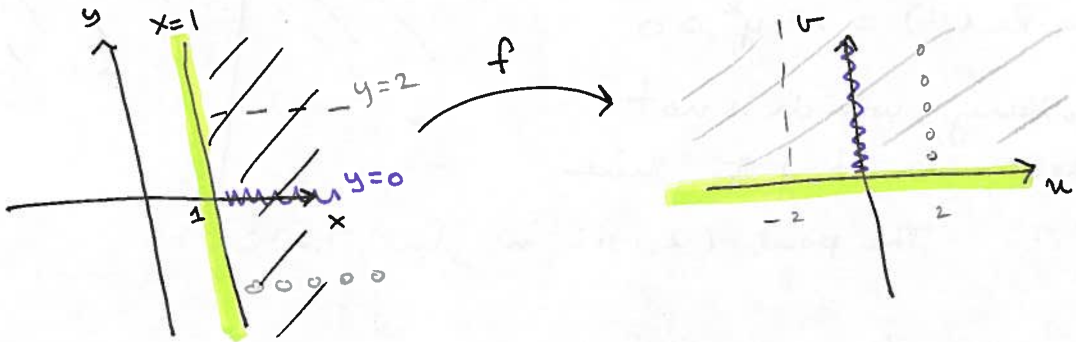
Math 350 Winter 2016

Assignment #2 - Solutions

#1 (a). $f(z) = iz - i = i(z-1)$

$\operatorname{Re}(z) \geq 1$



Let $z = x+iy$ then
$$\begin{aligned} f(z) &= i(z-1) \\ &= i(x+iy-1) \\ &= ix - y - i \\ &= -y + i(x-1) \end{aligned}$$

the line $x=1, y \in \mathbb{R}$ gets mapped to $-y$ - all pure real.

the line $y=0, x \geq 1$ gets mapped to $i(x-1)$ for $x \geq 1$ $\Rightarrow$ positive pure imaginary numbers.

The line $y=2, x \geq 1$ gets mapped to

$-2 + i(x-1)$

$u = -2$ $v \Rightarrow$ positive ~~imaginary~~.

the line $y=-2, x \geq 1$ gets mapped to

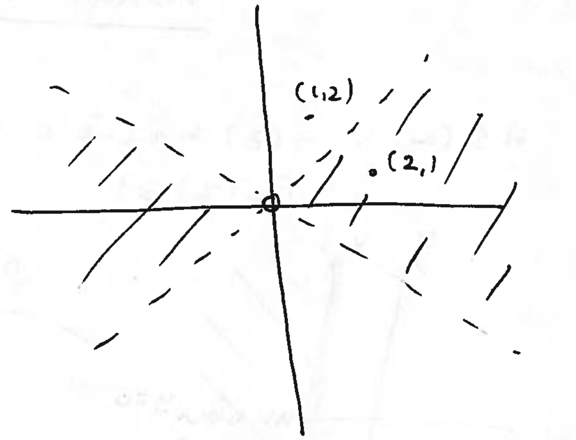
$2 + i(x-1)$

$u = 2$ v is positive

So the region $\operatorname{Re}(z) \geq 1$ gets mapped to

$\{z = x+iy \in \mathbb{C} \mid y \geq 0, x \in \mathbb{R}^{+}\}$

(b) $\{z \in \mathbb{C} \mid Re(z^2) > 0\}$



Set $z = x + iy$

$$z^2 = x^2 - y^2 + i2xy$$

So $Re(z^2) = x^2 - y^2 > 0$

the boundary, which is not included, consist of the lines $y = \pm x$. The point $(2,1)$ is in, but $(1,2)$ isn't

2. Use the formal definition of limits to show that
$$\lim_{z \to i} \frac{z^2+1}{z-i} = 2i$$

Proof: Let $\varepsilon > 0$, we need to show that there exists $\delta > 0$ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.
Here $z_0 = i$ and $w_0 = 2i$.

So
$$|f(z) - w_0| = \left| \frac{z^2+1}{z-i} - 2i \right| = \left| \frac{(z-i)(z+i)}{z-i} - 2i \right|$$
$$= |z + i - 2i| = |z - i| < \delta.$$

Choose $\delta = \varepsilon/2$

Then

$$|f(z) - w_0| = |z - i| < \delta < \varepsilon \quad \blacksquare$$

3. Use the definition of the derivative to show that $f(z) = \text{Im}(z)$ is nowhere analytic.

By defn

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\text{Im}(z_0 + \Delta z) - \text{Im}(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\text{Im}\, z_0 + \text{Im}\, \Delta z - \text{Im}(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\text{Im}\, \Delta z}{\Delta z}$$

We set $\Delta z = \Delta x + i\Delta y$ and take the limit along two different path.

