PHYS 232: Monte Carlo Integration

- useful for numerical integration

Monte Carlo method use repeated random sampling to obtain numerical results.

Example: Computing π.

\[ m_T : \text{total mass of all water} \]
\[ m_c : \text{mass of water in circle bucket} \]

\[ P_H = \frac{A_c}{A_s} = \frac{\pi a^2}{(2a)^2} = \frac{\pi}{4} \]

\[ P_H = \frac{m_c}{m_T} \]

\[ \frac{\pi}{4} = \frac{m_c}{m_T} \Rightarrow \pi = 4 \frac{m_c}{m_T} \]
Assume that we can produce random nos. uniformly drawn from the interval $[-1, 1]$

Repeatedly draw sample pts $\{x_i, y_i\}$ from $[-1, 1] \times [-1, 1]

Prob. that randomly drawn pt lands within a circle of radius 1 is:

$$P = \frac{\text{area of unit circle}}{\text{area of square}} = \frac{\pi (1)^2}{2 \times 2} = \frac{\pi}{4}$$
\[ \pi = 4 \frac{Z_n}{n} \]

Have expressed our desired quantity \( \pi \) as a fun of a prob.

Est. \( p \) using Monte Carlo simulation.
Spray \( n \gg 1 \) points onto square

\[ p = \frac{Z_n}{n} \]
where \( Z_n \) is no. of pts landing within the circle
\[ \rightarrow \text{no. of hits (}Z_n\text{)}\]

What is uncertainty in value of \( \pi \)? Expect \( \sigma_\pi \downarrow \) as \( n \uparrow \).

Recall the binomial distn where \( \mu = np \)
\( p \) is prob. of success in any individual trial

\[ \sigma^2 = n p (1-p) \]
\[
\begin{align*}
\bar{Z}_n &= np \\
\sigma^2_{\bar{Z}_n} &= np(1-p) \\
\bar{q} &= 4 \frac{\bar{Z}_n}{n} \Rightarrow \text{prop. of errors} \\
\sigma_{\bar{q}} &= \frac{4}{n} \sigma_{\bar{Z}_n} \\
&= \frac{4}{n} \sqrt{np(1-p)} \\
&= 4 \frac{\sqrt{p(1-p)}}{\sqrt{n}} \\
\text{\therefore obtain } p &= \frac{\bar{Z}_n}{n} \text{ via simulation} \\
\sigma_{\bar{q}} &\propto \frac{1}{\sqrt{n}} \quad \therefore \text{as expected } \sigma_{\bar{q}} \downarrow \text{ as } n \uparrow.
\end{align*}
\]
Monte Carlo "Hit & Miss" Integration.

Assume that we want to evaluate

\[ \int_0^1 f(x) \, dx \]

Eq. \( f(x) = \frac{1}{2^7} \left(-65536 x^8 + 262144 x^7 - 409600 x^6 + 311296 x^5 - 114688 x^4 + 16384 x^3\right) \)

on \( 0 \leq x \leq 1 \), this fits within a unit square.

Use the same raindrop simulation. Prob. that a drop falls below the curve is area under the curve divided by area of square.
\[ P = \frac{A_{\text{curve}}}{h(b-a)} \]

\[ = \int_a^b f(x) \, dx \cdot \frac{z_n}{h(b-a)} = \frac{z_n}{n} \]

\[ \therefore I = \int_a^b f(x) \, dx = h(b-a) \cdot \frac{z_n}{n} = \sqrt{h(b-a) \cdot p} \]

\[ \sigma_I = \frac{h(b-a)}{n} \cdot \sigma_{z_n} = \frac{h(b-a)}{n} \cdot \sqrt{np(1-p)} \]

\[ \sigma_I = \frac{b(b-a) \sqrt{p(1-p)}}{\sqrt{n}} \approx \frac{1}{\sqrt{n}} \]

\[ P = \frac{z_n}{n} \]