PHYS 232 March 25, 2024

- Assignment #6 due April 5 @ 11am
- Formal Report due Apr. 10 @ 11am
- Bring Exp't #5 notebooks to class on Wednesday, April 3 @ 11:00 am
- If needed, sign up for make up labs today

Last Time: Nonlinear fits

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

Goal is to minimize χ^2 by adjusting the parameters in $y(x_i; a_1, a_2, \ldots a_m)$

Acheived by:

① Selecting a range a values for each a_k

$$a_{k,min} < a_k < a_{k,max}$$

② Select a step size Δa_k for each parameter

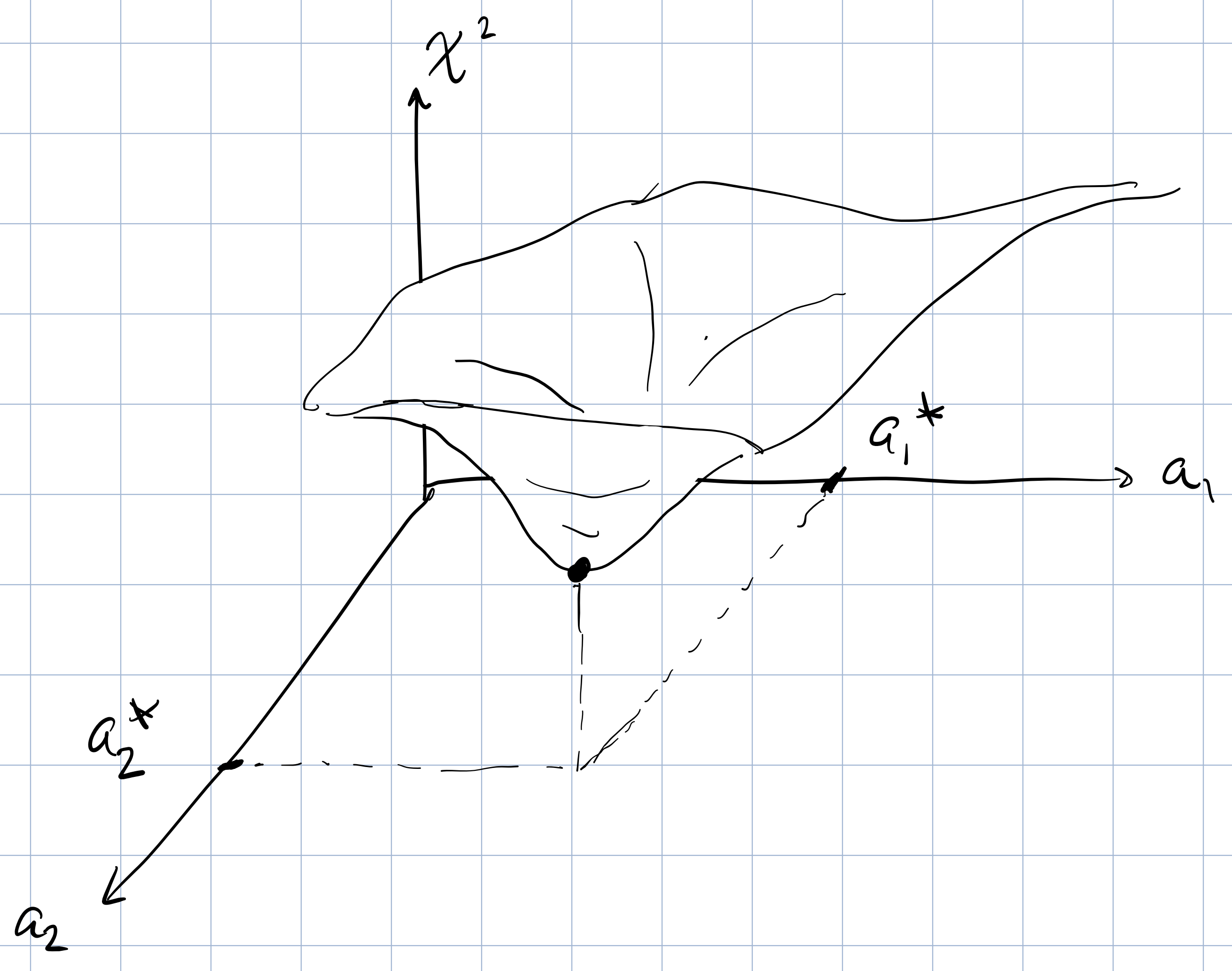
a_k values are then:

$$a_{k,min}, \quad a_{k,min} + \Delta a_k, \quad a_{k,min} + 2\Delta a_k$$

$$\ldots, \quad a_{k,max} - \Delta a_k, \quad a_{k,max}$$

③ Evaluate χ^2 at all combinations of a_k values

Eg. for a two-parameter fit.