(1) Along the horizontal line - $\Delta y = 0$, $\Delta z = \Delta x$

$$\lim_{\Delta z \to 0} \frac{\text{Im}\, \Delta z}{\Delta z} = \lim_{\Delta x \to 0} \frac{\cancelto{0}{\text{Im}\, \Delta x}}{\Delta x} = 0$$

(2) Along a vertical line $\Delta x = 0$ $\Delta z = i\Delta y$

$$\lim_{\Delta z \to 0} \frac{\text{Im}\, \Delta z}{\Delta z} = \lim_{\Delta y \to 0} \frac{\text{Im}(i\Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1$$

Since the limit along two different path is different the derivative doesn't exist.

Note that the limit above is independent of z_0

so the function is nowhere differentiable

4. ($\Rightarrow$) Assume $\lim_{z \to z_0} f(z) = w_0$

Consider

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = \lim_{z \to z_0} \operatorname{Re}(f(z))$$

$$= \lim_{z \to z_0} \frac{f(z) + \overline{f(z)}}{2} = \frac{1}{2} \lim_{z \to z_0} \left(f(z) + \overline{f(z)}\right)$$

Since $\lim_{z \to z_0} f(z) = w_0$ implies $\lim_{z \to z_0} \overline{f(z)} = \overline{w_0}$

both limits exists so

$$= \frac{1}{2} \lim_{z \to z_0} f(z) + \frac{1}{2} \lim_{z \to z} \overline{f(z)}$$

$$= \frac{1}{2} w_0 + \frac{1}{2} \overline{w_0} = \frac{1}{2}(w_0 + \overline{w_0}) =$$

$$= \operatorname{Re}(w_0) = u_0$$

Similarly, we use the def^{n} of $\operatorname{Im}(f(z))$ to show $v(x,y) \to v_0$

($\Leftarrow$) Assume $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$ & $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$

Consider $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} (u(x,y) + i v(x,y))$

by assumption Since both limits exists

$$= \lim_{z \to z_0} u(x,y) + i \lim_{z \to z_0} v(x,y)$$

$$= u_0 + i v_0 = w_0$$

$\blacksquare$

#5 - Use Cauchy-Riemann equations to show that the function $f(z) = x^2 + iy^2$ is nowhere analytic.

Where is f differentiable?

we have $u(x,y) = x^2$ $\Rightarrow$ $u_x = 2x$, $u_y = 0$

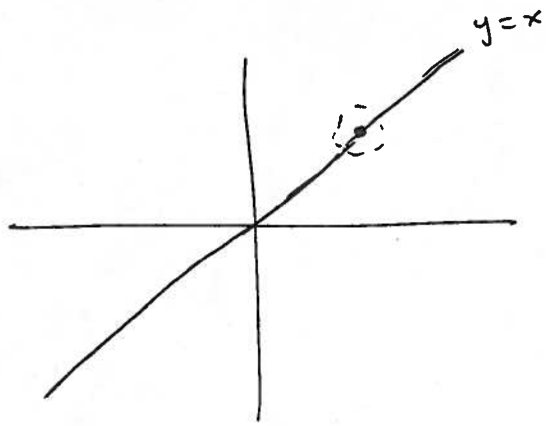
$v(x,y) = y^2$ $\qquad v_x = 0$, $v_y = 2y$

u, v, u_x, v_x, u_y, v_y are all continuous for all x, y.

We need $u_x = v_y \iff 2x = 2y$ or $x = y$

and $v_x = -u_y \iff 0 = 0$

so $f(z)$ is differentiable at points on the line $y = x$



but for any points on the line $y = x$, every neighborhood will contains points where f is not differentiable

So f is nowhere analytic.

#5. Use Cauchy Riemann equation To show that the function $f(z) = x^2 + iy^2$ is nowhere analytic. Where is f differentiable?

$u(x,y) = x^2$ $\quad u_x = 2x \;;\; u_y = 0$

$v(x,y) = y^2$ $\quad v_x = 0 \;;\; v_y = 2y$

u, v, u_x, u_y, v_x, v_y are all continuous for all x,y.

CR need $u_x = v_y$ so $2x = 2y$ or $x = y$

Similarly $-0 = 0$

so $f(z)$ is differentiable only on the line $y = x$