PHYS 232 Feb. 16, 2024

Last Time: "Derivation" of Gaussian distribution.

$\Rightarrow$ Found $P_G = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Prob. of a measurement falling between x_1 & x_2 is:

$$P(x_1, x_2) = \int_{x_1}^{x_2} P_G(x; \mu, \sigma)\, dx$$

<u>68-95-99.7 Rule</u>

Prob. of a measurement falling within one std. dev. of mean is:

$$\mu - \sigma < x < \mu + \sigma \Rightarrow 68\%$$

$$\mu - 2\sigma < x < \mu + 2\sigma \Rightarrow 95\%$$

$$\mu - 3\sigma < x < \mu + 3\sigma \Rightarrow 99.7\%$$

Summary of Prob. dist'ns:

Binomial:

$$P_B(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\mu = np \qquad \sigma^2 = np(1-p)$$

Poisson:

$p \ll 1$ limit of Binomial dist'n

(small prob. of success in any individual trial)

$$P_P(x; \mu) = \frac{\mu^x}{x!} e^{-\mu}$$

mean: μ $\qquad \sigma = \sqrt{\mu}$

Relevant for counting experiments

Gaussian:

n large & $np \gg 1$ limit of Binomial dist'n.

$$P_G(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

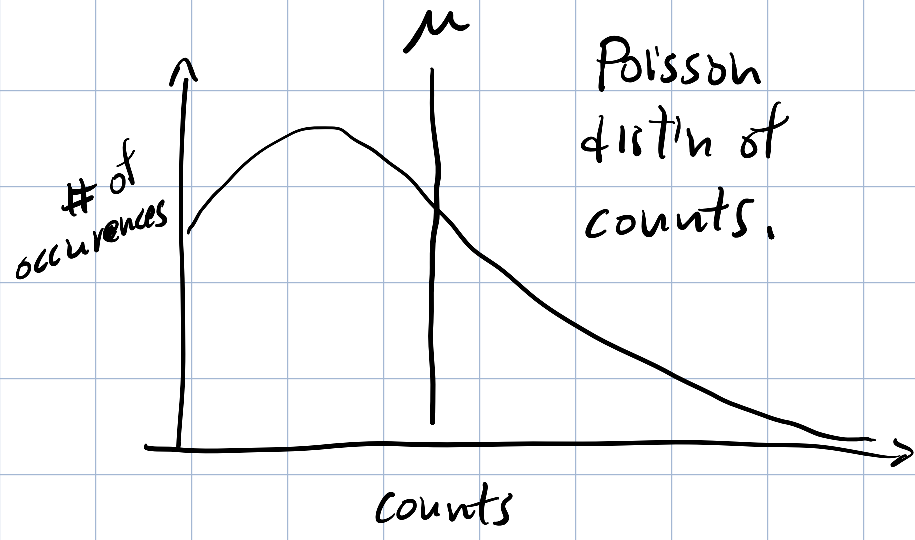
mean: μ std. dev.: σ

Uncertainty in Counting Exp'ts.

In principle can meas. no of events in a time interval Δt w/ absolute certainty.

However, if repeat meas. many times will get statistical fluctuations in the no. of counts.

trial	counts
1	18
2	24
3	21
4	19
⋮	⋮



Poisson dist'n of counts.

std. dev. of mean is $\sigma = \sqrt{\mu}$

In practice, often cannot make many repeated experiments to map out the full dist'n in order to properly estimate μ.

∴ in counting exp'ts often do the meas. once & assume that our result is a reasonable est. of μ.

Assume that $\mu = N$ & $\sigma = \sqrt{N}$

no. of counts.

Propagation of Errors

Suppose we want to determine some quantity y that depends on meas. values

$u \pm \sigma_u$ & $v \pm \sigma_v$.

If $y = f(u,v)$, what is σ_y?

Eg. In Boltzmann's Const.

$$\ln I \approx \ln I_0 + \left(\frac{e}{k_B T}\right) V$$

If you plot $\ln I$ vs V, slope is equal to

$$m = \frac{e}{k_B T}$$

Assume that you've determined $m \pm \sigma_m$ & $T \pm \sigma_T$. Now, you want to calc. k_B (assume e is known)

$$k_B = \frac{e}{mT}$$

now $k_B(m, T)$

know $m \pm \sigma_m$ & $T \pm \sigma_T$

What is σ_{k_B}?

$y = f(u)$ know $u \pm \sigma_u$

Want to find σ_y.

Recall that if we have a fcn of one variable, can approx $f(u)$ as a Taylor series

$$f(u) = f(\bar{u}) + (u-\bar{u})\, f'(u)\Big|_{u=\bar{u}}$$

Taylor series expansion about $u=\bar{u}$.

