PHYS 232 Mar. 8, 2024

- Assignment #4 due Wed., Mar. 13
- Sign up for Experiment #5
- Assignment #5 has also been poster
- Formal Lab Report due April 10 @ 11:00 am

Last Time: Weighted Linear Fit to $(x_i, y_i \pm \sigma_i)$ for $i = 1..N$

Eq'n of straight line:

$$y = a + bx$$

Best-fit values for a & b are:

$$\boxed{\begin{aligned} a &= \frac{1}{\Delta}\left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2}\right) \\ b &= \frac{1}{\Delta}\left(\sum \frac{1}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2}\right) \end{aligned}}$$

where: $$\boxed{\Delta = \sum \frac{1}{\sigma_i^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2}\right)^2}$$

Today: Apply prop. of errors to find σ_a & σ_b $\Rightarrow$ The uncertainties in the best-fit parameters.

Recall: $$\frac{\partial y_i}{\partial y_j} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and $$\sum_{i=1}^{N} f(x_i)\,\delta_{ij} = f(x_j)$$

What are the uncertainties in a & b?

$$a = a(y_1, y_2, \ldots, y_N)$$
$$b = b(y_1, y_2, \ldots, y_N)$$

$$\sigma_a^2 = \sum_{j=1}^{N} \left(\frac{\partial a}{\partial y_j} \sigma_j \right)^2 \quad (1)$$

$$\sigma_b^2 = \sum_{j=1}^{N} \left(\frac{\partial b}{\partial y_j} \sigma_j \right)^2$$

First, note that Δ is indep of y_j

Start by evaluating

$$\frac{\partial a}{\partial y_j} = \frac{1}{\Delta} \frac{\partial}{\partial y_j} \left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{y_i}{\sigma_i^2} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i y_i}{\sigma_i^2} \right)$$

$$= \frac{1}{\Delta} \left(\sum \frac{x_i^2}{\sigma_i^2} \sum \frac{1}{\sigma_i^2} \frac{\partial y_i}{\partial y_j} - \sum \frac{x_i}{\sigma_i^2} \sum \frac{x_i}{\sigma_i^2} \frac{\partial y_i}{\partial y_j} \right)$$

$$\frac{\partial a}{\partial y_j} = \frac{1}{\Delta}\left(\sum \frac{x_i^2}{\sigma_i^2} \underbrace{\sum \frac{1}{\sigma_i^2} \overset{\delta_{ij}}{\delta_{ij}}}_{\frac{1}{\sigma_j^2}} - \sum \frac{x_i}{\sigma_i^2} \underbrace{\sum \frac{x_i}{\sigma_i^2} \overset{\delta_{ij}}{\delta_{ij}}}_{\frac{x_j}{\sigma_j^2}} \right.$$

$$\sum \frac{1}{\sigma_i^2}\delta_{ij} = \frac{1}{\sigma_1^2}\cancelto{0}{\delta_{1j}} + \frac{1}{\sigma_2^2}\cancelto{0}{\delta_{2j}} + \dots + \frac{1}{\sigma_j^2}\cancelto{1}{\delta_{jj}}$$

$$+ \dots + \frac{1}{\sigma_N^2}\cancel{\delta_{Nj}}$$

$$= \frac{1}{\sigma_j^2}$$

$$\frac{\partial a}{\partial y_j} = \frac{1}{\Delta}\left(\frac{1}{\sigma_j^2} \sum \frac{x_i^2}{\sigma_i^2} - \frac{x_j}{\sigma_j^2} \sum \frac{x_i}{\sigma_i^2} \right)$$

Sub $\frac{\partial a}{\partial y_j}$ result back into ①

$$\sigma_a^2 = \sum_{j=1}^{N} \left(\frac{\partial a}{\partial y_j} \sigma_j \right)^2$$

$$= \frac{1}{\Delta^2} \sum_{j=1}^{N} \left(\frac{1}{\sigma_j} \sum \frac{x_i^2}{\sigma_i^2} - \frac{x_j}{\sigma_j} \sum \frac{x_i}{\sigma_i^2} \right)^2$$

$$= \frac{1}{\Delta^2} \sum_{j=1}^{N} \left[\frac{1}{\sigma_j^2} \left(\sum \frac{x_i^2}{\sigma_i^2} \right)^2 + \frac{x_j^2}{\sigma_j^2} \left(\sum \frac{x_i}{\sigma_i^2} \right)^2 - 2 \frac{x_j}{\sigma_j^2} \sum \frac{x_i^2}{\sigma_i^2} \sum \frac{x_i}{\sigma_i^2} \right]$$