The estimate of the best-fit parameters is given by the set of parameters (a_1^*, a_2^*) that gives the smallest value for χ^2.

Today: Develop a method for estimating the uncertainty in the best-fit parameters.

Imagine that perform an experiment & meas $(x_i, y_i \pm \sigma_i)$ $i = 1..N$.

Fit our data to a model $y = y(x; a_1, a_2, \dots a_m)$ that depends on parameters a_k $k = 1..m$.

Find set of parameters that minimizes

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

Recall that χ^2 came from fact that prob. of obtaining a set of data $(x_i, y_i \pm \sigma_i)$ from parameters $a_1, a_2, \ldots a_m$ is given by:

$$P(a_1, a_2, \ldots, a_m) = P_1 P_2 P_3 \cdots P_N$$

$$= \prod_{i=1}^{N} P_i = \prod_{i=1}^{N} \left[\frac{\Delta y}{\sigma_i \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2 \right\} \right]$$

$$= \prod_{i=1}^{N} \left[\frac{\Delta y}{\sigma_i \sqrt{2\pi}} \right] \exp\left\{ -\frac{1}{2} \underbrace{\sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2}_{\equiv \chi^2} \right\}$$

$$\boxed{P(a_1, a_2, \ldots, a_m) = \prod_{i=1}^{N} \left[\frac{\Delta y}{\sigma_i \sqrt{2\pi}} \right] \exp\left\{ -\frac{1}{2} \chi^2 \right\} \quad \#}$$

Try to determine the shape of χ^2 near its minimum.

Minimization of χ^2.

$$y = (a_1, a_2, \ldots, a_m; x)$$

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2$$

Taylor series expansion of χ^2 about its minimum w.r.t. one of the a parameters $\rightarrow$ a_j.

