l} \mu = n/2 \\ \sigma^2 = n/4 \end{array} \right)$

steps right $x = \frac{n}{2} + \frac{m}{2}$

steps left $n - x = \frac{n}{2} - \frac{m}{2}$

Want to find: $P_G = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Last time, after applying Stirling's approx for factorials, we found:

$$P(m, n) \approx \frac{n^n \sqrt{n}\; 2^{-n}}{\left[\left(\frac{n}{2}+\frac{m}{2}\right)^{\left(\frac{n}{2}+\frac{m}{2}\right)} \underline{\sqrt{\frac{n}{2}+\frac{m}{2}}}\right]\left[\sqrt{2\pi}\left(\frac{n}{2}-\frac{m}{2}\right)^{\left(\frac{n}{2}-\frac{m}{2}\right)} \underline{\sqrt{\frac{n}{2}-\frac{m}{2}}}\right]}$$

extra step right

total no. steps

$$\sqrt{\frac{n}{2}+\frac{m}{2}}\sqrt{\frac{n}{2}-\frac{m}{2}} = \sqrt{\frac{n^2}{4}-\frac{m^2}{4}}$$

$$P(m,n) = \frac{n^n\sqrt{n}\,2^{-n}}{\sqrt{2\pi}\sqrt{\frac{n^2}{4}-\frac{m^2}{4}}\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{n/2}\left(\frac{n+m}{n-m}\right)^{m/2}}$$

$$\left(\frac{n}{2}+\frac{m}{2}\right)^{\left(\frac{n}{2}+\frac{m}{2}\right)}\left(\frac{n}{2}-\frac{m}{2}\right)^{\left(\frac{n}{2}-\frac{m}{2}\right)}$$

$$\underbrace{\left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{n}{2}}\left(\frac{n}{2}-\frac{m}{2}\right)^{\frac{n}{2}}}_{\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{\frac{n}{2}}\quad 1}\underbrace{\left(\frac{n}{2}+\frac{m}{2}\right)^{\frac{m}{2}}\left(\frac{n}{2}-\frac{m}{2}\right)^{-\frac{m}{2}}}_{\left(\frac{\frac{n}{\cancel{2}}+\frac{m}{\cancel{2}}}{\frac{n}{\cancel{2}}-\frac{m}{\cancel{2}}}\right)^{m/2}\quad 2}$$

Consider $\left(\frac{n+m}{n-m}\right)^{m/2} = \left(\frac{1+\frac{m}{n}}{1-\frac{m}{n}}\right)^{m/2}$

$$= \left[\left(1+\frac{m}{n}\right)\left(1-\frac{m}{n}\right)^{-1}\right]^{m/2}$$

note: $\frac{m}{n}$ is small if we imagine a small no. of extra steps right.

$$\left(1-\frac{m}{n}\right)^{-1} \approx 1+(-1)\left(-\frac{m}{n}\right)$$

$$\approx 1+\frac{m}{n}$$

$(1+x)^p$

$\approx 1+px$

$|x| \ll 1$

Binomial approx

$$\left[\left(1+\frac{m}{n}\right)\left(1+\frac{m}{n}\right)\right]^{m/2}$$

$$= \left[1+2\frac{m}{n}+\left(\frac{m}{n}\right)^2\right]^{m/2}$$

If $\frac{m}{n}$ is small, then $\left(\frac{m}{n}\right)^2$ is very small

$$\approx \left[1+2\frac{m}{n}\right]^{m/2}$$

$$\therefore P(m,n) \approx \frac{n^n \sqrt{n}\, 2^{-n}}{\sqrt{2\pi}\sqrt{\frac{n^2}{4}-\frac{m^2}{4}}\left(\frac{n^2}{4}-\frac{m^2}{4}\right)^{n/2}\left(1+2\frac{m}{n}\right)^{m/2}}$$

$$= \frac{\sqrt{\frac{n}{2\pi}}\cancel{\left(\frac{n}{2}\right)^n}}{\frac{n}{2}\sqrt{1-\left(\frac{m}{n}\right)^2}\cancel{\left(\frac{n^2}{4}\right)^{\frac{n}{2}}}\left(1-\left(\frac{m}{n}\right)^2\right)^{n/2}\left(1+\frac{2m}{n}\right)^{m/2}}$$

$$P(m,n) \approx \frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\left(1-\frac{m^2}{n^2}\right)^{-\frac{1}{2}}\left(1-\frac{m^2}{n^2}\right)^{-\frac{n}{2}}\left(1+\frac{2m}{n}\right)^{-m/2}$$

