Assignment #3 - Solutions

#1 - Give the complex form of $e^{\sqrt{i}}$.

First consider $\sqrt{i} = i^{1/2} = e^{1/2 \log i}$

where $\log i = \cancel{\text{Log}\, 1}^{\,0} + i \,\text{Arg}\, i + 2k\pi i = \frac{i\pi}{2} + i2k\pi$

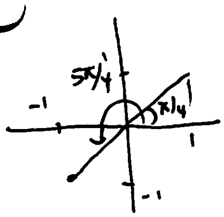
since $i^{1/2} = z^{\alpha}$ with $\alpha = \frac{1}{2} \in \mathbb{Q}$ we expect two distinct solutions so $k = 0, 1$.

For $k=0 \Rightarrow \sqrt{i} = e^{1/2(\frac{i\pi}{2})} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

For $k=1 \Rightarrow \sqrt{i} = e^{1/2(\frac{i\pi}{2} + 2\pi i)} = e^{i\pi/4} e^{i\pi} = e^{i5\pi/4}$

we can use the fact that $e^{i\pi} = \cos\pi + i\sin\pi = -1$

then $= -e^{i\pi/4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$



or evaluate $e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Then $e^{\sqrt{i}} = e^{1/\sqrt{2}} e^{i/\sqrt{2}} = e^{1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)$

and $e^{\sqrt{i}} = e^{-1/\sqrt{2}} e^{-i/\sqrt{2}} = e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)$

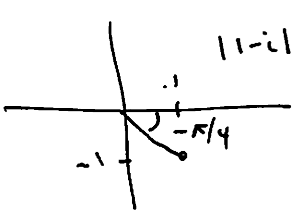
#2 Evaluate

(a) $(1-i)^{1+i}$ we use the formula $z^{\alpha} = e^{\alpha \log z}$

we have $(1-i)^{1+i} = e^{(1+i)\log(1-i)}$

where $\log(1-i) = \text{Log}\,|1-i| + i\,\text{Arg}(1-i) + 2k\pi i$

$= \text{Log}\,\sqrt{2} - i\frac{\pi}{4} + 2k\pi i$



and $(1+i)\log(1-i) = \text{Log}\sqrt{2} - \frac{i\pi}{4} + i2k\pi + i\,\text{Log}\sqrt{2} + \frac{\pi}{4} - 2k\pi$

$$= \text{Log}\sqrt{2} + \frac{\pi}{4} - 2k\pi + i\left(\text{Log}\sqrt{2} - \frac{\pi}{4} + 2k\pi\right)$$

So $e^{(1+i)\log(1-i)} = e^{\text{Log}\sqrt{2} + \frac{\pi}{4} - 2k\pi}\, e^{i\left(\text{Log}\,2 - \frac{\pi}{4} + 2k\pi\right)}$

note that $e^{i2k\pi} = 1$

so we can write

$(1-i)^{1+i} = e^{\text{Log}\sqrt{2} + \frac{\pi}{4} - 2k\pi}\, e^{i\left(\text{Log}\,2 - \frac{\pi}{4}\right)}$ for $k \in \mathbb{Z}$

there are ∞ many values!!

(b) Evaluate $\sinh(1+\pi i)$

We use the fact that $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$ with $z_1 = 1$ and $z_2 = \pi i$ (see text p115)

$$\sinh \pi i = \frac{e^{\pi i} - e^{-\pi i}}{2} = i\,\frac{e^{i\pi} - e^{-i\pi}}{2i} = i \sin\pi = 0$$

$$\cosh \pi i = \frac{e^{\pi i} + e^{-\pi i}}{2} = \cos\pi = -1$$

so $\sinh(1+\pi i) = -\sinh 1$.

alternatively, write $\sinh(1+\pi i) = \dfrac{e^{1+\pi i} - e^{-(1+\pi i)}}{2} = \dfrac{e^1 e^{\pi i} - e^{-1} e^{-\pi i}}{2}$

but $e^{\pm\pi i} = \cos\pi \pm i\sin\pi = -1$

so $\sinh(1+\pi i) = -\dfrac{(e^1 - e^{-1})}{2} = -\sinh 1$

(c) Evaluate i^{i^i}

we have $i^i = e^{i \log i} = e^{i(\cancel{\operatorname{Log}}^{0}|i| + i\pi/2 + 2k\pi)}$ $k \in \mathbb{Z}$

$$= e^{-\pi/2} e^{i2k\pi} = e^{-\pi/2}$$

so $i^{(i^i)} = e^{i^i \log i}$

where $i^i \log i = e^{-\pi/2}(\cancel{\operatorname{Log}}^{0}|i| + \frac{i\pi}{2} + 2k\pi)$

So $i^{i^i} = e^{e^{-\pi/2}(\frac{i\pi}{2} + 2k\pi)}$

and the principal value is $i^{i^i} = e^{\frac{i\pi}{2} e^{-\pi/2}}$

#3. (a) First we find where $\sin \bar{z}$ is differentiable using the Cauchy-Riemann Equations.
we need to express $\sin \bar{z}$ in the form $u + iv$.
Let $z = x + iy$, then $\sin \bar{z} = \sin(x - iy)$

use $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$ with $z_1 = x$, $z_2 = iy$.