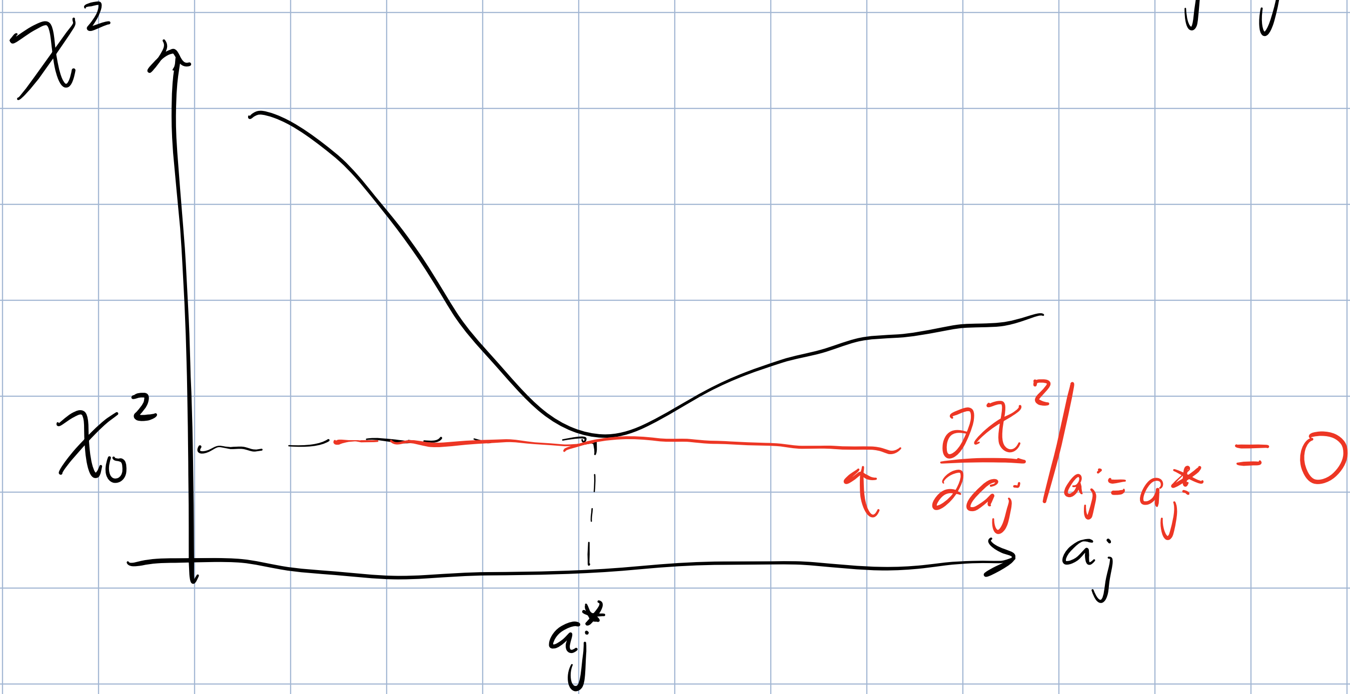
$j = 1 \ldots m$

$$\chi^2 \approx \overbrace{\chi^2 \Big|_{a_j = a_j^*}}^{\chi_0^2 \text{ (minimum of } \chi^2)} + \cancelto{0}{(a_j - a_j^*) \frac{\partial \chi^2}{\partial a_j}\Big|_{a_j = a_j^*}}$$

value of a_j that minimize χ^2

$$+ \frac{1}{2}(a_j - a_j^*)^2 \frac{\partial^2 \chi^2}{\partial a_j^2}\Big|_{a_j = a_j^*} + \ldots$$

At the minimum of χ^2, require $\left.\dfrac{\partial \chi^2}{\partial a_j}\right|_{a_j=a_j^*} = 0$