Assume that $f(\bar{u}) \approx \bar{f}$

$$f = \bar{f} + (u-\bar{u})\, f'(\bar{u})$$

$$\therefore \underline{f - \bar{f}} = (u-\bar{u})\, f'(\bar{u}) = \underbrace{(u-\bar{u})\frac{df}{du}\Big|_{\bar{u}}}$$

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\mu)^2}$$ Definition of std. dev.

In a similar way, we can find σ_f as:

$$\sigma_f^{\,2} = \frac{1}{N-1}\sum_{i=1}^{N}\left(f_i-\bar{f}\right)^2$$

$$\therefore \sigma_f^{\,2} = \frac{1}{N-1}\sum_{i=1}^{N}\left((u_i-\bar{u})\left.\frac{df}{du}\right|_{\bar{u}}\right)^2$$

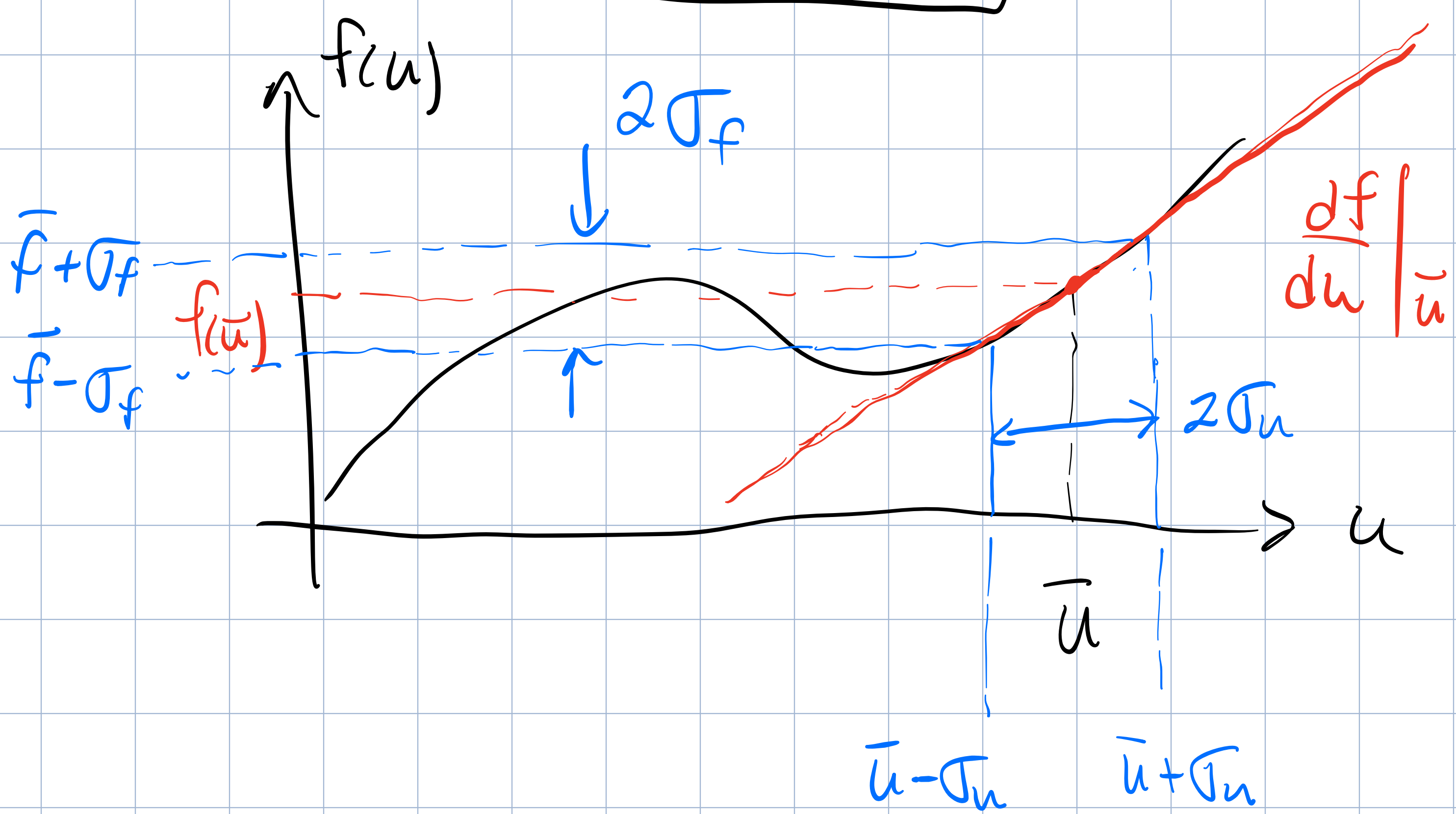
$$= \frac{1}{N-1}\sum_{i=1}^{N}\left[(u_i-\bar{u})^2\underbrace{\left(\left.\frac{df}{du}\right|_{\bar{u}}\right)^2}_{\text{indep. of } i}\right]$$

$$\therefore \sigma_f^{\,2} = \overbrace{\left(\frac{1}{N-1}\sum_{i=1}^{N}(u_i-\bar{u})^2\right)}^{\sigma_u^{\,2}}\left(\left.\frac{df}{du}\right|_{\bar{u}}\right)^2$$

$$\therefore \quad \boxed{\sigma_f^2 = \sigma_u^2 \left(\left. \frac{df}{du} \right|_{\bar{u}} \right)^2}$$

