Stirling's Approximation:

Desired result: \( \ln(N!) \approx N \ln N - N \) for large \( N \)

\[ \text{Eg:} \quad \ln(50!) = 148.4 \]

\[ 50 \ln(50) - 50 = 145.6 \]

Simple Derivation:

\[ \ln(N!) = \ln \left[ N(N-1)(N-2) \cdots (1) \right] \]
\[ = \ln(N) + \ln(N-1) + \ln(N-2) + \cdots + \ln(1) \]
\[ = \sum_{x=1}^{N} \ln(x) \]

Approximation sum as an integral

\[ \ln(N!) = \sum_{x=1}^{N} \ln(x) \approx \int_{1}^{N} \ln(x) \, dx \]

Integrate \( \ln(x) \) by parts. Recall \( \int u \, dv = uv - \int v \, du \)

\( u = \ln x \), \( dv = dx \)
\[ du = \frac{dx}{x} \quad \therefore \quad v = x \]
\[ uv = x \ln x \]
\[ vdu = x \frac{dx}{x} = dx \]

\[ \therefore \ln(N!) \approx \int_{1}^{N} \ln(x) \, dx = x \ln x \bigg|_{1}^{N} - \int_{1}^{N} \frac{dx}{x} \]
\[ \approx \ln(N) - \ln(1) = \ln(N) - 0 = \ln(N) \approx N \ln N - N \]

\[ \therefore \ln(N!) \approx N \ln N - N \]
Can do better with a little more work.

Can write factorials in terms of the $\Gamma$ function:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = (z-1)!$$

\[ \therefore \text{we first verify the expression in the box.} \]

Evaluate $\int_0^\infty e^{-t} t^2 dt$ using integration by parts.

Take $u = t^2$ \hspace{1cm} $dV = e^{-t} dt$

$dV = e^{-t} dt$ \hspace{1cm} $V = -e^{-t}$

$$\int_0^\infty e^{-t} t^2 dt = -e^{-t} t^2 \bigg|_0^\infty - \int_0^\infty (-e^{-t}) 2t \ dt$$

$e^{-t}$ goes to zero faster than $t^2 \to \infty$ for large $t$

(Use L'Hopital's rule)

This term goes to zero

$$\therefore \Gamma(z+1) = z \Gamma(z)$$
Integrate by parts again:

\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \]

\[ u = t^{z-1}, \quad dv = e^{-t} \, dt \]

\[ du = (z-1) t^{z-2} \, dt, \quad v = -e^{-t} \]

\[ \Gamma(z) = \left. -e^{-t} t^{z-1} \right|_0^\infty - \int_0^\infty (-e^{-t})(z-1) t^{z-2} \, dt \]

\[ = (z-1) \Gamma(z-1) \]

Have shown so far that

\[ \Gamma(z+1) = z(z-1) \Gamma(z-1) \]

can continue in this fashion. If take \( z \) to be integer, eventually get to:

\[ \Gamma(z+1) = z(z-1)(z-2) \cdots (2)(1) \Gamma(1) \]

\[ \Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t} \bigg|_0^\infty = -1 - (-1) = 1 \]
So we have managed to show that:

\[ \Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = z! \]

Now back to Stirling's Approx.

\[ N! = \int_0^\infty e^{-t} t^N dt \]

note that \( t^N = e^{N \ln t} \)

\[ \therefore N! = \int_0^\infty e^{N \ln t - t} dt \]

make a substitution \( t = N + N^{1/2} x \)

\[ \therefore dt = N^{1/2} dx \]

limits:

- when \( t = 0 \), \( x = -N^{1/2} \)
- \( t = \infty \), \( x = \infty \)

\[ N! = \int_{-\sqrt{N}}^{\infty} \exp \left\{ N \ln (N + \sqrt{N} x) - N - \sqrt{N} x \right\} \sqrt{N} dx \]

\[ = \int_{-\sqrt{N}}^{\infty} \exp \left\{ N \ln \left[ N \left(1 + \frac{x}{\sqrt{N}}\right)\right] - N \left(1 + \frac{x}{\sqrt{N}}\right) \right\} \sqrt{N} dx \]
\[ N! = \sqrt{N} \int_{-\frac{1}{\sqrt{N}}}^{\infty} \exp \left\{ N \ln N + N \ln \left(1 + \frac{x}{\sqrt{N}}\right) - N \left(1 + \frac{x^2}{2N}\right)^2 \right\} \, dx \]

\[ = \sqrt{N} N^N e^{-N} \int_{-\frac{1}{\sqrt{N}}}^{\infty} \exp \left\{ N \ln \left(1 + \frac{x}{\sqrt{N}}\right) - \sqrt{N} x^2 \right\} \, dx \]

Now we make our first approx.

Recall that \( \ln(1 + s) \approx s - \frac{s^2}{2} + \ldots \)

Since \( N \) is large (we can take \( N \) to be large) we can approx

\[ \ln \left(1 + \frac{x}{\sqrt{N}}\right) \approx \frac{x}{\sqrt{N}} - \frac{x^2}{2N} \]

\[ \text{s.t.} \quad N \ln \left(1 + \frac{x}{\sqrt{N}}\right) - \sqrt{N} x \]

\[ \approx \sqrt{N} x - \frac{x^2}{2} - \sqrt{N} x = -\frac{x^2}{2} \]

\[ \therefore N! \approx \sqrt{N} N^N e^{-N} \int_{-\frac{1}{\sqrt{N}}}^{\infty} e^{-x^2/2} \, dx \]

Our next approx is to take \( -\frac{1}{\sqrt{N}} \) as \( -\infty \) in integration limit. Not too bad of an approx since \( e^{-x^2/2} \) very quickly goes to zero away from \( x=0 \).
\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx \quad \text{is a Gaussian integral}
\]
\[
= \sqrt{2\pi}
\]
\[
\therefore \quad N! \approx \sqrt{2\pi N} N^N e^{-N} \quad \text{Stirling's Approx}
\]
\[
\ln(N!) \approx \ln \left( \sqrt{2\pi N} N^N e^{-N} \right)
\]
\[
\ln(N!) = N \ln N - N + \frac{1}{2} \ln(2\pi N)
\]

Check \( N = 50 \)
\[
50 \ln 50 - 50 + \frac{1}{2} \ln(2\pi \cdot 50) = 148.5 \quad \text{very close to actual answer!}
\]

Actually, this form of Stirling's approx. is pretty good even for small nos.
\[
\ln(3!) = \ln(6) = 1.79
\]
\[
3 \ln 3 - 3 + \frac{1}{2} \ln(2\pi \cdot 3) = 1.76
\]

For very large nos., as are common in Stat. Mech, the \( \ln(2\pi N) \) term is small c.f. the other two terms s.t.
\[
\ln(N!) \approx N \ln N - N
\]