$\sin \bar{z} = \sin x \cos iy - \sin iy \cos z$

with $\cos iy = \dfrac{e^{i(iy)} + e^{-i(iy)}}{2} = \dfrac{e^{-y} + e^{y}}{2} = \cosh y$

$\sin iy = \dfrac{e^{i(iy)} - e^{-i(iy)}}{2i} = \dfrac{e^{-y} - e^{y}}{2i} = \dfrac{-(e^{y} - e^{-y})}{2i}$

$$= i\frac{(e^{y} - e^{-y})}{2} = i \sinh y$$

so $\sin \bar{z} = \sin x \cosh y - i \sinh y \cos x$

It follows that

$u(x,y) = \sin x \cosh y \Rightarrow u_x = \cos x \cosh y$, $u_y = \sin x \sinh y$

$v(x,y) = -\sinh y \cos x \Rightarrow v_x = \sin x \sinh y$, $v_y = -\cos x \cosh y$

the Cauchy-Riemann Equations are

① $u_x = v_y$

② $u_y = -v_x$

and f is differentiable a z_0 if Ⓐ u, v & all partial derivatives are continuous at z_0

and Ⓑ u & v satisfies the C-R. Eqs

u, v, and all partial derivatives are continuous on $\mathbb{C}$.

① $\cos x \cosh y = -\cos x \cosh y \iff \cos x = 0$ (i.e $\cosh y \neq 0$ for all y)

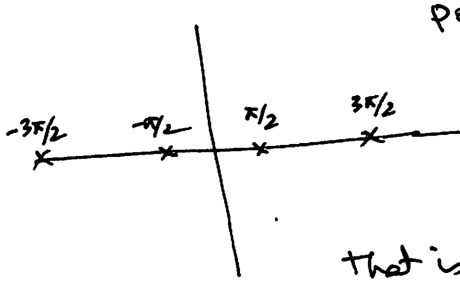
$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

② $\sin x \sinh y = -\sin x \sinh y \iff$ either $\sin x = 0$ or $\sinh y = 0$.

but $\sin x$ & $\cos x$ are never both zero

so $\sinh y = 0 \iff y = 0$

We conclude that $\sin z$ is differentiable at the points $y=0$ and $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$



(b) Since these are isolated points we conclude that $\sin z$ is nowhere analytic.

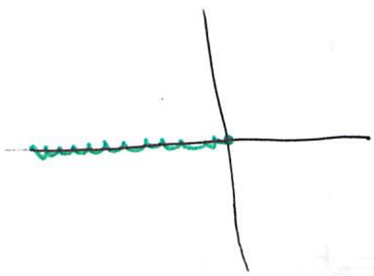
that is there are no open sets for which $\sin z$ is differentiable

#4 (a) Find the largest domain of analycity for $f(z) = \text{Log}(z^2)$

By def^n, $\text{Log}\, w$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$

i.e everywhere except on the branch cut.

The set determined by $\text{Re}\, w \leq 0$ & $\text{Im}\, w = 0$



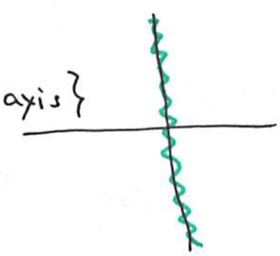
we need to determine the branch cut for $\text{Log}(z^2)$

We need to solve $z^2 = -c$ for $c \geq 0$, $c \in \mathbb{R}$

so $z = \pm\sqrt{-c} = \pm i\sqrt{|c|}$

this corresponds to every point on the imaginary axis.

the branch cut for $\text{Log}(z^2) = \mathbb{C} \setminus \{\text{Im axis}\}$



Alternatively, set $z = x + iy$.

Then $z^2 = (x+iy)^2$

$= x^2 + 2ixy - y^2$

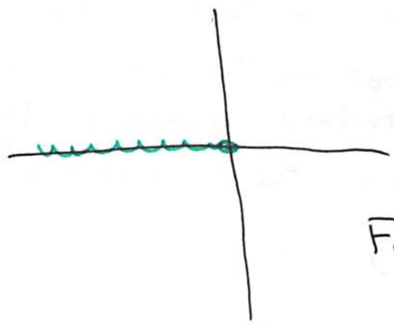
$= x^2 - y^2 + i\,2xy$

we need $x^2 - y^2 \leq 0$ and $2xy = 0$.