Prop. of errors for a function of one variable.

$$\boxed{\sigma_f = \sigma_u \left| \left. \frac{df}{du} \right|_{\bar{u}} \right|}$$



For the red line, slope is $\frac{\text{rise}}{\text{run}} = \frac{\cancel{2}\sigma_f}{\cancel{2}\sigma_u}$

& must also equal $\left.\frac{df}{du}\right|_{\bar{u}}$

$$\frac{\sigma_f}{\sigma_u} = \left.\frac{df}{du}\right|_{\bar{u}} \implies \boxed{\sigma_f = \sigma_u \left.\frac{df}{du}\right|_{\bar{u}}}$$

For a fcn of more than one variable

$$y = f(u, v, \ldots)$$

the first order Taylor series expansion is:

$$y = f(u,v,\ldots) = f(\bar{u}, \bar{v}, \ldots) + (u-\bar{u})\left.\frac{\partial f}{\partial u}\right|_{\bar{u}}$$

$$+ (v-\bar{v})\left.\frac{\partial f}{\partial v}\right|_{\bar{v}} + \ldots$$

Assume $f(\bar{u}, \bar{v}, \ldots) \approx \bar{f}$

$$\therefore \quad f - \bar{f} \approx (u-\bar{u}) \left.\frac{\partial f}{\partial u}\right|_{\bar{u}} + (v-\bar{v}) \left.\frac{\partial f}{\partial v}\right|_{\bar{v}} + \ldots$$

$$\sigma_f^2 \approx \frac{1}{N-1} \sum_{i=1}^{N} (f_i - \bar{f})^2$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \left[(u_i-\bar{u}) \left.\frac{\partial f}{\partial u}\right|_{\bar{u}} + (v_i-\bar{v}) \left.\frac{\partial f}{\partial v}\right|_{\bar{v}} + \ldots \right]^2$$

$$\therefore \sigma_f^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left[(u_i-\bar{u})^2 \left(\left.\frac{\partial f}{\partial u}\right|_{\bar{u}}\right)^2 + (v_i-\bar{v})^2 \left(\left.\frac{\partial f}{\partial v}\right|_{\bar{v}}\right)^2 + \ldots \right.$$

$$\left. + 2(u_i-\bar{u})(v_i-\bar{v}) \left(\left.\frac{\partial f}{\partial u}\right|_{\bar{u}}\right)\left(\left.\frac{\partial f}{\partial v}\right|_{\bar{v}}\right) + \ldots \right]$$

$$\frac{1}{N-1} \sum_{i=1}^{N} (u_i-\bar{u}) = \sigma_u^2 \qquad \frac{1}{N-1} \sum_{i=1}^{N} (v_i-\bar{v}) = \sigma_v^2$$

"Variance" of u

Define the covariance σ_{uv}^2

$$\sigma_{uv}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (u_i - \bar{u})(v_i - \bar{v})$$

$$\sigma_f^2 = \sigma_u^2 \left(\frac{\partial f}{\partial u} \bigg|_{\bar{u}} \right)^2 + \sigma_v^2 \left(\frac{\partial f}{\partial v} \bigg|_{\bar{v}} \right)^2 + \dots$$

$$+ 2\sigma_{uv}^2 \left(\frac{\partial f}{\partial u} \bigg|_{\bar{u}} \right) \left(\frac{\partial f}{\partial v} \bigg|_{\bar{v}} \right) + \dots$$

If the meas. of u & v follow Gaussian dist'n, then

$$\sum (u_i - \bar{u}) = 0$$

$$\sum (v_i - \bar{v}) = 0$$

$$\sum (u_i - \bar{u})(v_i - \bar{v}) = 0$$

$\Rightarrow$ covariance $\sigma_{uv}^2 = 0$.

$$\boxed{\sigma_f^2 = \sigma_u^2 \left(\frac{\partial f}{\partial u}\bigg|_{\bar{u}} \right)^2 + \sigma_v^2 \left(\frac{\partial f}{\partial v}\bigg|_{\bar{v}} \right)^2 + \dots}$$

Propagation of Errors.