Nonlinear Fits:

1. Find the best-fit parameters
   - Set up a grid of parameters
   - Evaluate $\chi^2$ at each grid point
   - The point that results in minimum $\chi^2$ values gives an estimate of best-fit parameters.
(2) Pick one of the parameters $\chi^2$ at 3 values of $a_i$ that bracket the minimum in $\chi^2$. Expect $\chi^2$ to vary quadratically near the minimum.

\[
\chi^2 \approx a_j^2
\]

\[
\begin{array}{c}
\chi^2 \\
\chi_1^2 \\
\chi_2^2 \\
\chi_3^2
\end{array}
\begin{array}{c}
a_j \\
a_j + \Delta a \\
a_j + 2\Delta a
\end{array}
\]

Estimate $\sigma_j = \sqrt{\chi^2 \Delta a / \sqrt{\chi_1^2 - 2\chi_2^2 + \chi_3^2}}$

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Intuitive understanding of meaning of $\chi^2$

... qualitative discussion

Definition: $\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - y_c(x_i)}{\sigma_i} \right)^2$
where \((x_i, y_i \pm \sigma_i)\) \(i = 1..N\).

\(y(x_i)\) is expected value of \(y\) calculated from theoretical model.

On average, expect that data should deviate from model prediction by \(\approx \sigma_i\)

→ assuming that

1. reasonable estimates of \(\sigma_i\) values.
2. Theoretical model used is valid.
Then should expect that

\[
\left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2 \sim \left( \frac{\sigma_i}{\sigma_i} \right)^2 = 1
\]

on average.

\[
\therefore \chi^2 = \sum_{i=1}^{n} \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2 = N \text{ no. of meas.}
\]

So assuming \( \sigma_i \) reasonable, if \( \chi^2 \approx N \) then model captures features of data

\( \Rightarrow \chi^2 \) is "goodness of fit" parameter.
In this scenario, pts within red box result in
\[ \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2 > 1 \]

\( \chi^2 \) will be greater than \( N \).

A \( \chi^2 \) value significantly greater than \( N \) indicates a poor fit to the data.

A \( \chi^2 \) value significantly less than \( N \)
\[ \chi^2 = \sum_i \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2 \]
usually indicates that the \( \sigma_i \) has overestimated the values of \( \sigma_i \).
Expecting $\chi^2 = N$ is correct when have $N$ large. However, consider the following extreme cases.

1 parameter fit with $N=1$ (single meas.)

$(x_1, y_1 \pm \sigma_1)$

Model $y = a$

Best-fit value of $a$ based on the data is $a = \bar{y}_1 \leftarrow$ unknown parameter estimated from our single measurement.
Calc. \[ \chi^2 = \frac{y_i - y(x_i)}{\sigma_i} = \frac{y_i - y_1}{\sigma_1} = 0 \]

If, on the other hand, we used the true value \( a^* \) to calc. \( \chi^2 = \chi_a^2 \) we would find

\[ \chi_a^2 = \left( \frac{y_i - a^*}{\sigma_i} \right)^2 \approx 1 \]

Find that \( \chi^2 \) is artificially low.

Issue is that the meas. data is artificially set to be close to the parameter value since the parameter was estimated using the experimental data.

2 parameter fit (straight line) w/ \( N=2 \).

\[ (x_1, y_i \pm \sigma_i) \] model is \( y = a + b \times \)

\[ (x_2, y_2 \pm \sigma_2) \]
would calculate \[ \chi^2 = \sum_{i=1}^{n} \left( \frac{y_i - y(x_i)}{\sigma_i} \right)^2 = 0 \]

We would then calculate \[ \chi^2_{\text{true}} = \sum_{i=1}^{n} \left( \frac{y_i - (a^* + b^* x_i)}{\sigma_i} \right)^2 \approx 2 \]

Recall standard dev. \[ s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2 \]

\[ \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \]
If fit $N$ data pts to a model w/ $m$ parameters, then only $N-m$ of the pts are independent.

$\nu = N-m$ is called the number of degrees of freedom.

Expected value of $\chi^2$ is

$$\chi^2 \sim \nu = N-m$$

or $\chi^2 = \frac{\chi^2}{\nu} = 1$ if have reasonable est. of $\bar{\sigma}_i$' and theoretical model properly describes data, expect $\chi^2 = 1$.

"reduced $\chi^2$"

If $\chi^2 > 1$ ... model does not properly capture all features of data

$\chi^2 < 1$ ... probably overest. the $\bar{\sigma}_i$ values.

$\Rightarrow \chi^2$ is goodness of fit parameter.