Now, take $\ln(P(m,n))$ & use

$\ln(1+x) \approx x$ when $|x| << 1$.

$$\ln P(m,n) \approx \ln\left(\frac{2}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln n - \frac{1}{2}\ln\overbrace{\left(1-\frac{m^2}{n^2}\right)}^{-m^2/n^2}$$

$$- \frac{n}{2}\ln\underbrace{\left(1-\frac{m^2}{n^2}\right)}_{-\frac{m^2}{n^2}} - \frac{m}{2}\ln\underbrace{\left(1+\frac{2m}{n}\right)}_{2\frac{m}{n}}$$

$$\ln P(m,n) \approx \ln\left(\frac{2}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln n + \boxed{\frac{1}{2}\frac{m^2}{n^2}} + \frac{1}{2}\frac{m^2}{n} - \frac{m^2}{n}$$

small c.t. m^2/n
→ ignore.

$$\ln P(m,n) \approx \ln\left(\frac{2}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln n - \frac{1}{2}\frac{m^2}{n}$$

$$\approx \ln\left(\frac{2}{\sqrt{2\pi}\sqrt{n}}\right) - \frac{m^2}{2n}$$

Now exponentiate :

$$P(m,n) \approx e^{\ln\left(\frac{2}{\sqrt{2\pi}\sqrt{n}}\right)} e^{-\frac{m^2}{2n}}$$