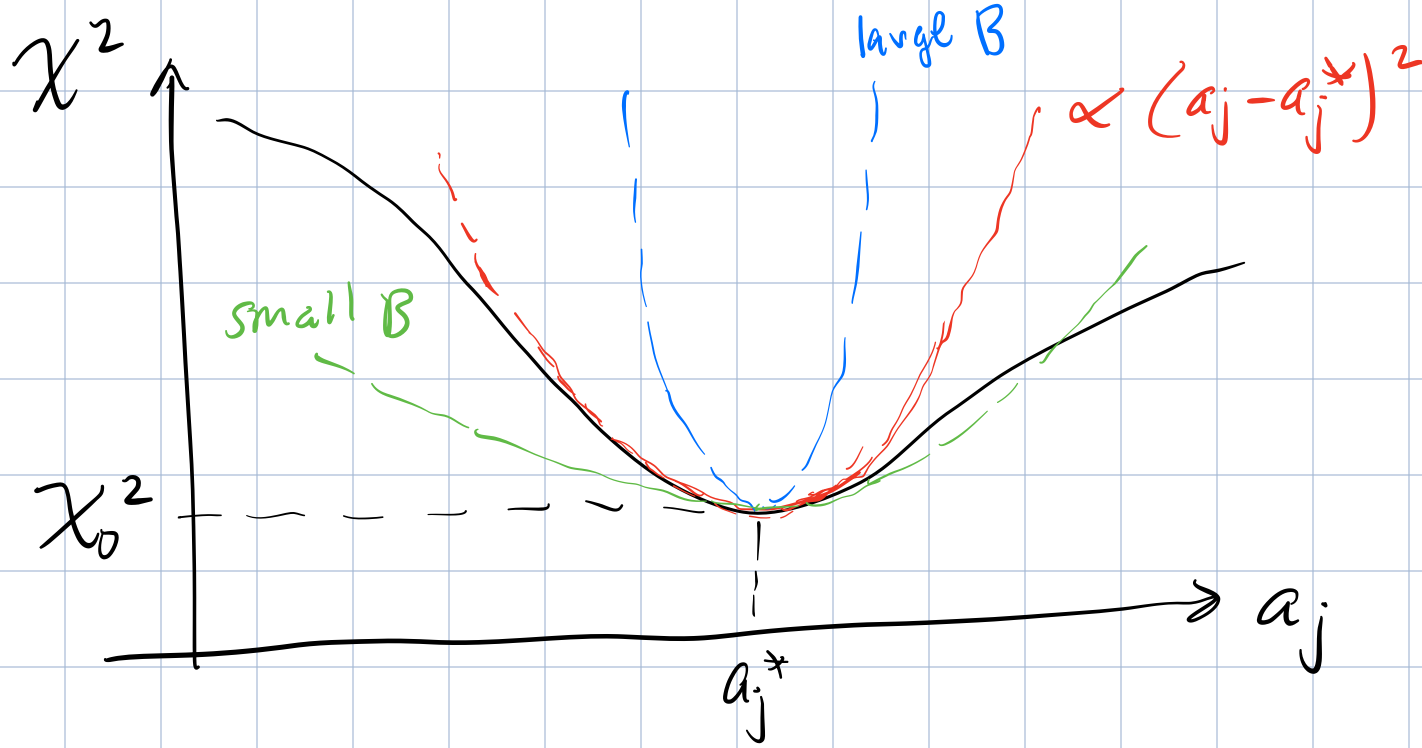


$$\therefore \chi^2 = \chi_0^2 + \underbrace{\left(\frac{1}{2}\left.\frac{\partial^2 \chi^2}{\partial a_j^2}\right|_{a_j=a_j^*}\right)}_{\equiv B} (a_j - a_j^*)^2$$

$$\boxed{\chi^2 \simeq \chi_0^2 + B(a_j - a_j^*)^2 \quad (*)}$$

unknowns.

χ^2 looks quadratic near its minimum.



When B is large, the range of a_j values that minimize χ^2 is narrow

$\rightarrow$ a_j^* precisely known

σ_{a_j} small.

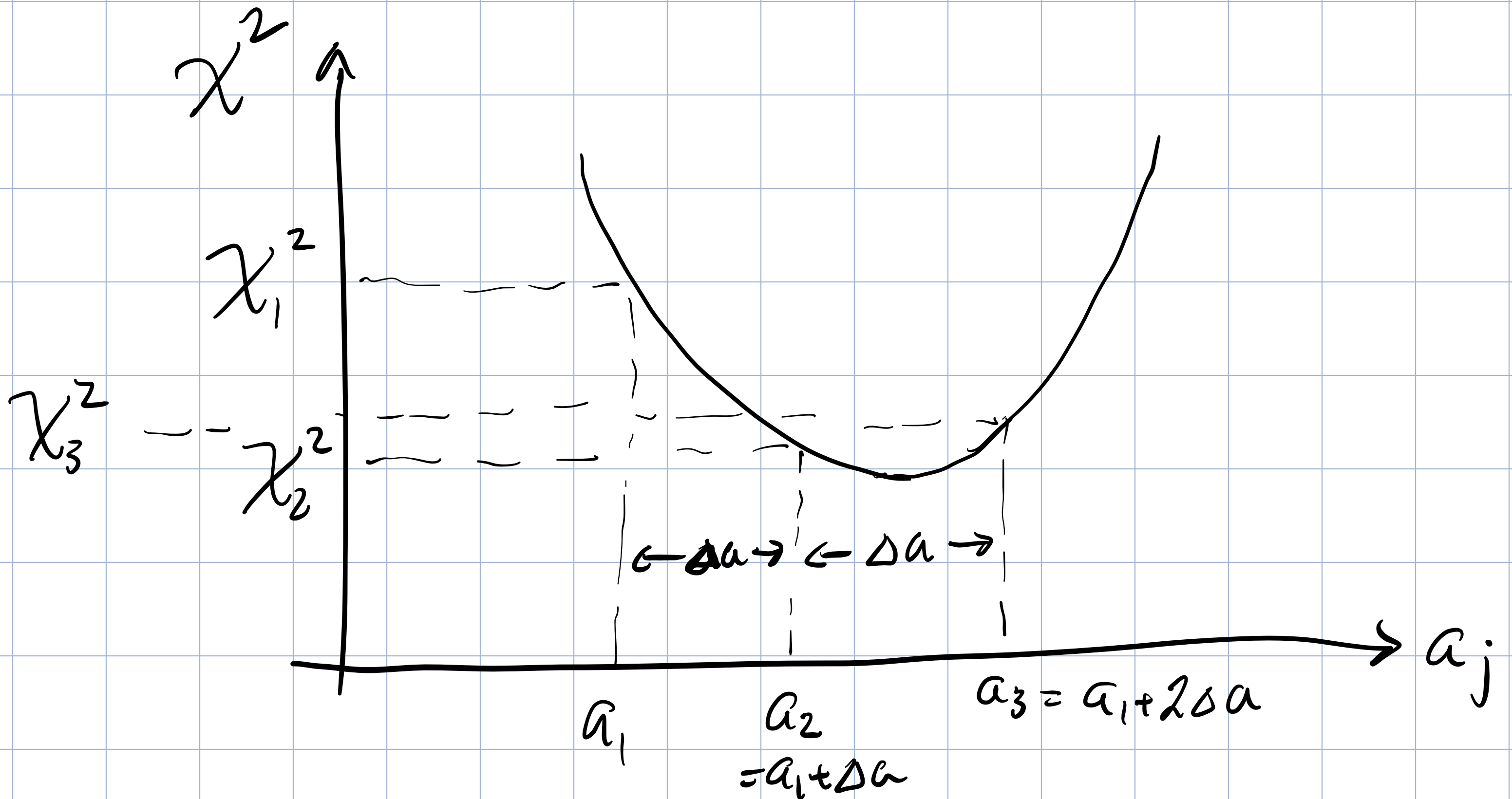
When B is small, large range of a_j values to come close to minimizing χ^2