Take $x = 0$ and $y^2 = c$ or $y = \pm\sqrt{c}$

4(b) $f(z) = \mathrm{Log}(\mathrm{Log}(z))$

again we use the fact that $\mathrm{Log}\, w$ is analytic in $\mathbb{C}$ everywhere except on the branch cut



For $\mathrm{Log}(\mathrm{Log}\, z)$

we need $\mathrm{Log}\, z$ to be analytic so point on the usual branch cut must be excluded.

and $\mathrm{Log}(w)$ with $w = \mathrm{Log}\, z$ to be analytic.

recall $w = \mathrm{Log}\, z = \mathrm{Log}|z| + i\,\mathrm{Arg}\, z$

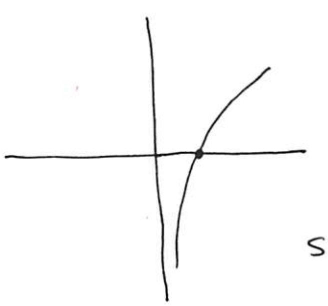
so $\mathrm{Re}\, w = \mathrm{Log}|z| \leq 0$ and $\mathrm{Im}\, w = \mathrm{Arg}\, z = 0$

this is the "real" natural log.

$\Downarrow$

if $z = x + iy \Rightarrow y = 0$ & $x \geq 0$

so z is pure real



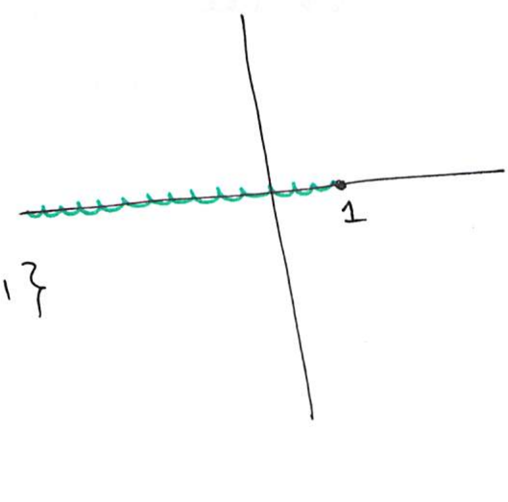
$\ln x \leq 0$ for $0 < x < 1$

so

$0 \leq |z| \leq 1$

so we must also exclude the set $y = 0$ and $0 \leq |x| \leq 1$

The new branch cut is



$D = \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Im}\, z = 0 \text{ and } \mathrm{Re}\, w \leq 1\}$

#5. Find all values of z for which $\cosh z = \frac{1}{2}$.

We write $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

then $\cosh z = \dfrac{e^{z}+e^{-z}}{2} = \dfrac{1}{2} \iff e^{z} + e^{-z} - 1 = 0$

since $e^{z} \neq 0$ $\iff e^{2z} + 1 - e^{z} = 0$

set $w = e^{z}$ $\iff w^2 - w + 1 = 0$

use the quadratic equation $w = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$

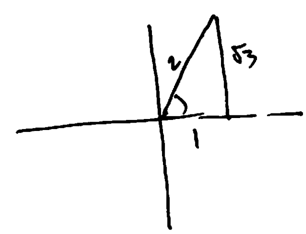
we need $e^{z} = w = \dfrac{1 \pm i\sqrt{3}}{2}$
To solve for z

apply log to both sides

$$z_1 = \log\left(\frac{1+i\sqrt{3}}{2}\right) = \operatorname{Log}\left|\frac{1+i\sqrt{3}}{2}\right| + i\operatorname{Arg}\left(\frac{1+i\sqrt{3}}{2}\right) + i2k\pi \quad k \in \mathbb{Z}$$

$$= \operatorname{Log} 1 + i\frac{\pi}{3} + i2k\pi$$

$$= i\left(\frac{\pi}{3} + 2k\pi\right)$$



$$z_2 = \log\left(\frac{1-i\sqrt{3}}{2}\right) = \operatorname{Log}\left|\frac{1-i\sqrt{3}}{2}\right| + i\operatorname{Arg}\left(\frac{1-i\sqrt{3}}{2}\right) + i2k\pi \quad k \in \mathbb{Z}$$

$$= -\frac{i\pi}{3} + i2k\pi$$

$$= i\left(-\frac{\pi}{3} + 2k\pi\right)$$