$$\therefore P(m,n) = \frac{2}{\sqrt{2\pi}\sqrt{n}} e^{-m^2/2n}$$

Recall that $\mu = \frac{n}{2}$ $\sigma = \frac{\sqrt{n}}{2}$

$$P(m,n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-m^2/2n}$$

Recall $x = \frac{n}{2} + \frac{m}{2} \Rightarrow m = 2x - n$

$$P(m,n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(2x-n)^2/2n}$$

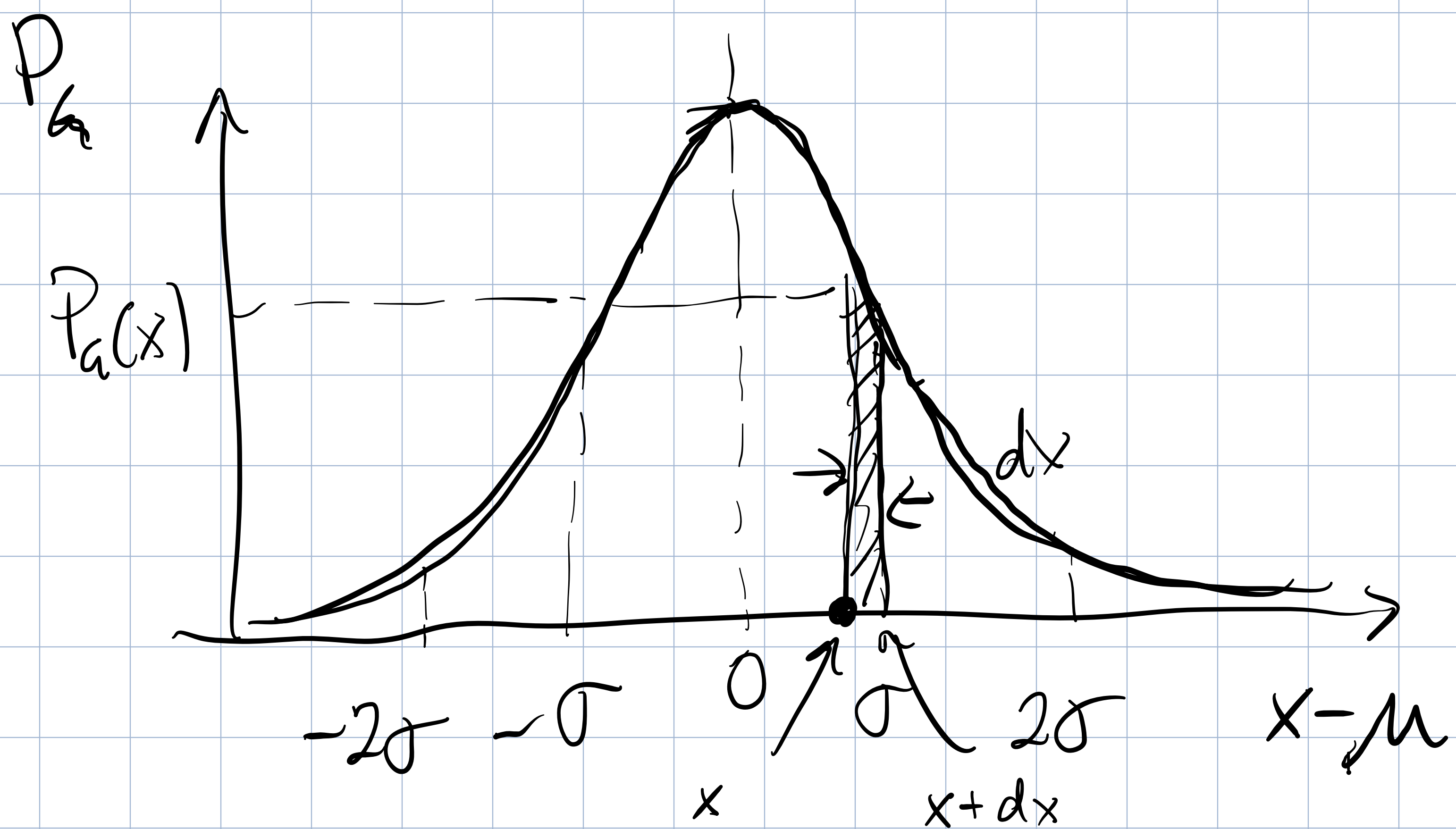
$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-4(x-\underbrace{\tfrac{n}{2}}_{\mu})^2/2n}$$

$$P(m,n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\underbrace{\boxed{\tfrac{n}{4}}}_{\sigma^2})}$$

<u>**Finally**</u>:

$$\boxed{P(m,n) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = P_G \quad \text{Gaussian distn}}$$

Properties of Gaussian distn.