$\rightarrow$ a_j^* poorly known

σ_{a_j} large.

Expect that σ_{a_j} is, in some way, inversly prop. to B.

In principle, (& in practice) if we know value of χ^2 at 3 points of a_j near the minimum, can determine the three unknowns: χ_0^2, a_j^*, B.



Pick 3 values of a_j.

$$a_1$$
$$a_2 = a_1 + \Delta a$$
$$a_3 = a_1 + 2\Delta a$$

Write $\chi_1^2 = \chi_0^2 + B(a_1 - a^*)^2$

$$= \chi_0^2 + B(a_3 - 2\Delta a - a^*)^2$$

(1). $$= \chi_0^2 + B((a_3 - a^*) - 2\Delta a)^2$$

$$\chi_2^2 = \chi_0^2 + B(a_2 - a^*)^2$$

(2) $$= \chi_0^2 + B((a_3 - a^*) - \Delta a)^2$$

(3) $$\chi_3^2 = \chi_0^2 + B(a_3 - a^*)^2$$

Can solve this system of 3 eq'ns for the 3 unknowns. Important result Bi

$$\boxed{B = \frac{\chi_1^2 - 2\chi_2^2 + \chi_3^2}{2(\Delta a)^2}}$$

Return to likelihood prob. (#):

$$P(a_1, a_2, \ldots, a_m) = \prod_{i=1}^{N} \left[\frac{\Delta y}{\sigma_i \sqrt{2\pi}}\right] \exp\left\{-\frac{1}{2}\chi^2\right\}$$

sub in (*) $\chi^2 = \chi_0^2 + B(a_j - a_j^*)^2$

near the minimum of χ^2.

$$P(a_1, \ldots a_m) \simeq \underbrace{\prod_{i=1}^{N} \left[\frac{\Delta y}{\sigma_i \sqrt{2\pi}}\right]}_{\equiv A} \exp\left\{-\frac{1}{2}\left[\chi_0^2 + B(a_j - a_j^*)^2\right]\right\}$$

$$\approx \underbrace{A \exp\left\{-\frac{1}{2}\chi_0^2\right\}}_{\equiv A'} \exp\left\{-\frac{B(a_j - a_j^*)^2}{2}\right\}$$

$$P(a_1, \ldots a_m) = A' \exp\left\{-\frac{(a_j - a_j^*)^2}{2B^{-1}}\right\}$$

Is of the form of a Gaussian dist'n

$$f_G = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Likelihood fcn becomes a Gaussian fcn of each parameter a_j.

$$\sigma_j^2 = B^{-1}$$

or

$$\boxed{\sigma_j = \frac{1}{\sqrt{B}}}$$

Uncertainty in parameter a_j

where:

$$\boxed{B = \frac{\chi_1^2 - 2\chi_2^2 + \chi_3^2}{2(\Delta a)^2}}$$