distribute the sum:

$$\sigma_a^2 = \frac{1}{\Delta^2}\left[\sum \frac{1}{\sigma_j^2}\left(\sum \frac{x_i^2}{\sigma_i^2}\right)^2 + \sum \frac{x_j^2}{\sigma_j^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2 - 2\underbrace{\sum_{j=1}^{N} \frac{x_j}{\sigma_j^2}\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}}_{\left(\sum \frac{x_i}{\sigma_i^2}\right)^2}\sum \frac{x_j^2}{\sigma_j^2}\right]$$

$$\sigma_a^2 = \frac{1}{\Delta^2}\left[\sum \frac{1}{\sigma_j^2}\left(\sum \frac{x_i^2}{\sigma_i^2}\right)^2 + \cancel{\sum \frac{x_j^2}{\sigma_j^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2} - \cancel{2}\sum \frac{x_i^2}{\sigma_i^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2\right]$$

$$\sigma_a^2 = \frac{1}{\Delta^2}\left[\sum \frac{1}{\sigma_j^2}\left(\sum \frac{x_i^2}{\sigma_i^2}\right)^2 - \sum \frac{x_i^2}{\sigma_i^2}\left(\sum \frac{x_i}{\sigma_i^2}\right)^2\right]$$

Factor out $\sum \frac{x_i^2}{\sigma_i^2}$

$$\sigma_a^2 = \frac{1}{\Delta^2} \sum \frac{x_i^2}{\sigma_i^2} \underbrace{\left[\sum \frac{1}{\sigma_j^2} \sum \frac{x_i^2}{\sigma_i^2} - \left(\sum \frac{x_i}{\sigma_i^2} \right)^2 \right]}_{\Delta}$$

$$\boxed{\therefore \sigma_a^2 = \frac{1}{\Delta} \sum \frac{x_i^2}{\sigma_i^2}}$$

Uncertainty in the best-fit value for the y-intercept.

Using the same methods, find uncertainty in the slope is given by:

$$\sigma_b^2 = \sum_{j=1}^{N} \left(\frac{\partial b}{\partial y_j} \sigma_j \right)^2$$

$$\Downarrow$$

$$\boxed{\sigma_b^2 = \frac{1}{\Delta}\sum \frac{1}{\sigma_i^2}}$$

uncertainty in the best-fit value of the slope.

From last time we had:

$$\therefore \quad \underbrace{\sum \frac{y_i}{\sigma_i^2}}_{u} = a\underbrace{\sum \frac{1}{\sigma_i^2}}_{u_a} + b\underbrace{\sum \frac{x_i}{\sigma_i^2}}_{u_b}$$

$$\underbrace{\sum \frac{x_i y_i}{\sigma_i^2}}_{v} = a\underbrace{\sum \frac{x_i}{\sigma_i^2}}_{v_a} + b\underbrace{\sum \frac{x_i^2}{\sigma_i^2}}_{v_b}$$

solve system of 2 eq'ns for 2 unknowns a & b.

Of the form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} u_a & u_b \\ v_a & v_b \end{pmatrix}}_{\equiv \underline{\underline{A}}} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underline{\underline{A}} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \underline{\underline{A}}^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \frac{1}{\det A} \begin{pmatrix} V_b & -U_b \\ -V_a & U_a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\frac{1}{\Delta} \begin{pmatrix} V_b & -U_b \\ -V_a & U_a \end{pmatrix}}_{\text{called the covariance matrix}} \begin{pmatrix} u \\ v \end{pmatrix}$$

For linear fits, the covariance matrix is given by:

$$\frac{1}{\Delta}\begin{pmatrix} \sum \frac{x_i^2}{\sigma_i^2} & -\sum \frac{x_i}{\sigma_i^2} \\ -\sum \frac{x_i}{\sigma_i^2} & \sum \frac{1}{\sigma_i^2} \end{pmatrix}$$

The diagonal elements of the co-variance matrix give us the square of the uncertainty in the best-fit parameters.

The 11 element gives σ_a^2

$$\sigma_a^2 = \frac{1}{\Delta}\sum \frac{x_i^2}{\sigma_i^2}$$

The 22 element gives σ_b^2

$$\sigma_b^2 = \frac{1}{\Delta} \sum \frac{1}{\sigma_i^2}$$