- Total area is 1

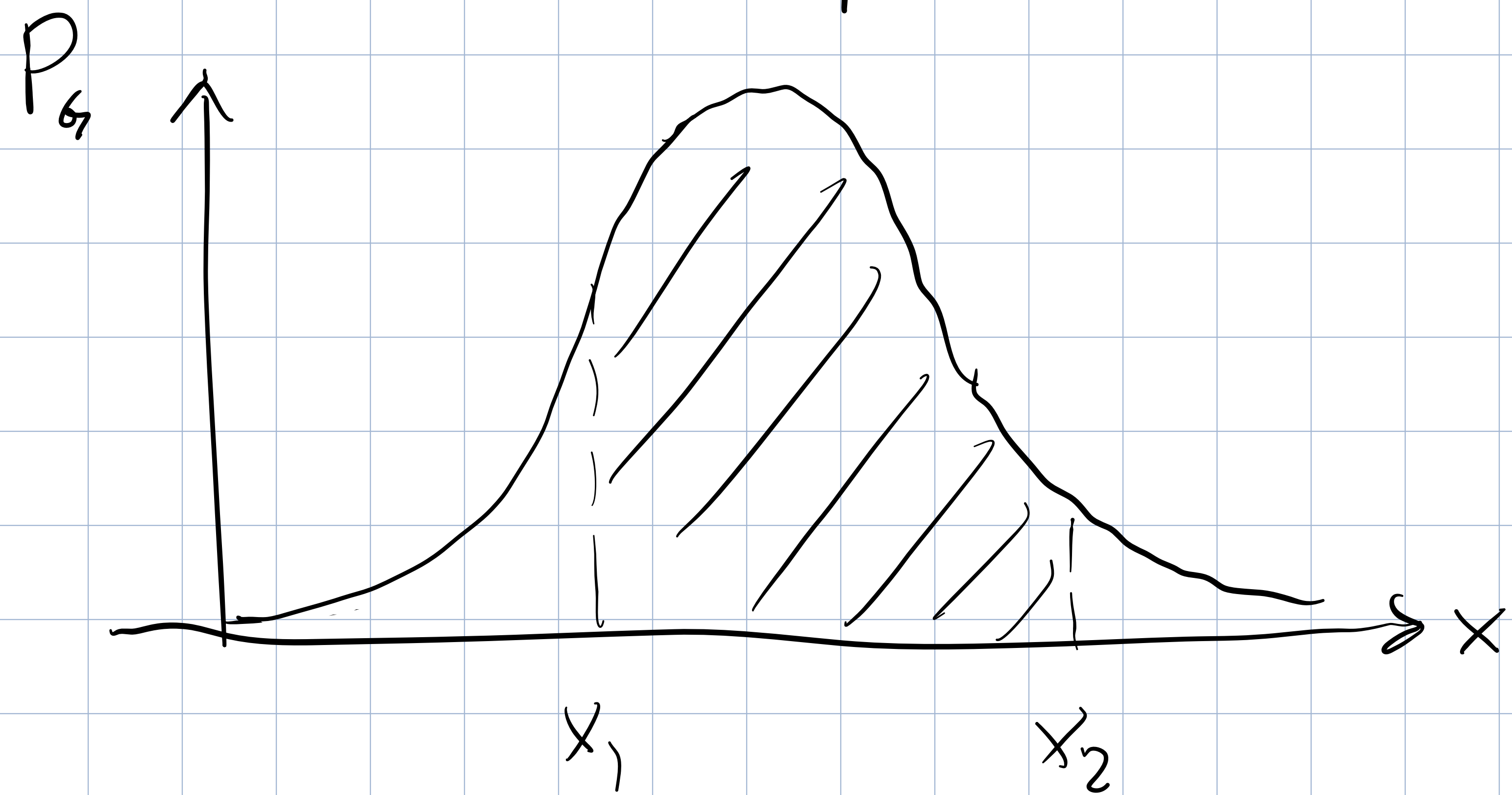
$$\int_{x=-\infty}^{x=+\infty} P_G \, dx = 1 .$$

- Prob. of a meas. falling between x & $x+dx$ is:

$$dP_G = P_G \, dx$$

- Prob. of meas. a value between x_1 & x_2 is:

$$P(x_1 \to x_2) = \int_{x_1}^{x_2} P_G(x; \mu, \sigma) \, dx$$



Eg. Find prob. of a meas. falling within Δx of the mean.

$$P(\mu-\Delta x, \mu+\Delta x) = \int_{\mu-\Delta x}^{\mu+\Delta x} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \cdot dx$$

make a substitution

$$z = \frac{x-\mu}{\sigma}$$

when $x = \mu - \Delta x$

$$z = -\frac{\Delta x}{\sigma}$$

$$dz = \frac{dx}{\sigma}$$

$x = \mu + \Delta x$

$$z = +\frac{\Delta x}{\sigma}$$

$$P(\mu-\Delta x, \mu+\Delta x) = \int_{-\frac{\Delta x}{\sigma}}^{+\frac{\Delta x}{\sigma}} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \exp\left[-\frac{z^2}{2}\right] \cancel{\sigma}\, dz$$

$$\boxed{= \frac{1}{\sqrt{2\pi}} \int_{-\frac{\Delta x}{\sigma}}^{\frac{\Delta x}{\sigma}} e^{-z^2/2}\, dz}$$

Cannot evaluate analytically, requires numerical methods.

$\Delta x / \sigma$	$P(\mu - \Delta x, \mu + \Delta x)$
1	0.683
2	0.954
3	0.997
4	0.999937
5	0.9999994